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Partial Differential Equations as Three-Dimensional Inverse Problem of Moments

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Abstract: We considerer partial differential equations of second order, for example the Klein-Gordon equation, the Poisson equation, on a region $E=(a_1,b_1)\times(a_2,b_2)\times(a_3,b_3)$. We will see that with a common procedure in all cases, we can write the equation in partial derivatives as an Fredholm integral equation of first kind and will solve this latter with the techniques of inverse problem moments. We will find an approximated solution and bounds for the error of the estimated solution using the techniques on problem of moments.

Keywords: Partial differential equations (PDEs), Freholm integral equations, generalized moment problem

1. Introduction

We considerer three-dimensional second order partial differential equations (PDE) of the general forms:

$$\left(f_1(w)w_x\right)_x+\left(f_2(w)w_y\right)_y+\left(f_3(w)w_t\right)_t \\ -h(w)-r(x,y,t)=0 \qquad (1)$$

$$\left(f_1(w)w_x\right)_x+\left(f_2(w)w_y\right)_y- \\ \left(f_3(w)w_t\right)_t-h(w)-r(x,y,t)=0 \qquad (2)$$

$$w_t-\left(f_1(w)w_x\right)_x-\left(f_2(w)w_y\right)_y-r(x,y,t)=0 \qquad (3)$$

where the unknown function $w(x,y,t)$ is defined in $E=\left(a_1,b_1\right)\times\left(a_2,b_2\right)\times\left(a_3,b_3\right)$. In all cases we will considerer Cauchy conditions on the boundary $S=\partial E$ and f_1, f_2, f_3, h, and r are known functions.

A lot of work has been done about the numerical solution of second order partial differential equations using diverse specific techniques mainly directed towards particular cases. We cite a few books that can be taken as representative of the subject [7],[12], [3], [14], [6].

We will show that, following in all the cases a common procedure, the partial differential equation can be transformed into a three-dimensional integral equation and that this one can be numerically solved using techniques normally employed with generalized moment problems. This approach was already suggested by Anget. al.[4] in relation with the heat conduction equation and we have applied to the nonlinear Klein-Gordon equation [10].

Next Section is devoted to show how the differential equations (1), (2) and (3) are transformed into integral equations of the first kind that can be seen as generalized moment problems as is shown in

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Section 3. There we also proof a theorem that guarantees under certain conditions the stability and convergence of the finite generalized moment problem. Finally in Section 4 we exemplify the general method by applying it to some PDEs which are particular cases of Eqs. (1), (2) and (3).

The d-dimensional generalized moment problem can be posed as follows: find a function u on a domain $\Omega \subset \mathbf{R}^d$ satisfying the sequence of equations

$$\int_\Omega u(x)g_n(x)dx = \mu_n \quad n \in \mathbf{N} \tag{4}$$

where (g_n) is a given sequence of functions lying in $\mathbf{L}^2(\Omega)$

Many inverse problems can be formulated as an integral equation of the first kind, namely,

$$\int_a^b K(x,y)u(y)dy = f(x) \quad x \in (a,b)$$

$K(x,y)$ and $f(x)$ are given functions and $u(y)$ is a solution to be determined, $f(x)$ is a result of experimental measurements and hence is given only at finite set of points. It follows that the above integral equation is equivalent to the following moment problem

$$\int_a^b K(x_n,y)u(y)dy = f(x_n) \quad n = 1,2,...$$

Also we considerer the multidimensional moment problems

$$\int_\Omega K(x_n,y)u(y)dy = f(x_n)$$

$$n = 1,2,... \quad \Omega \subset \mathbf{R}^d$$

Moment problem are usually ill-posed. There are various methods of constructing regularized solutions, that is, approximate solutions stable with respect to the given data. One of them is the *method of truncated expansion*.

The *method of truncated expansion* consists in approximating (4) by finite moment problems

$$\int_\Omega u(x)g_i(x)dx = \mu_i \quad i = 1,2,...,n \tag{5}$$

Solved in the subspace $\langle g_1, g_2, ..., g_n \rangle$ generated by $g_1, g_2, ..., g_n$ (5) is stable. Considering the case where the data $\mu = (\mu_1, \mu_2, ..., \mu_n)$ are inexact, convergence theorems and error estimates for the regularized solutions they are applied.

2. PDEs as Integral Equations of the First Kind

Let $F(w(x,y,t)) = 0$ be a partial differential equations such as (1), (2) o (3). The solution $w(x,y,t)$ is defined on the region $E = (a_1, b_1) \times (a_2, b_2 \times (a_3, b_3))$ and verifies Cauchy conditiones on the boundary $S = \partial E$:

$$w(a_1,y,t) = s_1(y,t) \qquad w(b_1,y,t) = s_2(y,t)$$

$$w(x,a_2,t) = s_3(x,t) \qquad w(x,b_2,t) = s_4(x,t)$$

$$w(x,y,a_3) = s_5(x,y) \qquad w(x,y,b_3) = s_6(x,y)$$

$$w_x(a_1,y,t) = s_7(y,t) \qquad w_x(b_1,y,t) = s_8(y,t)$$

$$w_y(x,a_2,t) = s_9(x,t) \qquad w_y(x,b_2,t) = s_{10}(x,t)$$

$$w_t(x,y,a_3) = s_{11}(x,y) \qquad w_t(x,y,b_3) = s_{12}(x,y)$$

Let $F^* = (F_1(w), F_2(w), F_3(w))$ be a vectorial field such that w verifies $div(F^*) = h^*(w)$ with h^* a known function and, reciprocally, if w verifies $div(F^*) = h^*(w)$ then $F(w(x,y,t)) = 0$

