

## Reply to “Comment on ‘Troublesome aspects of the Renyi-MaxEnt treatment’ ”

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This Reply is intended as a refutation of the preceding Comment [Oikonomou and Bagci, *Phys. Rev. E* **96**, 056101 (2017)] on our paper [Plastino *et al.*, *Phys. Rev. E* **94**, 012145 (2016)]. We show that the Tsallis probability distribution of our paper does not coincide with the Tsallis distribution studied by Oikonomou and Bagci. Consequently, their findings do not apply to our paper.

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### I. INTRODUCTION

The authors of [1], Oikonomou and Bagci (OB), are criticizing our previous paper [2]. Let us start with three statements regarding the essence of the discussion in Ref. [2].

(i) It was shown in Ref. [2] that the MaxEnt variational approach used in conjunction with Rényi’s entropy leads to inconsistencies.

(ii) These inconsistencies are due to a hidden relation between the concomitant Legendre multipliers ( $\lambda_1$  and  $\lambda_2$ ) discovered while dissecting the variational process that leads to the appropriate MaxEnt probability distribution.

(iii) Rényi’s entropy is not of trace form, while Tsallis’s entropy is of such nature. Thus, we can expect differences to arise in a MaxEnt treatment.

We now state the essence of the preceding Comment. (a) Oikonomou and Bagci do not question the first two above points. Their claim is that points (i) and (ii) apply also to Tsallis’s entropy. More precisely, they purport to discover that a hidden relation emerges in Tsallis’s MaxEnt treatment as well.

### II. THE TWO-DISTRIBUTION PROBLEM

Oikonomou and Bagci work with the pseudo-Tsallis probability distribution (PD)

$$P_{OB} = Z^{-1} [1 + (1 - q)\beta Z^{q-1}(U - \langle U \rangle)]^{1/(q-1)}, \quad (1)$$

$$Z = \left( \int P_{OB}^q d\mu \right)^{1/(1-q)}, \quad (2)$$

which does not coincide with Tsallis’s pioneering PD [3], which reads

$$P_T = \frac{[1 + \beta(1 - q)U]^{1/(q-1)}}{Z_T}, \quad (3)$$

$$Z_T = \int_M [1 + \beta(1 - q)U]^{1/(q-1)} d\mu. \quad (4)$$

Thus, we can rephrase statement (a) as follows: OB purport to discover that a hidden relation emerges in Tsallis’s MaxEnt treatment for their PD  $P_{OB}$ , which is not Tsallis’s PD. We

could finish our Reply right here. However, let us delve deeper into the issue in order to gain some insight into why OB get their peculiar PD distribution (1). Such is the subject of next section.

### III. THE BOLTZMANN-GIBBS EXPONENTIAL DISTRIBUTION ACCORDING TO OIKONOMOU AND BAGCI

#### A. Normal procedure

In order to better illustrate the Oikonomou-Bagci procedure, we apply it here to the Boltzmann-Gibbs (BG) exponential distribution. One maximizes in such an instance

$$F_{S_B}(P) = - \int P \ln P d\mu + \lambda_1 \left( \int P U d\mu - \langle U \rangle \right) + \lambda_2 \left( \int P d\mu - 1 \right). \quad (5)$$

The first variation becomes

$$F_{S_B}(P + h) - F_{S_B}(P) = - \int (\ln P - \lambda_1 U - \lambda_2 + 1) h d\mu + O(h^2). \quad (6)$$

Accordingly,

$$\ln P - \lambda_1 U - \lambda_2 + 1 = 0. \quad (7)$$

Here, as most people do, *but not OB*, one immediately deduces  $P$  and is immediately led to

$$P = e^{\lambda_1 U} e^{\lambda_2 - 1}. \quad (8)$$

Here  $P$  normalization entails

$$e^{\lambda_2 - 1} \int e^{\lambda_1 U} d\mu = 1 \quad (9)$$

and then

$$e^{\lambda_2 - 1} = \frac{1}{\int e^{\lambda_1 U} d\mu}. \quad (10)$$

In other words,

$$P = \frac{e^{\lambda_1 U}}{\int e^{\lambda_1 U} d\mu}. \quad (11)$$

Further, one finds

$$\langle U \rangle = \frac{\int U e^{\lambda_1 U} d\mu}{\int e^{\lambda_1 U} d\mu}. \quad (12)$$

