A note on Gaussian integrals over paragrassmann variables

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Abstract

We discuss the generalization of the connection between the determinant of an operator entering a quadratic form and the associated Gaussian path-integral valid for grassmann variables to the paragrassmann case [$\theta^{p+1} = 0$ with $p = 1$ ($p > 1$) for grassmann (paragrassamann) variables]. We show that the q-deformed commutation relations of the paragrassmann variables lead naturally to consider q-deformed quadratic forms related to multiparametric deformations of $GL(n)$ and their corresponding q-determinants. We suggest a possible application to the study of disordered systems.

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Using anticommuting functions as integration variables, Matthews and Salam showed that the path-integral for a system of relativistic fermions in an external field gives the determinant of the Dirac operator [1]. That is, the fermionic partition function does not yield a negative power of the determinant, as in the bosonic case, but a positive power, $p = +1$. Ten years later, Berezin completed his analysis of noncommutative algebras and fermion systems, making clear that the natural framework to define fermionic path-integrals was that of Grassmann algebras [2].

The main ingredient behind the result

$$\int \prod_{i=1}^{n} d\theta_i d\bar{\theta}_i \exp\left(\bar{\theta}_i A_{ij} \theta_j\right) = \det A \tag{1}$$

when one integrates over two sets of n Grassmann variables θ_i and $\bar{\theta}_i$ are the anticommutation rules

$$[\theta_i, \theta_j]_+ = 0 \ , \qquad [\bar{\theta}_i, \bar{\theta}_j]_+ = 0 \ , \tag{2}$$

for all pairs i,j, that imply $\theta_i^2 = \bar{\theta}_i^2 = 0$ for all i. In Eq. (1) we neglected an irrelevant factor related to the definition of the integration measure. The summation rule over repeated indices is used henceforth. Notice that the condition $[\bar{\theta}_i, \theta_j]_+ = 0$, which is usually imposed, is not necessary for the validity of Eq. (1). In fact, a relation of the form $\bar{\theta}_i \theta_j = \alpha \ \theta_j \bar{\theta}_i$, with α some c-number, yields the same result (modulo an irrelevant normalization).

In order to construct a path-integral representation of the p-th power of a determinant it seems natural to use p-paragrassmann variables such that

$$\theta_i^{p+1} = \bar{\theta}_i^{p+1} = 0 \ , \quad \text{for all } i \ . \tag{3}$$

Consistent integration rules for θ_i and $\bar{\theta}_i$ take the form (see for instance [3, 4, 5])

$$\int d\theta_i \ \theta_i^r = \mathcal{N} \ \delta_{r,p} \ , \qquad \int d\bar{\theta}_i \ \bar{\theta}_i^r = \bar{\mathcal{N}} \ \delta_{r,p} \ , \tag{4}$$

where $\mathcal{N}$ and $\bar{\mathcal{N}}$ are two complex numbers that, without loss of generality, we set to 1. Indeed, one can easily see that the Gaussian integral of a *diagonal* form, $\bar{\theta}_i A_{ij} \theta_j = \bar{\theta}_i \lambda_i \theta_i$, that is quadratic in the pair of p-paragrassmann variables θ_i and $\bar{\theta}_i$, leads to the p-th power of the product of the diagonal elements,

$$\int \prod_{i=1}^{n} d\theta_i d\bar{\theta}_i \exp\left(\bar{\theta}_i A_{ij} \theta_j\right) = \left(\prod_{i=1}^{n} \lambda_i\right)^p = (\det A)^p \ . \tag{5}$$

Here and in what follows we use the ordinary definition of the exponential, $e^x \equiv \sum_{m=0}^{\infty} x^m/m!$.

However, contrary to what seems to be accepted in the literature, it is not straightforward to obtain an analogous result whenever the quadratic form is not diagonal. The reason is that the change of variables needed to bring A to a diagonal form, spoils the commutation rules of the paragrassmann variables, unless $p = 1$. Thus, in order to define a consistent path-integral for paragrassmann variables, one has to take into account paragrassmann changes of variables (a fact that, to our knowledge, has not been discussed in the literature [4]-[11]).

It is the purpose of this work to fill this gap by developing a consistent framework to integrate paragrassmann variables that allows one to deal with Gaussian integrals of non-diagonal quadratic forms. We shall see that the quantum group $GL_{q,q'}(n)$ (with qq' a primitive root of unity) enters naturally into play.