Let $u(x,y,t,\tau,\xi,\gamma)$ be the auxiliary function such that

$$\nabla^2 u = u_{\tau\tau} + u_{\xi\xi} + u_{\gamma\gamma} = ug_1(x,y,t,\tau,\xi,\gamma) + ug_2(x,y,t,\tau,\xi,\gamma) + ug_3(x,y,t,\tau,\xi,\gamma)$$

Since

$$udiv(F^*) = uh^*(w)$$

we have

$$\iiint_E udiv(F^*)dV = \iiint_E uh^*(w)dV$$

Moreover, as

$$udiv(F^*) = div(uF^*) - F^* \cdot \nabla u$$

and

$$\iiint_E udiv(F^*)dV =$$

$$\iiint_E div(uF^*)dV - \iiint_E F^* \cdot \nabla u dV$$

we obtain by the divergence theorem

$$\iiint_E uh^*(w)dV = \iint_S (uF^*) \cdot nds - \iiint_E F^* \cdot \nabla u dV \tag{6}$$

where $\nabla u = \left(u_\tau, u_\xi, u_\gamma\right)$

If $F_1(w)$, $F_2(w)$ and $F_3(w)$ are non linear functions of w then (6) gives:

$$\iiint_E u\left(h(w) - \sum_{i=1}^{3} F_i^p(w) g_i\right) d\xi d\tau d\gamma =$$

$$G(x,y,t) - A(x,y,t) - B(x,y,t) - C(x,y,t) \tag{7}$$

$$-\iiint_E u(x,y,t,\tau,\xi,\gamma) r(\tau,\xi,\gamma) d\xi d\tau d\gamma$$

where

$$G(x,y,t) = \int_{a_3}^{b_3}\int_{a_2}^{b_2} (u(\tau = b_1)F_1(w(\tau = b_1))$$

$$-u(\tau = a_1)F_1(w(\tau = a_1)))d\xi d\gamma$$

$$+\int_{a_3}^{b_3}\int_{a_1}^{b_1} (u(\xi = b_2)F_2(w(\xi = b_2))$$

$$-u(\xi = a_2)F_2(w(\xi = a_2)))d\tau d\gamma$$

$$+\int_{a_2}^{b_2}\int_{a_1}^{b_1} (u(\gamma = b_3)F_3(w(\gamma = b_3))$$

$$-u(\gamma = a_3)F_3(w(\gamma = a_3)))d\tau d\xi$$

$u(\tau = b_1)$ reads $u(x,y,t,b_1,\xi,\gamma)$

$$A(x,y,t) = \int_{a_3}^{b_3}\int_{a_2}^{b_2} (u_\tau(\tau = b_1)F_1^p(w(\tau = b_1)$$

$$-u_\tau(\tau = a_1)F_1^p(w(\tau = a_1)) d\xi d\gamma$$

$$B(x,y,t) = \int_{a_1}^{b_1}\int_{a_3}^{b_3} (u_\xi(\xi = b_2)F_2^p(w(\xi = b_2))$$

$$-u_\xi(\xi = a_2)F_2^p(w(\xi = a_2)))d\gamma d\tau$$

$$C(x,y,t) = \int_{a_2}^{b_2}\int_{a_1}^{b_1} (u_\gamma(\gamma = b_3)F_3^p(w(\gamma = b_3))$$

$$-u_\gamma(\gamma = a_3)F_3^p(w(\gamma = a_3)))d\tau d\xi$$

also

$$F_1^p(w) = \int F_1(w)d\tau$$

$$F_2^p(w) = \int F_2(w)d\xi$$

$$F_3^p(w) = \int F_3(w)d\gamma$$

If $F_1(w)$, $F_2(w)$ and $F_3(w)$ are linear functions of w and we also assume that

$$\nabla u = (uk_1(x,y,t,\tau,\xi,\gamma),$$

$$uk_2(x,y,t,\tau,\xi,\gamma),$$

$$uk_3(x,y,t,\tau,\xi,\gamma))$$

then (6) gives:

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} u\left(h^*(w) + \sum_{i=1}^{3} F_i(w)k_i\right) d\xi d\tau$$

$$= G(x,y,t) - \int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} ur(\tau,\xi,\gamma) d\xi d\tau d\gamma \tag{8}$$