Well known physical arguments, as shown first by Gibbs himself [4], allow one to identify  $\lambda_1$ ,

$$\lambda_1 = -\beta = -\frac{1}{kT}. \quad (13)$$

**B. Oikonomou-Bagci procedure**

Starting with Eq. (7), OB follow a different trajectory so as to ascertain which is the proper  $P$ . They first multiply (7) by  $P$  and integrate, finding

$$\int P \ln P d\mu - \lambda_1 \langle U \rangle - \lambda_2 + 1 = 0, \quad (14)$$

so that

$$\lambda_2 = \int P \ln P d\mu + \beta \langle U \rangle + 1, \quad (15)$$

which OB would call a hidden relation between  $\lambda_2$  and  $\beta$ . Substituting (15) into (7), OB obtain

$$\ln P + \beta(U - \langle U \rangle) - \int P \ln P d\mu = 0 \quad (16)$$

or

$$P = \exp\left(-\beta(U - \langle U \rangle) + \int P \ln P d\mu\right). \quad (17)$$

Integrating once again OB are led to

$$\exp\left(\int P \ln P d\mu\right) \int e^{-\beta(U - \langle U \rangle)} d\mu = 1. \quad (18)$$

This is a critical stage. Oikonomou and Bagci choose to write  $Z$  not as

$$Z^{-1} = \exp\left(\int P \ln P d\mu + \beta \langle U \rangle\right), \quad (19)$$

but as

$$Z_{OB}^{-1} = \exp\left(\int P_{OB} \ln P_{OB} d\mu\right), \quad (20)$$

leading to

$$P_{OB} = \frac{e^{\beta(U - \langle U \rangle)}}{Z_{OB}}, \quad (21)$$

and then it follows that

$$S = \ln Z_{OB}, \quad (22)$$

which is obviously an incorrect result. This happens because of the Oikonomou-Bagci choice (20). With the selection (19) they would have reached instead

$$P = \frac{e^{-\beta U}}{Z}, \quad (23)$$

so that

$$\exp\left(-\int P \ln P d\mu\right) = Z e^{\beta \langle U \rangle} \quad (24)$$

and

$$S = \ln Z + \beta \langle U \rangle, \quad (25)$$

the correct result. We have clearly identified, with reference to the BG distribution, the origin of OB's troubles.

**IV. TSALLIS'S PD**

**A. Normal procedure**

The first variation's equation is [2]

$$\frac{q}{1-q} P^{q-1} + \lambda_1 U + \lambda_2 = 0, \quad (26)$$

$$\lambda_1 = -\beta q Z_q^{1-q}, \quad \lambda_2 = \frac{q}{q-1} Z_q^{1-q}, \quad (27)$$

leading to

$$P = \frac{[1 + (1-q)\beta U]^{1/(q-1)}}{Z_q}, \quad (28)$$

and, for  $S_q$ ,

$$S_q = \ln_q Z_q + Z_q^{1-q} \beta \langle U \rangle, \quad (29)$$

the correct result.

**B. Tsallis's PD according to Oikonomou and Bagci**

Oikonomou and Bagci multiply (26) by  $P$  and integrate

$$\frac{q}{1-q} \int P^q d\mu + \lambda_1 \langle U \rangle + \lambda_2 = 0, \quad (30)$$

which they call a hidden relation between two Lagrange multipliers entering the MaxEnt treatment. This is Eq. (1) in the preceding Comment, the core of their contribution. They now choose

$$\lambda_1 = -\frac{\beta q \int P^q d\mu}{1 + (1-q)\beta \langle U \rangle} \quad (31)$$