Let us start by fixing the commutation rules for the θ_i variables among themselves. For simplicity we consider just two variables θ_1 and θ_2 and impose the following q-commutation rule

$$\theta_1 \cdot \theta_2 = q\, \theta_2 \cdot \theta_1 \ , \tag{6}$$

with q a c-number. Usually, see for instance [10], this number is taken to be a primitive root of unity, $q^{p+1} = 1$, $q^m \neq 1$ for all $m < p+1$, but this condition is not necessary to define a consistent integral and we shall not impose it.

We now consider a linear change to new variables

$$\begin{aligned} \omega_1 &= a\, \theta_1 + b\, \theta_2 \ , \\ \omega_2 &= c\, \theta_1 + d\, \theta_2 \ , \end{aligned} \tag{7}$$

with a, b, c, d certain in principle non-commuting parameters that commute with θ_1 and θ_2 and that we encode in a 2×2 matrix

$$A = \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \ . \tag{8}$$

Suppose that

$$\begin{aligned} a \cdot b = q'\, b \cdot a \ , \\ c \cdot d = q'\, d \cdot c \ , \end{aligned}$$

with q' a second complex parameter. We want the ω's to have the same commutation properties as the θ's. It is easy to show that, defining

$$z = qq' \ , \tag{9}$$

we obtain

$$(a\,\theta_1 + b\,\theta_2)^m = \sum_{l=0}^{m} \binom{m}{l}_z b^{m-l} \cdot a^l \; \theta_2^{m-l} \cdot \theta_1^l \tag{10}$$

with

$$\binom{m}{l}_z \equiv \frac{(m)_z!}{(l)_z!(m-l)_z!} \ , \qquad (m)_z \equiv \frac{1-z^m}{1-z} \ ,$$

$$(m)_z! = (m)_z (m-1)_z! \ , \qquad 1_z = 1 \ .$$

Then, conditions

$$\omega_1^{p+1} = \omega_2^{p+1} = 0 \tag{11}$$

require z to be a primitive root of unity,

$$z^{p+1} = 1 \ , \qquad z^m \neq 1 \quad \text{for all} \;\; m < p+1 \ . \tag{12}$$

Note that q and q' are not fixed separately to be roots of unity.

If we further enforce the analogous to condition (6),

$$\omega_1 \cdot \omega_2 = q\, \omega_2 \cdot \omega_1 \ ,$$

the following commutation rules between the coefficients in the change of variables (7) should hold

$$\begin{aligned} i) \qquad & a \cdot c = q\, c \cdot a \ , \\ ii) \qquad & b \cdot d = q\, d \cdot b \ , \\ iii) \qquad & a \cdot d - d \cdot a - q'\, c \cdot b + 1/q\, b \cdot c = 0 \ . \end{aligned} \tag{13}$$

Already at this point we see that a, b, c, d cannot be ordinary c-numbers. Indeed, this would imply $q = q' = 1$ but condition (12) requires $(qq')^m \neq 1$ for any $m < p+1$.

Conditions (13) do not exhaust the commutation rules among the coefficients a, b, c, d. If we impose

$$b \cdot c = q/q'\; c \cdot b \ , \tag{14}$$

relation $iii)$ becomes

$$iii) \qquad a \cdot d - d \cdot a = (q' - 1/q)\; b \cdot c = (q - 1/q')\; c \cdot b\ . \tag{15}$$

The resulting commutation rules imply that A belongs to the quantum group $GL_{q,q'}(2)$ (some of its properties are discussed in Appendix A).

At this point we can compute the Jacobian J associated with the transformations (7) via

$$d\omega_1 \cdot d\omega_2 = J d\theta_1 \cdot d\theta_2\ . \tag{16}$$

Since ω_1 and ω_2 are p paragrassmann variables their integration yields

$$\int d\omega_1 \cdot d\omega_2\ \cdot \omega_1^r \cdot \omega_2^s = \delta_{rp}\delta_{sp} \tag{17}$$

where we have omitted the overall normalization constant. Consider now the integral

$$1 = \int d\omega_1 \cdot d\omega_2 \cdot \omega_1^p \cdot \omega_2^p = J \int d\theta_1 \cdot d\theta_2 \cdot (a\ \theta_1 + b\ \theta_2)^p \cdot (c\ \theta_1 + d\ \theta_2)^p\ . \tag{18}$$