2.1 Elliptic Partial Differential Equation

If

$$F(w(x,y,t)) =$$

$$(f_1(w)w_x)_x + (f_2(w)w_y)_y +$$

$$(f_3(w)w_t)_t - h(w) - r(x,y,t) = 0$$

We write

$$\left(f_1(w)w_\tau\right)_\tau + \left(f_2(w)w_\xi\right)_\xi +$$

$$\left(f_3(w)w_\gamma\right)_\gamma = h(w) + r(\tau,\xi,\gamma)$$

We take

$$F^* = (\underbrace{f_1(w)w_\tau}_{F_1(w)}, \underbrace{f_2(w)w_\xi}_{F_2(w)}, \underbrace{f_3(w)w_\gamma}_{F_3(w)})$$

so

$$div(F^*) = (f_1(w)w_\tau)_\tau + (f_2(w)w_\xi)_\xi$$

$$+(f_3(w)w_\gamma)_\gamma = h(w) + r(x,y,t)$$

and

$$h^*(w)=h(w)+r(\tau,\xi,\gamma)$$

besides

$$F_1^p(w)=\int F_1(w)d\tau=\int f_1(w)w_\tau d\tau=\int f_1(w)dw$$

$$F_2^p(w)=\int F_2(w)d\xi=\int f_2(w)w_\xi d\xi=\int f_2(w)dw$$

$$F_3^p(w)=\int F_3(w)d\gamma=\int f_3(w)w_\gamma d\gamma=\int f_3(w)dw$$

Functions $F_1(w)$, $F_2(w)$ and $F_3(w)$ are not linear of w then(6) reads:

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} u\left(h(w)-\sum_{i=1}^{3}F_i^p(w)g_i\right)d\xi d\tau d\gamma=\mu_{xyt}$$

where

$$\mu_{xyt}=G(x,y,t)-A(x,y,t)-B(x,y,t)$$
$$-C(x,y,t)-\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} ur(\tau,\xi,\gamma)d\gamma d\xi d\tau$$

Next we will considerer particular cases according with the values taken by $h(w)$, $F_1^p(w)$ and $F_2^p(w)$

2.1.1. Linear Poisson Equation

If $h(w)=w$ $F_1^p(w)=w$ $F_2^p(w)=w$ then

$$\left(w_\tau\right)_\tau+\left(w_\xi\right)_\xi+\left(w_\gamma\right)_\gamma=w+r(\tau,\xi,\gamma)$$

We choose

$$u(x,y,t,\tau,\xi,\gamma)=e^{-x\tau}e^{-y\xi}e^{-t\gamma}$$

then

$$\nabla u=(-xu,-yu,-tu)$$

y

$$\nabla^2 u=x^2u+y^2u+t^2u$$

so

$$g_1(x,y,t,\tau,\xi,\gamma)=x^2$$

$$g_2(x,y,t,\tau,\xi,\gamma)=y^2$$

$$g_3(x,y,t,\tau,\xi,\gamma)=t^2$$

Then

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} uw(1-x^2-y^2-t^2)d\gamma d\xi d\tau=\mu_{xyt}$$

Thus

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} e^{-x\tau}e^{-y\xi}e^{-t\gamma}wd\gamma d\xi d\tau$$
$$=\frac{\mu_{xyt}}{(1-x^2-y^2-t^2)}=\overline{\mu}_{xyt}$$

2.1.2. Non Linear Poisson Equation

If $F_1^p(w)=F_2^p(w)=F_3^p(w)=w$ we have the nonlinear Poisson equation or Hemlhotz equation

$$\left(w_\tau\right)_\tau+\left(w_\xi\right)_\xi+\left(w_\gamma\right)_\gamma=h(w)+r(\tau,\xi,\gamma)$$

We choose

$$u(x,y,t,\tau,\xi,\gamma)=$$
$$e^{-x\tau}e^{-y\xi}\cos(t\gamma)\Rightarrow\nabla^2u=x^2u+y^2-t^2u$$

$$g_1(x,y,t,\tau,\xi,\gamma)=x^2\quad g_2(x,y,t,\tau,\xi,\gamma)=y^2$$
$$g_3(x,y,t,\tau,\xi,\gamma)=-t^2$$

Therefore

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} u(h(w)-wx^2-wy^2+wt^2)d\gamma d\xi d\tau$$
$$=\mu_{xyt}$$

If $t^2=x^2+y^2$ then

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} e^{-x\tau}e^{-y\xi}\cos(\sqrt{x^2+y^2}\gamma)h(w)d\gamma d\xi d\tau$$
$$=\overline{\mu}_{xy}$$

Now we estimate $h(w)$ with $p_n(\tau,\xi,\gamma)$. Then the approximate solution for w is $h^{-1}\left(p_n(\tau,\xi,\gamma)\right)$ assuming that h^{-1} is Lipschitz in $\mathbf{R}^3$

2.2 Hyperbolic Partial Differential Equation

For the Klein-Gordon equation we can use the general form deduced for elliptic equations.

