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Finite-dimensional pointed Hopf algebras over finite simple groups of Lie type I. Non-semisimple classes in $\mathbf{PSL}_n(q)$ ☆

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ABSTRACT

We show that Nichols algebras of most simple Yetter–Drinfeld modules over the projective special linear group over a finite field, corresponding to non-semisimple orbits, have infinite dimension. We spell out a new criterium to show that a rack collapses.

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Racks
Finite groups of Lie type

Ph'nglui mglw'nafh Cthulhu R'lyeh wgah'nagl fhtagn

1. Introduction

This is the first article of a series intended to determine the finite-dimensional pointed Hopf algebras with group of group-likes isomorphic to a finite simple group of Lie type. We now give an Introduction to the whole series.

1.1. The general question we are dealing with is the classification of finite-dimensional complex pointed Hopf algebras H whose group of group-like elements is a finite simple group. We say that a finite group G *collapses* when every finite-dimensional pointed Hopf algebra H, with $G(H) \simeq G$ is isomorphic to $\mathbb{C}G$ [3]. Here are the antecedents of that question.

- If $G \simeq \mathbb{Z}/p$ is simple abelian, then the classification is known: for $p = 2$ by [22], see also [11]; for $p > 7$, by [9, Remark 1.10 (v)]; for $p = 5, 7$, combining [7, Theorem 1.3] and [10].
- If $G \simeq \mathbb{A}_m$, $m \geq 5$ is alternating, then G collapses [3].
- If G is a sporadic simple group, then G collapses, except for the groups $G = Fi_{22}$, B, M. For these groups, all irreducible Yetter–Drinfeld modules $M(\mathcal{O}, \rho)$ have infinite dimensional Nichols algebra, except for a short list appearing in [4, Table 1] and improved in [14, Appendix], of examples not known to be finite-dimensional.
- If $G = \mathbf{SL}_2(q)$, $\mathbf{GL}_2(q)$, $\mathbf{PGL}_2(q)$ or $\mathbf{PSL}_2(q)$, all irreducible Yetter–Drinfeld modules $M(\mathcal{O}, \rho)$ have infinite dimensional Nichols algebra, except for a list of examples not known to be finite-dimensional given in [15,16]. Particularly, $\mathbf{PSL}_2(q)$ collapses for $q > 2$ even.

1.2. In this series of papers we consider finite-dimensional pointed Hopf algebras with finite simple group of Lie type. We recall the description of such groups. Let p be a prime number, $m \in \mathbb{N}$, $q = p^m$ and $\mathbb{F}_q$ the field with q elements.

⋄ Let $\mathbb{G}$ be a semisimple algebraic group defined over $\mathbb{F}_q$. A Steinberg endomorphism $F : \mathbb{G} \to \mathbb{G}$ is an abstract group automorphism having a power equal to a Frobenius map [21, 21.3]. The subgroup $\mathbb{G}^F$ of fixed points by F is called a *finite group of Lie type* [21, 21.6].

⋄ Assume that $\mathbb{G}$ is a simple simply connected algebraic group. Then $\mathbb{G}/Z(\mathbb{G})$ is a simple abstract group [21, 12.5] but $\mathbb{G}^F$ is not simple in general. In fact $\boldsymbol{G} := \mathbb{G}^F/Z(\mathbb{G}^F)$ is a simple finite group except for 8 examples that appear in low rank and with $q = 2$ or 3 (Tits Theorem [21, 24.17]). These $\boldsymbol{G}$ are called *finite simple groups of Lie type* although they are *not* finite groups of Lie type in the sense above, in general.

There are three possible classes of Steinberg endomorphisms of simple algebraic groups [21, 22.5] and accordingly we consider three families of finite simple groups of Lie type:

Chevalley groups. In the terminology of [21], these correspond to $\mathbb{F}_q$-split Steinberg endomorphisms. That is, there exists an F-stable torus $\mathbb{T}$ such that $F(t) = t^q$ for all $t \in \mathbb{T}$. Then F is called a Frobenius map and $\mathbb{G}^F = \mathbb{G}(\mathbb{F}_q)$ is the finite group of $\mathbb{F}_q$-points. Explicitly, these are the groups:

$$\mathbf{PSL}_n(q), \quad n \geq 2 \ \big(\text{except } \mathbf{PSL}_2(2) \simeq \mathbb{S}_3 \text{ and } \mathbf{PSL}_2(3) \simeq \mathbb{A}_4 \text{ that are not simple}\big);$$
$$\mathbf{PSp}_{2n}(q), \quad n \geq 2; \qquad \mathbf{P\Omega}_{2n+1}(q), \quad n \geq 3, \ q \text{ odd};$$
$$\mathbf{P\Omega}^+_{2n}(q), \quad n \geq 4; \qquad G_2(q), \quad q \geq 3; \qquad F_4(q); \qquad E_6(q); \qquad E_7(q); \qquad E_8(q).$$

Also, $\mathbf{PSL}_2(4) \simeq \mathbf{PSL}_2(5) \simeq \mathbb{A}_5$; $\mathbf{PSL}_2(9) \simeq \mathbb{A}_6$; $\mathbf{PSL}_4(2) \simeq \mathbb{A}_8$, cf. [25].

Steinberg groups. These correspond to *twisted* Steinberg endomorphisms. A Steinberg endomorphism is twisted if it is not split and it is the product of an $\mathbb{F}_q$-split endomorphism with an algebraic automorphism of $\mathbb{G}$ [21]; we may assume that F is the product of a Frobenius endomorphism with an automorphism of $\mathbb{G}$ induced by a non-trivial Dynkin diagram automorphism. Explicitly, these are the groups:

$$\mathbf{PSU}_n(q), \quad n \geq 3 \ \big(\text{except } \mathbf{PSU}_3(2)\big); \qquad \mathbf{P\Omega}^-_{2n}(q), \quad n \geq 4; \qquad {}^3D_4(q); \qquad {}^2E_6(q).$$

Suzuki–Ree groups. Related to *very twisted* Steinberg endomorphisms [21]. Explicitly, these are the groups:

$${}^2B_2\big(2^{2h+1}\big), \quad h \geq 1; \qquad {}^2G_2\big(3^{2h+1}\big), \quad h \geq 1; \qquad {}^2F_4\big(2^{2h+1}\big), \quad h \geq 1.$$

1.3. The base field is $\mathbb{C}$. Let G be a finite group and let H be a pointed Hopf algebra with $G(H) \simeq G$. For details on the following exposition – not needed henceforth and included only for completeness, see [8,5].