The inverse of the Jacobian is then given by

$$J^{-1} = \int d\theta_1 \cdot d\theta_2 \cdot (a\ \theta_1 + b\ \theta_2)^p \cdot (c\ \theta_1 + d\ \theta_2)^p \equiv \mathcal{I}\ . \tag{19}$$

Using the expression (10) and the integration rules (4) it is easy to show that the only contribution to the integral $\mathcal{I}$ comes from terms having p powers of θ_1 and p powers of θ_2,

$$\begin{aligned} \mathcal{I} &= \sum_{l=0}^{p} \binom{p}{l}_z^2 b^{p-l} \cdot a^l \cdot d^l \cdot c^{p-l} \int d\theta_2 \cdot d\theta_1 \cdot \theta_2^{p-l} \cdot \theta_1^l \cdot \theta_2^l \cdot \theta_1^{p-l} \\ &= \sum_{l=0}^{p} \Gamma_l^p\ a^l \cdot d^l \cdot b^{p-l} \cdot c^{p-l} \end{aligned} \tag{20}$$

with

$$\Gamma_l^p \equiv \frac{1}{z^{l(p-l)}} \binom{p}{l}_z^2 q^{-(p-l)^2}\ . \tag{21}$$

Here we have used the commutation rules given above for a, b, c, d and normalized the integral of the θ and $\bar{\theta}$ variables as in Eq. (17). After some algebra, Eq. (20) can be accommodated as

$$\mathcal{I} = (a \cdot d - q'\ b \cdot c)^p \equiv \Delta^p\ . \tag{22}$$

Indeed, writing

$$(a \cdot d - q' \, b \cdot c)^p = \sum_{l=0}^{p} \Lambda_l^p \; a^l \cdot d^l \cdot b^{p-l} \cdot c^{p-l} \tag{23}$$

we can show that the coefficients Λ_l^p coincide with the coefficients Γ_l^p in Eqs. (20) and (21). To this end, using $\Delta \equiv a \cdot d - q' \, b \cdot c$ and Eq. (52) we have

$$\Delta^{p+1} = \Delta^p \cdot (a \cdot d - q' \, b \cdot c) = a \cdot \Delta^p \cdot d - q' \Delta^p \cdot b \cdot c \, . \tag{24}$$

and replacing Δ^p with the expression in Eq. (23) we obtain

$$\Delta^{p+1} = \sum_{l=0}^{p+1} a^l \cdot d^l \cdot b^{p+1-l} \cdot c^{p+1-l} \; z^{p+1-l} \; \left(\Lambda_{l-1}^p - q^{2l-2p-1} \Lambda_l^p \right) \tag{25}$$

where we defined $\Lambda_{-1}^p = \Lambda_{p+1}^p = 0$. Equation (25) determines the following recurrence relation for Λ:

$$\Lambda_l^{p+1} = z^{p+1-l} \left(\Lambda_{l-1}^p - q^{2l-2p-1} \Lambda_l^p \right) \, , \tag{26}$$

with the initial conditions $\Lambda_0^1 = -q'$, $\Lambda_1^1 = 1$ that is solved by

$$\Lambda_l^p = (-1)^{p-l} q^{-(p-l)^2} z^{(p-l)(p-l+1)/2} \binom{p}{l}_z \, . \tag{27}$$

When z is a primitive root of unity we have

$$\binom{p}{l}_z = (-1)^l z^{-l(l+1)/2} \, , \tag{28}$$

and Eq. (27) becomes

$$\Lambda_l^p = \frac{1}{z^{l(p-l)}} \binom{p}{l}_z^2 q^{-(p-l)^2} = \Gamma_l^p \tag{29}$$

completing the proof of the identity in Eq. (22).

Thus, the Jacobian of the linear change of variables (7) that is associated to an element $A \in GL_{q,q'}(2)$, is given by the inverse p-th power of the "q-determinant" of A:

$$J = (\det A)^{-p} \tag{30}$$

with the generalized determinant defined as

$$\det A \equiv a \cdot d - q' \, b \cdot c \ . \tag{31}$$

We shall now discuss the calculation of the Gaussian integral for a non diagonal quadratic form. In order to do it, we need to fix the complete paragrassmann algebra by defining the commutation rules between the $\bar{\theta}$'s and the θ's and those among different $\bar{\theta}$'s.