If $F_1^p(w)=w$, $F_2^p(w)=w$ and $F_3^p(w)=-w$ we have:

$$\left(w_\tau\right)_\tau+\left(w_\xi\right)_\xi-\left(w_\gamma\right)_\gamma=h(w)+r(\tau,\xi,\gamma)$$

Take

$$u(t,x,\tau,\xi,\gamma)=e^{-x\tau}e^{-y\xi}e^{-t\gamma}$$
$$\Rightarrow\nabla^2u=x^2u+y^2u+t^2u$$

Then

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} u(h(w)-wx^2$$

$$-wy^2+wt^2)d\gamma d\xi d\tau=\mu_{xyt}$$

If $t^2=x^2+y^2$ then

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} e^{-x\tau}e^{-y\xi}e^{-\sqrt{x^2+y^2}\gamma}h(w)d\gamma d\xi d\tau=\overline{\mu}_{xy}$$

Now we estimate $h(w)$ with $p_n(\tau,\xi,\gamma)$. Then the approximate solution for w is $h^{-1}(p_n(\tau,\xi,\gamma))$ assuming that h^{-1} is Lipschitzin $\mathbf{R}^3$.

2.3 Parabolic Partial Differential Equation

$$(f_1(w)w_x)_x+(f_2(w)w_y)_y-w_\gamma-h(w)-r(x,y,t)=0$$

We write

$$(f_1(w)w_\tau)_\tau+(f_2(w)w_\xi)_\xi-w_\gamma=h(w)+r(\tau,\xi,\gamma) \tag{9}$$

Take

$$F^*=(F_1(w),F_2(w),F_3(w))$$

$$=(f_1(w)w_\tau,f_2(w)w_\xi,-w)$$

Then

$$div(F^*)=(f_1(w)w_\tau)_\tau+$$

$$(f_2(w)w_\xi)_\xi-w_\gamma=h(w)+r(\tau,\xi,\gamma)$$

and

$$h^*(w)=h(w)+r(\tau,\xi,\gamma)$$

Besides

$$F_1^p(w)=\int F_1(w)d\tau=\int f_1(w)w_\tau d\tau=\int f_1(w)dw$$

$$F_2^p(w)=\int F_2(w)d\xi=\int f_2(w)w_\xi d\xi=\int f_2(w)dw$$

In this case $F_3(w)$ is a linear function of w. $F_1(w)$ and $F_2(w)$ are not linear function of w then (6) gives:

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} u(h(w)-F_1^p(w)g_1-$$

$$F_2^p(w)g_2+F_3(w)k_3)d\gamma d\xi d\tau=\mu_{xyt}$$

with

$$\mu_{xyt}=G(x,y,t)-A(x,y,t)-$$

$$B(x,y,t)-\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} ur(\tau,\xi,\gamma)d\gamma d\xi d\tau$$

2.3.1

If $F_1^p(w)=F_2^p(w)=w$ then

$$(w_\tau)_\tau+(w_\xi)_\xi-w_\gamma=h(w)+r(\tau,\xi,\gamma)$$

We take again

$$u(t,x,\tau,\xi,\gamma)=e^{-x\tau}e^{-x\xi}e^{-t\gamma}\Rightarrow$$

$$\nabla u=(-xu,-yu,-tu)\quad y\quad \nabla^2u=x^2u+y^2u+t^2u$$

therefore

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} e^{-x\tau}e^{-y\xi}e^{-t\gamma}(h(w)-$$

$$x^2w-y^2w+tw)d\gamma d\xi d\tau=\mu_{xyt}$$

If $t=x^2+y^2$ then

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} e^{-x\tau}e^{-y\xi}e^{-(x^2+y^2)\tau}(h(w))d\gamma d\xi d\tau=\overline{\mu}_{xy}$$

$h(w)$ is estimated by using $p_n(\tau,\xi,\gamma)$. Then the approximate solution for w is $h^{-1}(p_n(\tau,\xi,\gamma))$ assuming that h^{-1} is Lipschitz in $\mathbf{R}^3$

2.3.2

If $h(w)=0$ and $F_1^p(w)=F_2^p(w)=w$ then

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} e^{-x\tau}e^{-y\xi}e^{-t\tau}(-x^2w$$

$$-y^2w+tw)d\gamma d\xi d\tau=\mu_{xyt}$$

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} e^{-x\tau}e^{-y\xi}e^{-t\tau}w(-x^2-y^2+t)d\gamma d\xi d\tau=$$

$$=\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} uwd\gamma d\xi d\tau=\frac{\mu_{xyt}}{(-x^2-y^2+t)}=\overline{\mu}_{xyt}$$

3. Solution of Generalized Moment Problems

If (7) and (8) can be written in the form:

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} F(w(\tau,\xi,\gamma))K(x,y,t,\tau,\xi,\gamma)d\tau d\xi d\gamma$$

$$=\phi(x,y,t)$$

with $\phi(x,y,t)\in L^2(E)$, then taking a basis $\{\psi_m(x,y,t)\}_m$ of $L^2(E)$ this Fredholm integral equation of the first kind can be transformed into a three-dimensional generalized moment problem

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} F(w(\tau,\xi,\gamma))K_m(\tau,\xi,\gamma)d\tau d\xi d\gamma = \mu_m \qquad m=0,1,2,... \tag{10}$$

where

$$K_m(\tau,\xi,\gamma)= \int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} K(x,y,t,\tau,\xi,\gamma)\psi_m(x,y,t)dxdydt \tag{11}$$

and the moments μ_m are

$$\mu_m=\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} \phi(x,y,t)\psi_m(x,y,t)dxdydt \tag{12}$$