Let $0 = H_{-1} \subset H_0 = \mathbb{C}G(H) \subset H_1 \subset \dots$ be the coradical filtration of H and $\operatorname{gr} H = \bigoplus_{n\in\mathbb{N}_0} H_n/H_{n-1} \simeq R\#\mathbb{C}G(H)$ be the associated graded Hopf algebra. Here $R = \bigoplus_{n\in\mathbb{N}_0} R^n$ is a graded Hopf algebra in the braided tensor category ${}^{\mathbb{C}G}_{\mathbb{C}G}\mathcal{YD}$ of Yetter–Drinfeld modules over $\mathbb{C}G$. Also, the subalgebra of R generated by $V := R^1$ is isomorphic to the Nichols algebra $\mathfrak{B}(V)$ of V. Hence $\dim H < \infty \iff \dim R < \infty \implies \dim \mathfrak{B}(V) < \infty$. Thus we need to address the question: *Determine all* $V \in {}^{\mathbb{C}G}_{\mathbb{C}G}\mathcal{YD}$ *with* $\dim \mathfrak{B}(V) < \infty$.

In particular, the following are equivalent [3, Lemma 1.4]:

- G collapses.
- For every $V \in {}^{\mathbb{C}G}_{\mathbb{C}G}\mathcal{YD}$, $\dim \mathfrak{B}(V) = \infty$.
- For every *irreducible* $V \in {}^{\mathbb{C}G}_{\mathbb{C}G}\mathcal{YD}$, $\dim \mathfrak{B}(V) = \infty$.

Now all irreducible Yetter–Drinfeld modules over $\mathbb{C}G$ are of the form $M(\mathcal{O},\rho) = \operatorname{Ind}_{C_G(g)}^G \rho$, where $\mathcal{O}$ is a conjugacy class of G and $\rho \in \operatorname{Irr} C_G(g)$ for $g \in \mathcal{O}$ fixed. Set $\mathfrak{B}(\mathcal{O},\rho) := \mathfrak{B}(M(\mathcal{O},\rho))$. Then the initial question about the classification of finite-dimensional pointed Hopf algebras with finite simple group of Lie type G relies on the consideration of the following:

Question 1. *For such G, determine all pairs $(\mathcal{O},\rho)$ with* $\dim \mathfrak{B}(\mathcal{O},\rho) < \infty$.

1.4. A crucial observation is that the algebra $\mathfrak{B}(\mathcal{O},\rho)$ does not depend on the Yetter–Drinfeld module structure of $M(\mathcal{O},\rho)$ but only on the underlying braided vector space $(\mathbb{C}\mathcal{O}, c^\rho)$. In other words, the algebra $\mathfrak{B}(\mathcal{O},\rho)$ depends only on the rack $\mathcal{O}$ and the non-principal 2-cocycle arising from ρ, see Section 2 for definitions, or [5] for more details. In fact, to solve Question 1 for every finite group G is tantamount to solve

Question 2. *(See [3, Question 2].) Determine all pairs (X,q), where X is a finite rack and q is a non-principal 2-cocycle, such that* $\dim \mathfrak{B}(X, c^q) < \infty$.

The meaning of the next definition relies on the existence of some criteria for a rack to collapse, cf. Section 1.5, Section 2.2.

Definition 1.1. (See [3, 2.2].) A rack X *collapses* when $\dim \mathfrak{B}(X,q) = \infty$ for every finite faithful 2-cocycle q.

Therefore, we tackle the initial question about the classification of finite-dimensional pointed Hopf algebras with finite simple group of Lie type G (rephrased as Question 1) in the following way:

- Determine all conjugacy classes in G that collapse.
- If $\mathcal{O}$ is a conjugacy class in G that does not collapse, then for any ρ as above, compute the restriction c_X^ρ of the braiding c^ρ to a suitable abelian subrack X of $\mathcal{O}$. If the Nichols algebra $\mathfrak{B}(\mathbb{C}X, c_X^\rho)$ has infinite dimension (and this is checked by inspection of the list in [17]), then so has $\mathfrak{B}(\mathbb{C}\mathcal{O}, c^\rho)$.

1.5. In principle, to solve Question 2 one would need first to compute all possible non-principal 2-cocycles for a fixed rack X, before starting to deal with the corresponding Nichols algebras. A remarkable fact is the existence of criteria that dispense of this computation. The first such criterium is about racks of type D [3], see Section 2.2: *If X is a finite rack of type D, then X collapses*. In Section 2.2 we introduce the notion rack of type F, and prove an analogous criterium. To distinguish the setting where neither of these criteria apply, we shall say that a rack is *cthulhu*[1] when it is neither of type D nor

[1] See http://en.wikipedia.org/wiki/Cthulhu for spelling and pronunciation.

of type F. Also a rack is *sober* if every subrack is either abelian or indecomposable; this is stronger than being cthulhu. See Section 2.3 for examples.

1.6. We need the description of the conjugacy classes in finite simple groups of Lie type. Let $\mathbb{G}$ be a simple algebraic group, $\mathbb{G}_{\mathrm{sc}}$ its simply connected cover with $\boldsymbol{\pi} : \mathbb{G}_{\mathrm{sc}} \to \mathbb{G}$ the natural projection, F a Steinberg endomorphism of $\mathbb{G}_{\mathrm{sc}}$, cf. Section 1.2, $\boldsymbol{G} = \mathbb{G}_{\mathrm{sc}}^F/Z(\mathbb{G}_{\mathrm{sc}}^F)$, $\pi : \mathbb{G}_{\mathrm{sc}}^F \to \boldsymbol{G}$ the natural projection. Often F descends to $\mathbb{G}$, and then there is a projection $\pi : [\mathbb{G}^F, \mathbb{G}^F] \to [\mathbb{G}^F, \mathbb{G}^F]/\boldsymbol{\pi}(Z(\mathbb{G}_{\mathrm{sc}}^F)) \simeq \boldsymbol{G}$. Every $x \in \mathbb{G}_{\mathrm{sc}}$ has a Chevalley–Jordan decomposition $x = x_s x_u = x_u x_s$, with x_s semisimple and x_u unipotent. This decomposition boils down to $\mathbb{G}$ and to the finite groups $\mathbb{G}_{\mathrm{sc}}^F$ and $\boldsymbol{G}$, where it agrees with the decomposition in the p-part, namely x_u, and the p-regular part, namely x_s. We state a well-known fact referred to as the *isogeny argument*. Let $\mathcal{G}$ be a semisimple algebraic, respectively finite, group and $\mathcal{G}_u$ the set of unipotent, respectively p-elements, in $\mathcal{G}$.