and obtain

$$\lambda_2 = \frac{q}{q-1} \int P^q d\mu + \frac{\beta q \int P^q d\mu}{1 + (1-q)\beta \langle U \rangle}, \quad (32)$$

so that

$$P^{q-1} = \frac{\beta q \int P^q d\mu}{1 + (1-q)\beta \langle U \rangle} [1 + (1-q)\beta U] \quad (33)$$

or

$$P = \left(\frac{\beta q \int P^q d\mu}{1 + (1-q)\beta \langle U \rangle}\right)^{1/(q-1)} [1 + (1-q)\beta U]^{1/(q-1)}. \quad (34)$$

Here OB are at a critical stance. Had they selected

$$Z_q = \left(\frac{\beta q \int P^q d\mu}{1 + (1-q)\beta \langle U \rangle}\right)^{1/(1-q)}, \quad (35)$$

they would have found for  $P$  the expression (28), the right Tsallis result, obtaining the hidden relation (30) notwithstanding. We see that, contrary to OB's claim, the hidden relation

impedes nothing. However, at this crucial stage OB chose to write

$$\lambda_1 = -\beta q, \tag{36}$$

leading to

$$P = \left( \int P^q d\mu \right)^{1/(q-1)} \left[ 1 + \frac{1+(1-q)\beta(U-\langle U \rangle)}{\int P^q d\mu} \right]^{1/(q-1)}, \tag{37}$$

which is Eq. (1) above. According to the Oikonomou-Bagci choice (36) above we have now

$$\mathcal{Z}_q = \left( \int P^q d\mu \right)^{1/(1-q)}, \tag{38}$$

and

$$\mathcal{S}_q = \ln_q \mathcal{Z}_q, \tag{39}$$

an incorrect result, arising because  $\lambda_1$  was incorrectly chosen.

### V. MaxEnt RECIPROCITY RELATIONS

A word of caution is needed. We have above used words like “choosing” or “selection.” This is speaking in a rather loose fashion. In fact, MaxEnt prescribes a definite recipe to find the Lagrange multipliers  $\lambda$ . MaxEnt asserts that, if the *a priori* known information concerns  $N$  expectation values  $\langle A_k \rangle$  and then  $N + 1$  (accounting for normalization) Lagrange multipliers  $\lambda_k$ , then the entropy  $S$  acquires the form (the MaxEnt version of  $S$ )

$$S = \lambda_0 + \sum_{k=1}^N \lambda_k \langle A_k \rangle. \tag{40}$$

The  $\lambda_k$ 's are obtained via so-called reciprocity relations (see, for example, [5])

$$\lambda_k = \frac{\partial S}{\partial \langle A_k \rangle}. \tag{41}$$

In practice, however, instead of solving Eq. (41) one often makes educated guesses for the  $\lambda_k$ 's, as reported above in this paper.

How is such an educated guess made in the Tsallis instance? A main criterion is to choose Tsallis's  $\lambda_1$  in such a manner that, in the limit  $q \rightarrow 1$ , it should coincide with the Boltzmann-Gibbs  $\lambda_1$ . In that case, from such a correct  $\lambda_1$  one immediately derives a Tsallis  $\lambda_2$  that yields then the usual Tsallis distribution. This  $\lambda_1$  criterion is satisfied by both OB's guess (31) and Tsallis's guess (27). One can appeal then to Ockham's razor to select (27). A word of caution seems appropriate. Guessing is an art, not science. If inspiration fails in guessing the Lagrange parameter, one can always appeal to Eq. (41), which never fails.

### VI. CONCLUSION

The preceding Comment to which we have replied here serves the useful purpose of highlighting issues related to Tsallis's statistics, but does not invalidate our paper [2]. Oikonomou and Bagci are not using Tsallis's PD distribution but one deduced by them. The main result of the preceding Comment is Eq. (1) therein, which we showed does not prevent one from arriving at the correct Tsallis PD. The error of the preceding Comment lies in an injudicious choice of Lagrange multipliers. This invalidates its conclusions.

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