If the functions $\{K_m(\tau,\xi,\gamma)\}_m$ are linearly independent then the generalized moment problem defined by equations (10), (11) and (12) can be solved considering the correspondent finite problem

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} F(w(\tau,\xi,\gamma))K_m(\tau,\xi,\gamma)d\tau d\xi d\gamma = \mu_m \quad m=0,1,2,...,n \quad n\in N \tag{13}$$

whose solution we denote

$$p_n(\tau,\xi,\gamma)\approx\beta(\tau,\xi,\gamma)=F(w(\tau,\xi,\gamma))$$

If $F(w)$ has continuous inverse, then $F^{-1}(p_n(\tau,\xi,\gamma))=w_n(\tau,\xi,\gamma)$ is an estimation of $w(\tau,\xi,\gamma)$.

To reach this result let considerer the basis $\{\varphi_i(\tau,\xi,\gamma)\}_{i=0}^{\infty}$ obtained from the sequence $\{K_m(\tau,\xi,\gamma)\}_{m=0}^{n}$ by Gram-Schmidt method and addition of the necessary functions in order to have an orthonormal basis.

We then approximate the solution $\beta(\tau,\xi,\gamma)=F(w(\tau,\xi,\gamma))$ de (13) with

$$p_n(\tau,\xi,\gamma)=\sum_{i=0}^{n}\lambda_i\varphi_i(\tau,\xi,\gamma)$$

with

$$\lambda_i=\sum_{j=0}^{i}C_{ij}\mu_j \qquad i=0,1,...,n$$

where the coefficients C_{ij} verifies

$$C_{ij}= \left(\sum_{k=j}^{i-1}(-1)\frac{\langle K_i(\tau,\xi,\gamma)\,|\,\varphi_k(\tau,\xi,\gamma)\rangle}{\|\varphi_k(\tau,\xi,\gamma)\|^2}C_{kj}\right).\|\varphi_i(\tau,\xi,\gamma)\|^{-1}$$

$$1<i\le n;1\le j<i \tag{14}$$

$$C_{ii}=\|\varphi_i(\tau,\xi,\gamma)\|^{-1} \qquad i=0,1,...,n. \tag{15}$$

We extend to the tridimensional case the arguments of reference [8] to proof the

Theorem *Let* $\{\mu_m\}_{m=0}^{n}$ *be a set of real numbers and let* ε *and* E *be two positive nembers such that*

$$\sum_{m=0}^{n}\left|\int_{a_3}^{b_3}\int_{a_2}^{b_2}\int_{a_1}^{b_1} K_m(\tau,\xi,\gamma)\beta(\tau,\xi,\gamma)d\tau d\xi d\gamma-\mu_m\right|^2\le\varepsilon^2 \tag{16}$$

y

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_1}^{b_1}[(b_1-a_1)^2\beta_\tau^2+(b_2-a_2)^2\beta_\xi^2 +(b_3-a_3)^2\beta_\gamma^2]d\tau d\xi d\gamma\le E^2$$

then

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_1}^{b_1}|\beta(\tau,\xi,\gamma)|^2 d\tau d\xi d\gamma\le \min_m\left\{\|CC^T\|^2\varepsilon^2+\frac{E^2}{12(m+1)^2};m=0,1,...,n\right\} \tag{17}$$

where C *is the triangular matriz with elements* C_{ij} $(1<i\le n;1\le j<i)$.

and

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_1}^{b_1}|p_n(\tau,\xi,\gamma)-\beta(\tau,\xi,\gamma)|^2 d\tau d\xi d\gamma \le\|CC^T\|^2\varepsilon^2+\frac{E^2}{12(n+1)^2} \tag{18}$$

If $F^{-1}(x)$ *is Lipschitzin* R^3 *, i.e.* $\|F^{-1}(x)-F^{-1}(y)\|\le\lambda\|x-y\|$ *for some* λ *and*

$\forall x, y \in R^3$ then

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_1}^{b_1}\left|w_n(\tau,\xi,\gamma)-w(\tau,\xi,\gamma)\right|^2 d\tau d\xi d\gamma \le \lambda\left(\left\|CC^T\right\|^2\varepsilon^2+\frac{E^2}{12(n+1)^2}\right) \tag{19}$$

Dem.) The demonstration is similar to that we have done for the unidimensional generalized moment problem [8], which is based in results of Talenti [13] for the Hausdorff moment problem. Here we simply introduce the necessary modification for the three-dimensional case.

Without loss of generality we take $\{\mu_m = 0\}_{m=0}^{n}$ in (16).