Lemma 1.2. *Let $\mathcal{Z}$ be a central (algebraic) subgroup of $\mathcal{G}$ consisting of semisimple, respectively p-regular elements. Then the quotient map $\pi : \mathcal{G} \to \mathcal{G}/\mathcal{Z}$ induces a rack isomorphism $\pi : \mathcal{G}_u \to (\mathcal{G}/\mathcal{Z})_u$ and a bijection between the set of $\mathcal{G}$-conjugacy classes in $\mathcal{G}_u$ and that of $\mathcal{G}/\mathcal{Z}$-conjugacy classes in $(\mathcal{G}/\mathcal{Z})_u$.*

If $\mathcal{G}$ is semisimple algebraic, then $\mathcal{Z}$ is finite because it consists of semisimple elements. Hence $\mathcal{G}/\mathcal{Z}$ is again a semisimple algebraic group.

Proof. Clearly $\pi(\mathcal{G}_u) \subset (\mathcal{G}/\mathcal{Z})_u$. Let $g \in \mathcal{G}$ with $\pi(g) \in (\mathcal{G}/\mathcal{Z})_u$ and let $g = g_s g_u$ be its Chevalley–Jordan decomposition (respectively, the decomposition in the p-regular and the p-part). Then $\pi(g) = \pi(g_s)\pi(g_u)$, hence $\pi(g) = \pi(g_u)$ by uniqueness of the decomposition. Thus $\pi : \mathcal{G}_u \to (\mathcal{G}/\mathcal{Z})_u$ is surjective. Let now $g, h \in \mathcal{G}_u$ with $\pi(g) = \pi(h)$. Then $g = hz = zh$ for some $z \in \mathcal{Z}$; but this turns out to be the decomposition of g, hence $g = h$ and $\pi : \mathcal{G}_u \to (\mathcal{G}/\mathcal{Z})_u$ is a rack isomorphism. Finally, let again $g, h \in \mathcal{G}_u$. If $\mathcal{O}_g^{\mathcal{G}} = \mathcal{O}_h^{\mathcal{G}}$, then clearly $\mathcal{O}_{\pi(g)}^{\mathcal{G}/\mathcal{Z}} = \mathcal{O}_{\pi(h)}^{\mathcal{G}/\mathcal{Z}}$. Conversely, if $\mathcal{O}_{\pi(g)}^{\mathcal{G}/\mathcal{Z}} = \mathcal{O}_{\pi(h)}^{\mathcal{G}/\mathcal{Z}}$, then there exist $u \in \mathbb{G}$ and $z \in \mathcal{Z}$ such that $ugu^{-1} = hz = zh$; this is the decomposition of $ugu^{-1} \in \mathcal{G}_u$, hence $ugu^{-1} = h$ and $\mathcal{O}_g^{\mathcal{G}} = \mathcal{O}_h^{\mathcal{G}}$. □

Let $x \in \boldsymbol{G}$; pick $\mathtt{x} \in \mathbb{G}_{\mathrm{sc}}^F$ such that $\pi(\mathtt{x}) = x$. If $\mathtt{x} = \mathtt{x}_s \mathtt{x}_u$ is its Chevalley–Jordan decomposition, then $x_s = \pi(\mathtt{x}_s)$, $x_u = \pi(\mathtt{x}_u)$ is the Chevalley–Jordan decomposition of x, with x_s semisimple and x_u unipotent. Now $\mathtt{x}_u$ belongs to $\mathcal{K} := C_{\mathbb{G}_{\mathrm{sc}}^F}(\mathtt{x}_s)$, thus $x_u \in K := \pi(\mathcal{K})$ and there are morphisms of racks

$$\mathcal{O}_{\mathtt{x}_u}^{\mathcal{K}} \simeq \mathcal{O}_{x_u}^{K} \hookrightarrow \mathcal{O}_x^{\boldsymbol{G}}, \tag{1.1}$$

the first by the isogeny argument and the second by Remark 2.9 (c). Now the centralizer $C_{\mathbb{G}_{\mathrm{sc}}^F}(\mathtt{x}_s)$ is a reductive subgroup of $\mathbb{G}_{\mathrm{sc}}$ by [19, Theorem 2.2], and $\mathcal{K} = C_{\mathbb{G}_{\mathrm{sc}}}(\mathtt{x}_s) \cap \mathbb{G}_{\mathrm{sc}}^F$. Strictly, $C_{\mathbb{G}_{\mathrm{sc}}^F}(\mathtt{x}_s)$ is not of Lie type in the sense above, but close enough to allow some inductive procedure. So, we are reduced to investigate the conjugacy classes

Table 1
Unipotent classes in $\mathbf{PSL}_n(q)$, with (n,q) different from $(2,2),(2,3),(2,4),(2,5),(2,9)$, not known to collapse.

n	Type	q	Remark
2	(2)	even or not a square	sober, Lemma 3.6
3	(3)	2	sober, Lemma 3.7 (b)
	(2, 1)	2	cthulhu, Lemma 3.7 (a)
		even ≥ 4	cthulhu, Proposition 3.13, 3.16
4	(2, 1, 1)	2	cthulhu, Lemma 3.12
		even ≥ 4	not of type D, Proposition 3.13, open for type F

- x semisimple (the case $x = x_s$), or
- x unipotent, and from this try to catch the general case.