The simplest possibility is to demand that *independent* $GL_{q,q'}(2)$ transformations for the θ's and $\bar{\theta}$'s preserve the commutation relations, leading to the $\bar{\theta}$'s and θ's commuting with each other independently of their indices (up to a factor which for simplicity we take to be 1). That is

$$\bar{\theta}_i \cdot \theta_j = \theta_j \cdot \bar{\theta}_i \ , \quad \forall \, i, j \tag{32}$$

(notice that these conditions are not those imposed in Ref. [10]).

Regarding the commutation relations for the $\bar{\theta}_i$'s, we have two possible choices,

$$C1: \qquad \bar{\theta}_1 \cdot \bar{\theta}_2 = q^{-1} \, \bar{\theta}_2 \cdot \bar{\theta}_1 \ , \tag{33}$$

$$C2: \qquad \bar{\theta}_1 \cdot \bar{\theta}_2 = q \, \bar{\theta}_2 \cdot \bar{\theta}_1 \ . \tag{34}$$

Let us start by analyzing $C1$. Consider the integral

$$I = \int d\bar{\theta}_1 \cdot d\bar{\theta}_2 \cdot d\theta_1 \cdot d\theta_2 \cdot e^{\bar{\theta}_i A_{ij} \theta_j} \tag{35}$$

where A is a matrix belonging to $GL_{q,q'}(2)$. Changing the integration variables as

$$\theta_i \to \omega_i = A_{ij}\theta_j \ , \qquad \bar{\theta}_i \to \bar{\theta}_i \ , \tag{36}$$

the result (30) for the Jacobian and the fact that ω_i and $\bar{\theta}_j$ commute lead to

$$I = (\det A)^p \int d\bar{\theta}_1 \cdot d\bar{\theta}_2 \cdot d\omega_1 \cdot d\omega_2 \cdot e^{\bar{\theta}_i \cdot \omega_i} = (\det A)^p \tag{37}$$

whenever $qq' = 1$ is a primitive root of unity.

Since for the rules $C1$ the parameters q and q^{-1} play a dual role regarding the θ's and $\bar{\theta}$'s, it is natural to consider, apart from the case discussed above, the one corresponding to quadratic forms $A \in GL_{q'^{-1},q^{-1}}(2)$ (notice that

$q'^{-1}q^{-1}$ is also a primitive root of unity; the reason why the order of the deformation parameters is transposed will become clear immediately). The appropriate change of variables is in this case

$$\theta_i \to \theta_i \ , \qquad \bar{\theta}_i \to \bar{\omega}_i = \bar{\theta}_j A_{ji} \ . \tag{38}$$

This is a consistent change of variables only for $A^T \in GL_{q^{-1},q'^{-1}}(2)$ or $A \in GL_{q'^{-1},q^{-1}}(2)$ with $q'q$ a primitive root of unity. An argument similar to the one leading to Eq. (37) yields also in this case

$$\int d\bar{\theta}_1 \cdot d\bar{\theta}_2 \cdot d\theta_1 \cdot d\theta_2 \cdot e^{\bar{\theta}_i \cdot A_{ij} \cdot \theta_j} = (\det A)^p. \tag{39}$$

Now, let us consider the case $C2$ in which the commutation relations for the $\bar{\theta}$'s are identical to those among the θ's The integral of the diagonal form

$$\int d\bar{\theta}_1 \cdot d\bar{\theta}_2 \cdot d\theta_1 \cdot d\theta_2 \; e^{\bar{\theta}_1 \cdot \theta_1 + \bar{\theta}_2 \cdot \theta_2} \tag{40}$$

vanishes. Moreover, with an argument similar to the one used in the case $C1$ one proves that any integral of the form (35) with $A \in GL_{q,q'}$ (with qq' a primitive root of the unity) also vanishes. We can however find a non trivial result by noticing that $C2$ differs from $C1$ only by the exchange

$$\bar{\theta}_1 \to \bar{\theta}_2 \ , \qquad \bar{\theta}_2 \to \bar{\theta}_1 \ . \tag{41}$$