We write

$$\beta(\tau,\xi,\gamma) = h_n(\tau,\xi,\gamma) + t_n(\tau,\xi,\gamma)$$

where $h_n(\tau,\xi,\gamma)$ is the orthogonal projection of $\beta(\tau,\xi,\gamma)$ on the linear space that the set $\{K_m(\tau,\xi,\gamma)\}_{m=0}^{n}$ generates and $t_n(\tau,\xi,\gamma) = \beta(\tau,\xi,\gamma) - h_n(\tau,\xi,\gamma)$ is the orthogonal projection of $\beta(\tau,\xi,\gamma)$ on the orthogonal complement. In terms of the basis $\{\varphi_i(\tau,\xi,\gamma)\}_{i=0}^{\infty}$ the functions $h_n(\tau,\xi,\gamma)$ and $t_n(\tau,\xi,\gamma)$ reads

$$h_n(\tau,\xi,\gamma) = \sum_{i=0}^{n} \lambda_i \varphi_i(\tau,\xi,\gamma)$$

$$t_n(\tau,\xi,\gamma) = \sum_{i=n+1}^{\infty} \lambda_i \varphi_i(\tau,\xi,\gamma)$$

with

$$\lambda_i = \sum_{j=0}^{i} C_{ij}\mu_j \quad i = 0,1,\ldots$$

and the matrix elements C_{ij} given by (14) and (15).

In matricial notation:

$$\lambda = \begin{pmatrix}\lambda_1\\ \lambda_2\\ \vdots\\ \lambda_n\end{pmatrix} \qquad \mu = \begin{pmatrix}\mu_1\\ \mu_2\\ \vdots\\ \mu_n\end{pmatrix} \qquad \lambda = C\mu$$

Besides

$$\lambda_i = \int_{a_3}^{b_3}\int_{a_2}^{b_2}\int_{a_1}^{b_1} \beta(\tau,\xi,\gamma)\varphi_i(\tau,\xi,\gamma)d\tau d\xi d\gamma$$

$$\mu_i = \int_{a_3}^{b_3}\int_{a_2}^{b_2}\int_{a_1}^{b_1} \beta(\tau,\xi,\gamma)K_i(\tau,\xi,\gamma)d\tau d\xi d\gamma$$

Therefore

$$\int_{a_3}^{b_3}\int_{a_2}^{b_2}\int_{a_1}^{b_1} |h_n|^2 d\tau d\xi d\gamma = \langle\lambda,\lambda\rangle = = \langle C^T C\mu,\mu\rangle \le \left\|C^T C\right\|\left\|\mu\right\|^2 \le \left\|C^T C\right\|\varepsilon^2$$

To estimate the norm of $t_n(\tau,\xi,\gamma)$ we observe that each element of the orthonormal basis $\{\varphi_i(\tau,\xi,\gamma)\}_{i=0}^{\infty}$ can be written as a function of the elements of another orthonormal basis, in particular the set $\{P_{klr}(\tau,\xi,\gamma)\}_{k,l,r=0}^{\infty}$ con $P_{klr}(\tau,\xi,\gamma) = L_{1k}(\tau)L_{2l}(\xi)L_{3r}(\gamma)$ with $L_{1k}(\tau)$ Legendre polynomial in (a_1, b_1), $L_{2l}(\xi)$ Legendre polynomial in (a_2, b_2), $L_{3r}(\xi)$ Legendre polynomial in (a_3, b_3)

$$\varphi_i(\tau,\xi,\gamma) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{r=0}^{\infty} \gamma_{klr,i} P_{klr}(\tau,\xi,\gamma)$$

The Legendre polynomials $L_{1k}(\tau)$ verify

$$\frac{d}{d\tau}[(a_1-\tau)(b_1-\tau)L_{1k}(\tau)] = k(k+1)L_{1k}(\tau) \quad k = 0,1,2,\ldots$$

and analogous property for the polynomials $L_{2l}(\xi)$ and $L_{3r}(\gamma)$

Defining $\lambda_{klr}^{*} = \sum_{i=n+1}^{\infty} \lambda_i \gamma_{klr,i}$ we can demonstrate that

$$\int_{a_3}^{b_3}\int_{a_2}^{b_2}\int_{a_1}^{b_1} |t_n|^2 d\tau d\xi d\gamma \le \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{r=0}^{\infty} k(k+1)\lambda_{klr}^{*2} \le \frac{1}{4(n+1)^2}\int_{a_3}^{b_3}\int_{a_2}^{b_2}\int_{a_1}^{b_1} (b_1-a_1)^2 \beta_\tau^2(\tau,\xi,\gamma)d\tau d\xi d\gamma$$

and

$$\int_{a_3}^{b_3}\int_{a_2}^{b_2}\int_{a_1}^{b_1} |t_n|^2 d\tau d\xi d\gamma \le \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{r=0}^{\infty} l(l+1)\lambda_{klr}^{*2} \le \frac{1}{4(n+1)^2}\int_{a_3}^{b_3}\int_{a_2}^{b_2}\int_{a_1}^{b_1} (b_2-a_2)^2 \beta_\xi^2(\tau,\xi,\gamma)d\tau d\xi d\gamma$$

$$\int_{a_3}^{b_3}\int_{a_2}^{b_2}\int_{a_1}^{b_1}|t_n|^2 d\tau d\xi d\gamma \leq \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{r=0}^{\infty} r(r+1)\lambda_{klr}^{*2} \leq$$

$$\frac{1}{4(n+1)^2}\int_{a_3}^{b_3}\int_{a_2}^{b_2}\int_{a_1}^{b_1}(b_3-a_3)^2\beta_\gamma^2(\tau,\xi,\gamma)d\tau d\xi d\gamma$$