1.7. In the first paper of the series, we deal with non-semisimple classes in $\boldsymbol{G} = \mathbf{PSL}_n(q)$, except $\mathbf{PSL}_2(q)$ with $q = 2, 3, 4, 5, 9$ which is either solvable or was treated in [3]; see Section 1.2. To state our results, we start with some terminology. By the classical theory of the Jordan form, unipotent conjugacy classes in $\mathbf{GL}_n(q)$ are classified by their type; $u \in \mathbf{GL}_n(q)$ is of type $\lambda = (\lambda_1, \dots, \lambda_k)$ when the elementary factors of its characteristic polynomial equal $(X-1)^{\lambda_1}$, $(X-1)^{\lambda_2}$, $\dots$, $(X-1)^{\lambda_k}$, where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k$; thus u is unipotent.

Theorem 1.3. *Let $x \in \boldsymbol{G}$ and pick $\mathtt{x} \in \mathbf{SL}_n(q)$ such that $\pi(\mathtt{x}) = x$, with Jordan decomposition $\mathtt{x} = \mathtt{x}_s\mathtt{x}_u$. Assume that $\mathtt{x}_u \neq e$. Then either $\mathcal{O} = \mathcal{O}_x^{\boldsymbol{G}}$ collapses or else $\mathtt{x}_s$ is central and $\mathcal{O}$ is a unipotent class listed in Table 1.*

Semisimple classes require different tools and are treated in work in progress.

We deal with the Nichols algebras associated to the unipotent classes in Table 1 in Lemma 3.18, concluding the following result.

Theorem 1.4. *Let $\mathcal{O}$ be the conjugacy class of $x \in \boldsymbol{G} = \mathbf{PSL}_n(q)$ non-semisimple. Assume that either $\boldsymbol{G} \neq \mathbf{PSL}_3(2)$, or else that x is not of type (3). Then $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$, for every $\rho \in \operatorname{Irr} C_{\boldsymbol{G}}(x)$.*

The unipotent class $\mathcal{O}$ of type (3) in $\mathbf{PSL}_3(2)$ is sober and the centralizer of $x \in \mathcal{O}$ is cyclic of order 4. Hence any abelian subrack has at most two elements. If $\rho \in \operatorname{Irr} C_{\boldsymbol{G}}(x)$ is given by $\rho(x) = -1$, then it is not possible to decide whether the dimension of the Nichols algebra $\mathfrak{B}(\mathcal{O}, \rho)$ is finite or not by looking at subracks.

Section 2 is devoted to racks and Section 3 to unipotent classes: we prove Theorem 1.3 for them in Section 3.5. In Section 4 we prove the theorem for non-semisimple classes, see Proposition 4.4.

Notation. We denote the cardinal of a set X by $|X|$. If ℓ is a positive integer, then we set $\mathbb{I}_\ell = \{i \in \mathbb{N} : 1 \leq i \leq \ell\}$.

Let $e_{i,j} \in \Bbbk^{N\times P}$ be the matrix with 1 in the position (i,j) and 0 elsewhere. We denote by $\mathrm{id}_N \in \Bbbk^{N\times N}$ the identity matrix, and omit the subscript N when clear from the context.

Let G be a group and $x_1,\dots,x_N \in G$. Then $\langle x_1,\dots,x_N\rangle$ denotes the subgroup generated by them.

2. Racks

2.1. A rack is a set $X \neq \emptyset$ with an operation $\triangleright : X\times X \to X$ satisfying (a) $\varphi_x := x\triangleright\_\_$ is a bijection for every $x \in X$, and (b) the self-distributivity axiom $x\triangleright(y\triangleright z) = (x\triangleright y)\triangleright(x\triangleright z)$ for all $x,y,z \in X$. Let $\operatorname{Inn} X$ be the subgroup of $\mathbb{S}_X$ generated by φ_x, $x \in X$. All racks in this paper are finite, unless explicitly stated. The archetypical example of a rack is a conjugacy class $\mathcal{O}$ in a finite group G with the operation $x\triangleright y = xyx^{-1}$, $x,y \in \mathcal{O}$. We denote by $\mathcal{O}_x^G$ (or $\mathcal{O}_x$ when no confusion arises) the conjugacy class of x in G. Conjugacy classes are racks of a special sort, namely crossed sets, as they satisfy (c) $x\triangleright x = x$ for all $x \in X$ and (d) $x\triangleright y = y$, iff $y\triangleright x = x$ for all $x,y \in X$, see e.g. [5]. But this distinction is not relevant for the purposes of this paper, so we assume that *all the racks appearing here are crossed sets*. The following statement will be used along the paper.

Remark 2.1. Let N be a normal subgroup of a finite group G, $x \in N$. Then there exists $x = x_1,\dots,x_s \in N$ such that

$$\mathcal{O}_x^G = \coprod_{1\le i\le s} \mathcal{O}_{x_i}^N, \tag{2.1}$$

and $\mathcal{O}_x^N \simeq \mathcal{O}_{x_i}^N$ as racks for all $i \in \mathbb{I}_s$.

Indeed, $\mathcal{O}_x^G \subset N$, since N is normal, hence (2.1) holds. Now, if $g_i \in G$ satisfies $g_i\triangleright x = x_i$, then $g_i\triangleright\mathcal{O}_x^N = \mathcal{O}_{x_i}^N$ and the last claim follows.

A rack X is *abelian* when $x\triangleright y = y$, for all $x,y \in X$. A rack is *indecomposable* when it is not a disjoint union of two proper subracks, or equivalently when it is a single $\operatorname{Inn} X$ orbit. Any rack is the disjoint union of maximal indecomposable subracks (in a unique way), called its indecomposable components [5, 1.17].

A rack X is *simple* when for any projection of racks $\pi : X \to Y$, either π is an isomorphism or Y has only one element. The classification of finite simple racks is known [5, 3.9, 3.12], [20]; one of the main parts consists of conjugacy classes in a finite simple non-abelian group.

2.2. Racks of type D, F

We discuss criteria to decide that a rack collapses, see Definition 1.1. We start by the relevant definitions. Let G be a group and let X be a finite rack.