If we construct a quadratic form $\bar{\theta}_i \cdot K_{ij}\theta_j$ with a 2×2 matrix K with entries satisfying

$$\begin{aligned}
a \cdot b &= q'^{-1} \, b \cdot a \ , \\
a \cdot c &= q \, c \cdot a \ , \\
b \cdot d &= q \, d \cdot b \ , \\
c \cdot d &= q'^{-1} \, d \cdot c \ , \\
a \cdot d &= q/q' \, d \cdot a \ , \\
b \cdot c - c \cdot b &= (q' - 1/q) \, a \cdot d \ ,
\end{aligned} \tag{42}$$

we have

$$\int d\bar{\theta}_1 \cdot d\bar{\theta}_2 \cdot d\theta_1 \cdot d\theta_2 \cdot e^{\bar{\theta}_i \cdot K_{ij} \theta_j} = (\det K)^p \tag{43}$$

where

$$\det K \equiv a \cdot d - q \; c \cdot b \ . \tag{44}$$

The elements of the matrices K satisfy commutation relations preserved under simultaneous left $GL_{qq'}(2)$ and right $GL_{q^{-1}q'^{-1}}(2)$ rotations:

$$K_{ij} \to K'_{ij} = M_{il}\bar{M}_{sj}K_{ls} \ , \quad M \in GL_{qq'}(2), \ \bar{M} \in GL_{q^{-1}q'^{-1}}(2) \tag{45}$$

where $[M_{ij}, \bar{M}_{lm}] = [M_{ij}, K_{lm}] = [\bar{M}_{ij}, K_{lm}] = 0$ [14]. Notice nevertheless that, unlike the A matrices in $GL_{q,q'}(2)$, the K matrices cannot be diagonal. This is consistent with our previous statement that commutation rules $C2$ are incompatible with integration of diagonal quadratic forms.

So far we have considered an algebra with two paragrassmann variables. As already mentioned in [12] the analysis of these algebras becomes rather involved as n increases and, to our knowledge, a complete classification is still missing. Nevertheless, we believe that the generalization of our result to $n > 2$ will lead us to consider q-commutation of the form

$$\theta_i \cdot \theta_j = R^{(1)}_{ij,kl} \, \theta_k \cdot \theta_l \tag{46}$$

$$\bar{\theta}_i \cdot \bar{\theta}_j = R^{(2)}_{ij,kl} \, \bar{\theta}_k \cdot \bar{\theta}_l \tag{47}$$

where $R^{(1)}$ and $R^{(2)}$ are matrices related to multiparametric deformations of $GL(n)$. We expect to report on this issue in the future.

In conclusion, in this paper we showed how to introduce consistent Gaussian integrals over paragrassmann variables. Surprisingly, one is obliged to introduce elements of quantum groups in the quadratic forms to allow for linear changes of variables in the integration.

Even if rather abstract at face value this result is a first step in our program to extend the supersymmetric approach to disordered systems in a way that its relation with the replica method becomes more general and transparent. In brief, many interesting problems are represented with "disordered" field theories in which some parameters are taken from a probability distribution (these can be random exchanges, masses, fields, etc.) In general, one is interested in knowing their averaged properties, *i.e.* the behavior of observables averaged over disorder

$$[O] \equiv \left[-\frac{1}{Z}\frac{\partial Z}{\partial h}\bigg|_{h=0} \right] \tag{48}$$

with $Z \equiv \int dfields \ \exp\left(-S[fields, disorder] - hO\right)$ and the square brackets representing the average over the distribution of random parameters. A typical example is given by the calculation of the averaged spectral properties of random matrices. The replica and supersymmetric methods allow one to represent the normalization $1/Z$ in exponential form. In the former one *replicates* the system by making $p-1$ identical copies of it and writes [15]

$$\frac{1}{Z} = \lim_{p\to 0} Z^{p-1} = \lim_{p\to 0} (\det A)^{-(p-1)/2} \, , \tag{49}$$

with the latter identity holding for a Gaussian model. In the latter, for a Gaussian problem, one writes [16]

$$\frac{1}{Z} = (\det A)^{1/2} = \frac{\det A}{(\det A)^{1/2}} = \int \prod_i d\phi_i d\bar{\theta}_i d\theta_i \; e^{\bar{\theta}_i A_{ij}\theta_j + \phi_i A_{ij}\phi_j} \tag{50}$$