From these equations we deduce that

$$\leq \frac{1}{12(n+1)^2}\int_{a_3}^{b_3}\int_{a_2}^{b_2}\int_{a_1}^{b_1}\left[(b_1-a_1)^2\beta_\tau^2(\tau,\xi,\gamma) + (b_2-a_2)^2\beta_\xi^2(\tau,\xi,\gamma) + (b_3-a_3)^2\beta_\gamma^2(\tau,\xi,\gamma)\right]d\tau d\xi d\gamma$$

$$\int_{a_3}^{b_3}\int_{a_2}^{b_2}\int_{a_1}^{b_1}|t_n(\tau,\xi,\gamma)|^2 d\tau d\xi d\gamma \leq \frac{E^2}{12(n+1)^2}$$

Adding the expressions for the two standards $\|h_n(\tau,\xi,\gamma)\|^2$ y $\|t_n(\tau,\xi,\gamma)\|^2$ result (18) is reached. An analogous demonstration proves inequality (19).

4. Examples

In each case we choose $u(x,y,t,\tau,\xi,\gamma)$ such that $A(x,y,t)$, $B(x,y,t)$, $C(x,y,t)$ and $G(x,y,t)$ are well defined.

4.1. Klein-Gordon Equation

We apply the method to the equation

4.1.1.

$$w_{\tau\tau}+w_{\xi\xi}-w_{\gamma\gamma}=0$$

in the domain $E=[0,0.5]\times[0,0.5]\times[0,0.5]$ and boundary condition on ∂E given by

$$w(0,\xi,\gamma)=ln\left(\frac{1}{(1+\gamma)^2}\right)+ln\left(\frac{1}{(1+\gamma+\xi)^2}\right)$$

$$w(0.5,\xi,\gamma)=ln\left(\frac{1}{(1.5+\gamma)^2}\right)+ln\left(\frac{1}{(1+\gamma+\xi)^2}\right)$$

$$w(\tau,0,\gamma)=ln\left(\frac{1}{(1+\gamma+\tau)^2}\right)+ln\left(\frac{1}{(1+\gamma)^2}\right)$$

$$w(\tau,0.5,\gamma)=ln\left(\frac{1}{(1+\gamma+\tau)^2}\right)+ln\left(\frac{1}{(1.5+\gamma)^2}\right)$$

$$w(\tau,\xi,0)=ln\left(\frac{1}{(1+\tau)^2}\right)+ln\left(\frac{1}{(1+\xi)^2}\right)$$

$$w(\tau,\xi,0.5)=ln\left(\frac{1}{(1.5+\tau)^2}\right)+ln\left(\frac{1}{(1.5+\xi)^2}\right)$$

$$w_\tau(0,\xi,\gamma)=-\frac{2}{1+\gamma}$$

$$w_\tau(0.5,\xi,\gamma)=-\frac{2}{1.5+\gamma}$$

$$w_\xi(\tau,0,\gamma)=-\frac{2}{1+\gamma}$$

$$w_\xi(\tau,0.5,\gamma)=-\frac{2}{1.5+\gamma}$$

$$w_\gamma(\tau,\xi,0)=-\frac{2}{1+\tau}-\frac{2}{1+\xi}$$

$$w_\gamma(\tau,\xi,0.5)=-\frac{2}{1.5+\tau}-\frac{2}{1.5+\xi}$$

We take

$$u(x,y,t,\tau,\xi,\gamma)=e^{-(1+x)\tau-0.6(1+y)\xi-2(1+t)\gamma}$$

We compare our numerical solution, obtained with $n=12$ moments with the known exact solution

$$w(\tau,\xi,\gamma)=ln\left(\frac{1}{(1+\gamma+\tau)^2}\right)+ln\left(\frac{1}{(1+\gamma+\xi)^2}\right)$$

In this case the accuracy of the approximate solution $w_{12}(\tau,\xi,\gamma)$ is

$$\int_0^{0.5}\int_0^{0.5}\int_0^{0.5}|w_{12}-w|^2 d\tau d\xi d\gamma = 0.0369106$$

4.1.2.