Definition 2.2. (See [3, 3.5].) X is *of type D* when it has a decomposable subrack $Y = R \amalg S$ with elements $r \in R$, $s \in S$ such that

$$r \triangleright \big(s \triangleright (r \triangleright s)\big) \neq s. \tag{2.2}$$

Remark 2.3. If $\mathcal{O}$ is a finite conjugacy class in G, then the following are equivalent:

(1) The rack $\mathcal{O}$ is of type D.
(2) There exist r, $s \in \mathcal{O}$ such that $\mathcal{O}_r^{\langle r,s\rangle} \neq \mathcal{O}_s^{\langle r,s\rangle}$ and

$$(rs)^2 \neq (sr)^2. \tag{2.3}$$

Proof. Notice that (2.2) and (2.3) are equivalent in this setting. If (2) holds, then $Y = \mathcal{O}_r^{\langle r,s\rangle} \amalg \mathcal{O}_s^{\langle r,s\rangle}$ is the desired decomposable subrack. Conversely if (1) holds with $Y = R \amalg S$ and $r \in R$, $s \in S$, then $\mathcal{O}_r^{\langle r,s\rangle} \subset R$, $\mathcal{O}_s^{\langle r,s\rangle} \subset S$. □

Definition 2.4. X is *of type F* if it has a family of subracks $(R_a)_{a\in\mathbb{I}_4}$ and a family $(r_a)_{a\in\mathbb{I}_4}$ with $r_a \in R_a$, and for $a \neq b \in \mathbb{I}_4$, $R_a \cap R_b = \emptyset$, $R_a \triangleright R_b = R_b$,

$$r_a \triangleright r_b \neq r_b. \tag{2.4}$$

Here F stands for a rack with a family of *four* mutually disjoint subracks.

Remark 2.5. If $\mathcal{O}$ is a finite conjugacy class in G, then the following are equivalent:

(1) The rack $\mathcal{O}$ is of type F.
(2) There exist $r_a \in \mathcal{O}$, $a \in \mathbb{I}_4$, such that $\mathcal{O}_{r_a}^{\langle r_a:a\in\mathbb{I}_4\rangle} \neq \mathcal{O}_{r_b}^{\langle r_a:a\in\mathbb{I}_4\rangle}$, $a \neq b$ in $\mathbb{I}_4$, and

$$r_a r_b \neq r_b r_a, \quad a \neq b \in \mathbb{I}_4. \tag{2.5}$$

Proof. Notice that (2.4) and (2.5) are equivalent in this setting. If (2) holds, then $R_a = \mathcal{O}_{r_a}^{\langle r_a:a\in\mathbb{I}_4\rangle}$, $a \in \mathbb{I}_4$ is the desired family of subracks. Conversely if (1) holds, then $\mathcal{O}_{r_a}^{\langle r_a:a\in\mathbb{I}_4\rangle} \subset R_a$, for all $a \in \mathbb{I}_4$ and we have (2). □

The rack formulations (1) in Remark 2.3, respectively 2.5, are more effective for applications to the classification of Hopf algebras, see Remark 2.9; the equivalent formulations (2) are useful in proofs.

Remark 2.6. Let $\mathcal{O}$ be a finite conjugacy class in G. If $\mathcal{O}$ is of type D, respectively F, then there is a maximal $K < G$ such that $\mathcal{O} \cap K$ is of type D, respectively F.

Indeed, let r, $s \in \mathcal{O}$ such that $\mathcal{O}_r^{\langle r,s\rangle} \neq \mathcal{O}_s^{\langle r,s\rangle}$; then $\langle r, s\rangle \neq G$, so there is a maximal K containing $\langle r, s\rangle$. Same for type F.

The following remark, a variation of [14, Lemma 2.5], is useful to check when the conditions in Remarks 2.3 or 2.5 hold.

Remark 2.7. Let G be a finite group and let $r, s \in G$ be involutions such that $[s, r] \neq 1$. Then $\mathcal{O}_r^{\langle r,s\rangle} \neq \mathcal{O}_s^{\langle r,s\rangle}$ if and only if $|rs|$ is even > 2.

Theorem 2.8. *A rack X of type D (respectively, F) collapses.*

Proof. Type D: This is [3, Theorem 3.6], cf. [18, Theorem 8.6].

Type F: Let $q : X \times X \to \mathbf{GL}(n, \mathbb{C})$ be a finite faithful 2-cocycle on X. We need to check that the Nichols algebra associated to the braided vector space $(V, c) := (\mathbb{C}X \otimes \mathbb{C}^n, c^q)$ attached to X and q has infinite dimension. By hypothesis there is a subrack $Y = \coprod_{a\in\mathbb{I}_4} R_a$ with $R_a \triangleright R_b = R_b$ as in Definition 2.4. Without loss of generality, we may assume that $Y = X$. By [5, 6.14], cf. also [3, Theorem 2.1], (V, c) can be realized as Yetter–Drinfeld module over a finite group G. Actually we may choose the subgroup G of $\mathbf{GL}(V)$ generated by $g_x : V \to V$, $g_x(e_y \otimes v) = e_{x\triangleright y} \otimes q_{xy}(v)$, $x, y \in X$, $v \in \mathbb{C}^n$. Let $V_a := \mathbb{C}R_a \otimes \mathbb{C}^n$, a Yetter–Drinfeld submodule of V; clearly $V = \bigoplus_{a\in A} V_a$. Now we may replace V_a by a simple Yetter–Drinfeld submodule U_a with $r_a \in \operatorname{supp} U_a = \{g \in G : U_{a,g} \neq 0\}$, where $U_a = \bigoplus_{g\in G} U_{a,g}$ is the grading coming from the Yetter–Drinfeld module structure. Then $c^2 \neq \mathrm{id}$ on $U_a \otimes U_b$ for $a \neq b \in \mathbb{I}_4$ by (2.4). This means that the Weyl groupoid $\mathcal{W}$ of $U = \bigoplus_{a\in\mathbb{I}_4} U_a$, see [6], has rank at least 4 and the Dynkin diagram of one of his objects would then have an edge between any two distinct vertices. Now if $\dim \mathfrak{B}(X, q) < \infty$, then $\mathcal{W}$ is finite. But this contradicts the classification of finite Weyl groupoids in [13, Theorem 1.1]. □

The proof for type F uses stronger facts than the proof for type D, as it relies on the classification from [13]. By this reason, our order of preference for application of these criteria is first type D, then F.

Remark 2.9. Being open conditions (i.e., expressed by inequalities), these notions enjoy some favorable properties.

(a). *If a rack X contains a subrack of type D (respectively, F), then X is of type D (respectively, F). If a rack X projects onto a rack of type D (respectively, F), then X is of type D (respectively, F).*

Let K be a subgroup of G, $\tau \in K$, $C_G(K)$ the centralizer of K in G.