with $\bar{\theta}_i$ and θ_i Grassmann and ϕ_i real bosonic variables. In both cases one takes advantage of the thermodynamic large n limit to analyze the effective replicated real bosonic and supersymmetric field theories. The connection between the two methods has not been fully clarified yet. However, "mappings" between the replica expressions when $p \to 0$ and the supersymmetric ones are easy to construct [17]. A trivial example is $\lim_{p\to 0}\sum_{k=1}^{p} 1 = 0 = \int d\theta d\bar{\theta}$. (Indeed, one can trace the relation to the properties of the 0-dimensional replica space and superspace.) A clue to the connection between the two approaches might come from the development of an extended supersymmetric treatment that relates to the replica one *for finite p*. This may also make possible the computation of some interesting properties that need manipulations of the finite p replica expressions (sample-to-sample fluctuations being one such example). A natural way of representing the $(p-1)$-th power of a determinant is to introduce copies of the fermionic variables. Another, as we showed here, is to use variables with extended statistics. We expect to report on progress in the development of an extended supersymmetric approach to the study of problems with quenched disorder in a forthcoming publication.

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Appendix A: Quantum group $GL_{q,q'}(2)$

Let us recall some properties of the quantum group $GL_{q,q'}(2)$:

1. *Closure under co-multiplication*
   If
   $$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} , \qquad g' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$
   are elements of $GL_{q,q'}(2)$ such that the entries a, b, c, d *commute* with the entries a', b', c', d', then the product
   $$g \cdot g' = \begin{pmatrix} a \cdot a' + b \cdot c' & a \cdot b' + b \cdot d' \\ c \cdot a' + d \cdot c' & c \cdot b' + d \cdot d' \end{pmatrix}$$
   also belongs to $GL_{q,q'}(2)$

2. *Existence of the inverse (antipode)*
   The element
   $$g^{-1} = \begin{pmatrix} d & -bq'^{-1} \\ -cq' & a \end{pmatrix} \cdot \Delta^{-1}$$
   where
   $$\Delta \equiv a \cdot d - q' \; b \cdot c \tag{51}$$
   is the inverse of g and is an element of $GL_{q^{-1},q'^{-1}}(2)$.

3. *Determinant*
   The object $\Delta = a{\cdot}d - q' \; b{\cdot}c = a{\cdot}d - q \; c{\cdot}b = d{\cdot}a - q^{-1} \; b{\cdot}c = d{\cdot}a - q'^{-1} \; c{\cdot}b$ is defined as the *determinant* of g. It satisfies
   $$\Delta \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & q/q'b \\ c & q'/qd \end{pmatrix} \cdot \Delta \tag{52}$$

4. The element g^m belongs to $GL_{q^m,q'^m}(2)$.

References

[1] P. T. Matthews and A. Salam, Il Nuovo Cim. **XII** (1954) 563, *ibid* **2** (1955) 120.

[2] F. A. Berezin, *The Method of Second Quantization*, (New York, Academic Press, 1966).

[3] L. Baulieu and E. Floratos, Phys. Lett. B **258** (1991) 271.

[4] J. L. Matheus-Valle and M. A. Monteiro, Phys. Lett. B **300** (1993) 66.

[5] S. Durand, Mod. Phys. Lett. A **8** (1993) 2323.

[6] A. T. Filippov, A. P. Isaev and A. B. Kurdikov, Mod. Phys. Lett. A **7** (1992) 2129.

[7] S. Durand, Phys. Lett. B **312** (1993) 115.

[8] S. Durand, Mod. Phys. Lett. A **8** (1993) 1795.

[9] N. Fleury and M. Rausch de Traubenberg, Mod. Phys. Lett. A **11** (1996) 899.

[10] A. P. Isaev, Int. J. Mod. Phys. A **12** (1997) 201.

[11] For a review and reference to the original papers see M. Rausch de Traubenberg, arXiv:hep-th/9802141.

[12] A. T. Filippov and A. B. Kurdikov, hep-th 9312081

[13] M. Chaichian and A. P. Demichev, *Introduction To Quantum Groups*, (Singapore, World Scientific, 1996).

[14] J. A. Azcarraga, P.P. Kulish and F. Rodenas, Czech. J. Phys. **44** (1994) 981 and Fortsch. Phys. **44** (1996) 1-40.

[15] M. Mézard, G. Parisi and M. A. Virasoro, *Spin-glass theory and beyond*, (World Scientific, Singapore, 1987).

[16] K. B. Efetov, Adv. Phys. **32** (1983) 53. *Supersymmetry in disorder and chaos* (Cambridge Univ. Press, Cambridge, 1997).

[17] See, for instance, J. Kurchan *Supersymmetric, replica and dynamic treatments of disordered systems: a parallel presentation*, to appear in Journal of Markov processes and related fields, cond-mat/0209399.