Now we apply the method to the equation

$$w_{\tau\tau}+w_{\xi\xi}-w_{\gamma\gamma}=-e^{-\gamma-\tau}Sin(\xi)$$

in the domain $E=[0,1]\times[0,1]\times[0,1]$ and boundary condition on ∂E given by

and boundary condition on ∂E given by

$$w(0,\xi,\gamma)=e^{-\gamma}Sin(\xi)$$

$$w(1,\xi,\gamma)=e^{-1-\gamma}Sin(\xi)$$

$$w(\tau,0,\gamma)=0$$
$$w(\tau,1,\gamma)=e^{-\tau-\gamma}Sin(1)$$
$$w(\tau,\xi,0)=e^{-\tau}Sin(\xi)$$
$$w(\tau,\xi,1)=e^{-\tau-1}Sin(\xi)$$
$$w_\tau(0,\xi,\gamma)=-e^{-\gamma}Sin(\xi)$$
$$w_\tau(1,\xi,\gamma)=-e^{-\gamma-1}Sin(\xi)$$
$$w_\xi(\tau,0,\gamma)=e^{-\gamma-\tau}$$
$$w_\xi(\tau,1,\gamma)=e^{-\gamma-\tau}Cos(1)$$
$$w_\gamma(\tau,\xi,0)=-e^{-\tau}Sin(\xi)$$
$$w_\gamma(\tau,\xi,1)=-e^{-1-\tau}Sin(\xi)$$

We take

$$u(x,y,t,\tau,\xi,\gamma)=e^{-(1+x)\tau-(1+y)\xi-(1+t)\gamma}$$

We compare our numerical solution, obtained with $n=12$ moments with the known exact solution

$$w(\tau,\xi,\gamma)=e^{-\tau-\gamma}Sin(\xi)$$

In this case the accuracy of the approximate solution $w_{12}(\tau,\xi,\gamma)$ is

$$\int_0^1\int_0^1\int_0^1|w_{12}-w|^2 d\tau d\xi d\gamma=0.096371$$

4.2. Non linear parabolic equation

Let consider the non linear hyperbolic equation

$$w_{\tau\tau}+w_{\xi\xi}-w_\gamma=w^2+(3e^{\tau+\xi+\gamma}-1)e^{-2(\tau+\xi+\gamma)}$$

in the domain $E=[0,1]\times[0,1]\times[0,1]$ and boundary condition on ∂E given by

$$w(0,\xi,t)=e^{-\xi-\gamma} \qquad w(1,\xi,\gamma)=e^{-1-\xi-\gamma}$$
$$w(\tau,0,\gamma)=e^{-\tau-\gamma} \qquad w(\tau,1,\gamma)=e^{-\tau-1-\gamma}$$
$$w(\tau,\xi,0)=e^{-\tau-\xi} \qquad w(\tau,\xi,1)=e^{-\tau-\xi-1}$$
$$w_\tau(0,\xi,\gamma)=-e^{-\gamma-\xi} \qquad w_\tau(1,\xi,\gamma)=-e^{-\gamma-1-\xi}$$
$$w_\xi(\tau,0,\gamma)=-e^{-\gamma-\tau} \qquad w_\xi(\tau,1,\gamma)=-e^{-\gamma-\tau-1}$$
$$w_\gamma(\tau,\xi,0)=-e^{-\tau-\xi} \qquad w_\gamma(\tau,y,1)=-e^{-1-\tau-\xi}$$

We take

$$u(x,y,t,\tau,\xi,\gamma)=e^{-(1+x)(1+\tau)-(1+y)(\xi+1)-(1+t)(1+\gamma)}$$

We compare our numerical solution, obtained with $n=4$ moments with the known exact solution

$$w(\tau,\xi,\gamma)=e^{-\tau-\xi-\gamma}$$

In this case the accuracy of the approximate solution $w_4(\tau,\xi,\gamma)$ is

$$\int_0^1\int_0^1\int_0^1|w_4-w|^2 d\tau d\xi d\gamma=0.0420354$$

5. Conclusions

Let $F(w(\tau,\xi,\gamma))=0$ be a partial differential equations. The solution $w(\tau,\xi,\gamma)$ is defined on the region $E=(a_1,b_1)\times(a_2,b_2\times(a_3,b_3))$ and verifies Cauchy conditiones on the boundary $S=\partial E$. Let $F^*=(F_1(w),F_2(w),F_3(w))$ be a vectorial field such that w verifies $div(F^*)=h^*(w)$ with h^* a known function and, reciprocally, if w verifies $div(F^*)=h^*(w)$ then $F(w(\tau,\xi,\gamma))=0$ Let $u(x,y,t,\tau,\xi,\gamma)$ be the auxiliary function $u(x,y,t,\tau,\xi,\gamma)$ such that

$$\nabla^2 u=u_{\tau\tau}+u_{\xi\xi}+u_{\gamma\gamma}=ug_1(x,y,t,\tau,\xi,\gamma)$$
$$+ug_2(x,y,t,\tau,\xi,\gamma)+ug_3(x,y,t,\tau,\xi,\gamma)$$

Then we can write the equation in partial derivatives as an Fredholm integral equation of first kind. Can solve the partial differential equation as a generalized inverse moment problem if the integral equation is of the form

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}F(w(\tau,\xi,\gamma))K(x,y,t,\tau,\xi,\gamma)d\gamma d\xi d\tau$$
$$=\phi(x,y,t)$$

where $K(x,y,t,\tau,\xi,\gamma)$ is such that

$$K_m(\tau,\xi,\gamma)=$$
$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}K(x,y,t,\tau,\xi,\gamma)\psi_m(x,y,t))dtdydx$$
$$m=0,1,2,\ldots$$

are linearly independent with $\psi_m(x, y, t)$ a basis of $L^2(E)$

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