(b). *If $\mathcal{O}_\tau^K$ is of type D (respectively, F), then so is $\mathcal{O}_\tau^G$.*

(c). *Let $\kappa \in C_G(K)$. The right multiplication by κ identifies $\mathcal{O}_\tau^K$ with a subrack of $\mathcal{O}_{\tau\kappa}^G$; if $\mathcal{O}_\tau^K$ is of type D (respectively, F), then so is $\mathcal{O}_{\tau\kappa}^G$.*

(d). *Assume that $G = G_1 \times \ldots \times G_r \ni x = (x_1, \ldots, x_r)$. Then $\mathcal{O}_x^G = \mathcal{O}_{x_1}^{G_1} \times \ldots \times \mathcal{O}_{x_r}^{G_r}$; hence, if $\mathcal{O}_{x_j}^{G_j}$ is of type D (respectively, F) for some j, then so is $\mathcal{O}_x^G$.*

Now an indecomposable rack Z always admits a rack epimorphism onto a simple rack X. Therefore, any indecomposable rack having a quotient simple rack of type D collapses. Hence it is natural to ask for the classification of all *simple* racks of type D or F. See [1] for the present status of this problem, in the case of type D.

Lemma 2.10. *Let X and Y be racks.*

- (i) *Assume that there are $y_1 \neq y_2 \in Y$, $x_1 \neq x_2 \in X$ such that $y_1 \triangleright y_2 = y_2$, $x_1 \triangleright (x_2 \triangleright (x_1 \triangleright x_2)) \neq x_2$. Then $X \times Y$ is of type D.*
- (ii) *Assume that there are $y_1, \dots, y_4 \in Y$ all different, $x_1 \dots, x_4 \in X$ such that $y_i \triangleright y_j = y_j$, $x_i \triangleright x_j \neq x_j$ for $i \neq j \in \mathbb{I}_4$. Then $X \times Y$ is of type F.*
- (iii) *Let X_i be disjoint sets provided with bijections $\varphi_i : X \to X_i$, $i \in \mathbb{I}_2$; $X^{(2)} := X_1 \amalg X_2 \simeq X \times \mathbb{I}_2$ is a rack with $\varphi_i(x) \triangleright \varphi_j(y) = \varphi_j(x \triangleright y)$, $i, j \in \mathbb{I}_2$. If there are $x_1 \neq x_2 \in X$ satisfying (2.2), then $X^{(2)}$ is of type D.*

Proof. Take $R = X \times \{y_1\}$, $S = X \times \{y_2\}$, $r = (x_1, y_1)$, $s = (x_2, y_2)$ in (i); $R_j = X \times \{y_j\}$, $r_j = (x_j, y_j)$, $j \in \mathbb{I}_4$, in (ii). Now (i) implies (iii). □

2.3. Cthulhu racks

Recall that a rack is *cthulhu* when it is neither of type D nor of type F; and that it is *sober* if every subrack is either abelian or indecomposable. A sober rack is cthulhu. More than this:

Remark 2.11. If all subracks *generated by two elements* of a rack X are either abelian or indecomposable, then X is cthulhu.

Here are some examples of these notions.

Example 2.12. The rack $\mathcal{O}_3^{\mathbb{S}_4}$ of 3-cycles in $\mathbb{S}_4$, also known as the cube rack, is the union of two tetrahedral racks (conjugacy classes in $\mathbb{A}_4$) not commuting with each other. It is neither of type D nor of type F.

Example 2.13. Every abelian rack is sober. The tetrahedral rack is sober. The conjugacy class of non-trivial unipotent elements in $\mathbf{PSL}_2(q)$, where either q is even, or odd but not a square, is sober, cf. Lemma 3.5.

Example 2.14. The rack of transpositions in $\mathbb{S}_n$ is cthulhu for $n \geq 2$ but not sober for $n \geq 4$; see [3, Remark 4.2] for other examples of conjugacy classes in symmetric groups that are cthulhu.

Example 2.15. Let $\mathcal{D}_n$ be the affine rack $(\mathbb{Z}_n, T)$ where T is the inversion; when n is odd, it is the class of involutions in the dihedral group $\mathbb{D}_n$ of order $2n$. If $n > 4$ is even, then

$\mathcal{D}_n$ is of type D [2, Lemma 2.2]. If n is odd, then $\mathcal{D}_n$ is sober. For, observe that every subgroup of $\mathbb{D}_n$ is either cyclic of order d or isomorphic to a dihedral group $\mathbb{D}_d$, for some $d|n$. Let X be a subrack of $\mathcal{D}_n$ and $H = \langle X \rangle$. Since X consists of involutions, $H \simeq \mathbb{D}_d$ for some $d|n$; hence X is the class of involutions in H, that is indecomposable.

3. Unipotent classes in $\mathbf{SL}_n(q)$

Let $n \in \mathbb{N}$, $n \geq 2$. In this section, we consider $G = \mathbf{SL}_n(q)$ and investigate when a unipotent conjugacy class collapses. By the isogeny argument, the result carries over $\boldsymbol{G} = \mathbf{PSL}_n(q)$. We deal with unipotent classes of type D in Section 3.3, with those of type F in Section 3.4. We summarize in Section 3.5.

Before starting we state an observation useful not only in the unipotent context. Let $u \in G$ with block decomposition

$$u = \begin{pmatrix} u_1 & 0 & \dots & 0 \\ 0 & u_2 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & u_k \end{pmatrix}, \tag{3.1}$$

where $u_j \in \mathbf{SL}_{\lambda_j}(q)$, $j \in \mathbb{I}_k$. By Remark 2.9, we have:

Lemma 3.1. *If $\mathcal{O}_{u_i}^{\mathbf{SL}_{\lambda_i}(q)}$ is of type D (respectively F) for some $i \in \mathbb{I}_k$, then so is $\mathcal{O}_u^G$.* □

3.1. Unipotent classes

Recall that a unipotent $u \in \mathbf{GL}_n(q)$ is of type $\lambda = (\lambda_1, \dots, \lambda_k)$ when the elementary factors of its characteristic polynomial equal $(X-1)^{\lambda_1}$, $(X-1)^{\lambda_2}$, ..., $(X-1)^{\lambda_k}$, where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k$. A (unipotent) $x \in \mathbf{GL}_n(q)$ (or its conjugacy class) is *regular* if it is of type (n), i.e. if its characteristic and minimal polynomials coincide. Every element of type $\lambda = (\lambda_1, \dots, \lambda_k)$ in $\mathbf{GL}_n(q)$ is conjugate to a u with block decomposition as in (3.1) with $u_i = \begin{pmatrix} 1 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 1 & 1 \\ 0 & \dots & 0 & 1 \end{pmatrix} \in \mathbf{SL}_{\lambda_i}(q)$. To describe unipotent conjugacy classes in $G = \mathbf{SL}_n(q)$ and other purposes we set some notation. For $\mathbf{a} = (a_1, \dots, a_{n-1}) \in \mathbb{F}_q^{n-1}$, define $r_{\mathbf{a}}$ and the set $R_{\mathbf{a}} \subset G$ by:

$$r_{\mathbf{a}} = \begin{pmatrix} 1 & a_1 & 0 & \dots & 0 \\ 0 & 1 & a_2 & \dots & 0 \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 1 & a_{n-1} \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix} \in R_{\mathbf{a}} = \left\{ \begin{pmatrix} 1 & a_1 & * & \dots & * \\ 0 & 1 & a_2 & \dots & * \\ \vdots & & \ddots & \ddots & * \\ 0 & \dots & \dots & 1 & a_{n-1} \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix} \right\}. \tag{3.2}$$

If $\mathbf{a} = (a, 1, \dots, 1)$, $a \in \mathbb{F}_q^{\times}$, then we simply write $r_{\mathbf{a}} = r_a$.

The sets $R_{\mathbf{a}}$ enjoy the following properties: $R_{\mathbf{a}}R_{\mathbf{b}} \subset R_{\mathbf{a}+\mathbf{b}}$, $R_{\mathbf{a}}^{-1} = R_{-\mathbf{a}}$, hence $R_{\mathbf{a}} \triangleright R_{\mathbf{b}} \subset R_{\mathbf{b}}$. Thus, $\coprod_{\mathbf{a}\in\mathcal{F}} R_{\mathbf{a}}$ is a subrack of G for every subset $\mathcal{F}$ of $\mathbb{F}_q^{n-1}$. We shall need more precise formulae. For $\mathbf{a} = (a_1, \dots, a_{n-1})$, $\mathbf{b} = (b_1, \dots, b_{n-1}) \in \mathbb{F}_q^{n-1}$, set

$$\theta^k_{\mathbf{a},\mathbf{b}} = a_k b_{k+1} - a_{k+1} b_k, \qquad 1 \le k \le n-2, \tag{3.3}$$

$$\gamma^k_{\mathbf{a},\mathbf{b}} = 2a_k b_{k+1} + (a_k + b_k)(a_{k+1} + b_{k+1}), \qquad 1 \le k \le n-2, \tag{3.4}$$

$$\nu^k_{\mathbf{a},\mathbf{b}} = a_k b_{k+1}(a_{k+2} + b_{k+2}) + a_{k+1} b_{k+2}(a_k + b_k) \qquad 1 \le k \le n-3. \tag{3.5}$$

Then

$$r_{\mathbf{a}} \triangleright r_{\mathbf{b}} = \begin{pmatrix} 1 & b_1 & \theta^1_{\mathbf{a},\mathbf{b}} & -a_3\theta^1_{\mathbf{a},\mathbf{b}} & \dots & (-1)^{n-1} a_3 \cdots a_{n-1} \theta^1_{\mathbf{a},\mathbf{b}} \\ 0 & 1 & b_2 & \theta^2_{\mathbf{a},\mathbf{b}} & \dots & (-1)^{n-2} a_4 \cdots a_{n-1} \theta^2_{\mathbf{a},\mathbf{b}} \\ \vdots & & \ddots & \ddots & & \\ 0 & \dots & \dots & & b_{n-2} & \theta^{n-2}_{\mathbf{a},\mathbf{b}} \\ 0 & \dots & \dots & \dots & 1 & b_{n-1} \\ 0 & \dots & \dots & \dots & 0 & 1 \end{pmatrix}. \tag{3.6}$$

Thus $r_{\mathbf{a}} \triangleright r_{\mathbf{b}} \neq r_{\mathbf{b}}$ if $\theta^k_{\mathbf{a},\mathbf{b}} \neq 0$ for some $1 \le k \le n-2$. Analogously,

$$(r_{\mathbf{a}} r_{\mathbf{b}})^2 = \begin{pmatrix} 1 & 2(a_1+b_1) & \gamma^1_{\mathbf{a},\mathbf{b}} & \nu^1_{\mathbf{a},\mathbf{b}} & \dots & * \\ 0 & 1 & 2(a_2+b_2) & \gamma^2_{\mathbf{a},\mathbf{b}} & \nu^2_{\mathbf{a},\mathbf{b}} & * \\ 0 & 0 & 1 & 2(a_3+b_3) & \ddots & * \\ \vdots & & & \ddots & \ddots & \gamma^{n-2}_{\mathbf{a},\mathbf{b}} \\ 0 & \dots & \dots & 0 & 1 & 2(a_{n-1}+b_{n-1}) \\ 0 & \dots & \dots & 0 & 0 & 1 \end{pmatrix};$$

hence $(r_{\mathbf{a}} r_{\mathbf{b}})^2 = (r_{\mathbf{b}} r_{\mathbf{a}})^2$ implies that

$$\gamma^k_{\mathbf{a},\mathbf{b}} = \gamma^k_{\mathbf{b},\mathbf{a}} \iff 2\theta^k_{\mathbf{a},\mathbf{b}} = 0, \quad \forall 1 \le k \le n-2 \quad \text{and}$$
$$\nu^k_{\mathbf{a},\mathbf{b}} = \nu^k_{\mathbf{b},\mathbf{a}} \quad \forall 1 \le k \le n-3. \tag{3.7}$$

Now every element in G of type $\lambda = (\lambda_1, \dots, \lambda_k)$ is conjugate to one of the form (3.1) with $u_i = r_{a_i} \in \mathbf{SL}_{\lambda_i}(q)$ for some $a_i \in \mathbb{F}_q^\times$.

Indeed, assume that $V \in \mathbf{SL}_n(q)$ admits $C \in \mathbf{GL}_n(q)$ such that CVC^{-1} is of the form (3.1) with regular unipotent blocks. Consider the diagonal matrix $D = (\det C^{-1}, 1, \dots, 1) \in (\mathbb{F}_q^\times)^n$. Then $E = DC \in \mathbf{SL}_n(q)$ and EVE^{-1} is of the form (3.1) with regular unipotent blocks.

Remark 3.2. To study the unipotent conjugacy classes in G, it suffices to consider classes of elements of the form (3.1) with $u_i = r_1$, cf. Remark 2.1.

For further purposes, we shall need the following well-known description of the regular unipotent conjugacy classes in G.

Lemma 3.3. *Let $d := \mathrm{gcf}(q-1, n)$. There are d regular unipotent conjugacy classes in G, all isomorphic as racks. Explicitly, they are of the form $\mathcal{O}_{r_a}$, for some $a \in \mathbb{F}_q^\times$, and $\mathcal{O}_{r_a} = \mathcal{O}_{r_b}$ if and only if $\theta^n a = b$ for some $\theta \in \mathbb{F}_q^\times$. If $\mathbf{a} = (a_1, \dots, a_{n-1}) \in (\mathbb{F}_q^\times)^{n-1}$, then $R_{\mathbf{a}} \subseteq \mathcal{O}_{r_a}$ for $a = a_1 a_2^2 \cdots a_{n-1}^{n-1}$.*

Proof. Let $x \in G$ be a regular unipotent element; we may assume that

$$x = \begin{pmatrix} 1 & x_{12} & x_{13} & \dots & x_{1n} \\ 0 & 1 & x_{23} & \dots & x_{2n} \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 1 & x_{n,n-1} \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix}, \quad x_{i,i+1} \in \mathbb{F}_q^\times,\ 1 \le i \le n-1. \tag{3.8}$$

Let $a \in \mathbb{F}_q^\times$; we claim that $x \in \mathcal{O}_{r_a}$ if and only if

$$\theta^n a = x_{12} x_{23}^2 x_{34}^3 \cdots x_{n-1,n}^{n-1} \quad \text{for some } \theta \in \mathbb{F}_q^\times. \tag{3.9}$$

Indeed, $x \in \mathcal{O}_{r_a}$ if and only if there exists $C = (c_{ij}) \in \mathbf{SL}_n(q)$ such that $Cr_a = xC$ which holds if and only if the following linear equations hold

$$c_{n,j} = 0 \qquad \text{for all } 1 \le j < n, \tag{3.10}$$

$$c_{i,j} = \sum_{k=i+1}^{n} x_{i,k} c_{k,j+1} \qquad \text{for all } 1 \le i < n,\ 2 \le j < n, \tag{3.11}$$

$$a c_{i,1} = \sum_{k=i+1}^{n} x_{i,k} c_{k,2} \qquad \text{for all } 1 \le i < n, \tag{3.12}$$

$$0 = \sum_{k=i+1}^{n} x_{i,k} c_{k,1} \qquad \text{for all } 1 \le i < n. \tag{3.13}$$

By a direct computation using (3.10), (3.11) and (3.13), $c_{ij} = 0$ for all $1 \le j < i \le n$, i.e. C is upper triangular. Thus, $ac_{11} = x_{12} c_{22}$ from (3.12), and $c_{ii} = x_{i,i+1} c_{i+1,i+1}$ for all $1 < i < n$, from (3.11). Since $\det C = 1$,

$$a = ac_{11} \cdots c_{nn} = x_{12} x_{23}^2 \cdots x_{n-1,n}^{n-1} c_{n,n}^n. \tag{3.14}$$

Thus, if $x \in \mathcal{O}_{r_a}$, then it is conjugated to r_a by an upper triangular matrix C and (3.9) holds with $\theta = c_{n,n}^{-1}$. Conversely, if (3.9) is satisfied, then define an upper triangular matrix C by $c_{n,n} = \theta^{-1}$, $c_{ii} = x_{i,i+1} c_{i+1,i+1}$ for $1 < i < n$, $c_{11} = a^{-1} x_{12} c_{2,2}$ and use Eqs. (3.11) to find the remaining elements. Consequently, $\mathcal{O}_{r_a} = \mathcal{O}_{r_b}$ if and only if

$\theta^n a = b$ for some $\theta \in \mathbb{F}_q^\times$; i.e. the set of regular unipotent classes in G is parameterized by the quotient of the cyclic group $\mathbb{F}_q^\times$ by the image of the map by $x \mapsto x^n$. Since the kernel of this map has order $d = \operatorname{gcf}(n, q-1)$, we get d different classes. □

3.2. Unipotent conjugacy classes in $\mathbf{PSL}_2(q)$

We start with unipotent classes in $\mathbf{PSL}_2(q)$; here $q \neq 2, 3, 4, 5, 9$, see Section 1.2. First we recall Dickson's classification of all subgroups of $\mathbf{PSL}_2(q)$. Let $d = (2, q-1)$.

Theorem 3.4. *(See [24, Theorems 6.25, p. 412; 6.26, p. 414].) A subgroup of* $\mathbf{PSL}_2(q)$ *is isomorphic to one of the following groups.*

(a) *The dihedral groups of order* $2(q \pm 1)/d$ *and their subgroups. There are always such subgroups.*
(b) *A group* H *of order* $q(q-1)/d$ *and its subgroups. It has a normal* p*-Sylow subgroup* Q *that is elementary abelian and the quotient* H/Q *is cyclic of order* $(q-1)/d$*. There are always such subgroups.*
(c) $\mathbb{A}_4$*, and there are such subgroups except when* $p = 2$ *and* m *is odd.*
(d) $\mathbb{S}_4$*, and there are such subgroups if and only if* $q^2 \equiv 1 \bmod 16$*.*
(e) $\mathbb{A}_5$*, and there are such subgroups if and only if* $q(q^2-1) \equiv 0 \bmod 5$*.*
(f) $\mathbf{PSL}_2(t)$ *for some* t *such that* $q = t^h$*,* $h \in \mathbb{N}$*. There are always such subgroups.*
(g) $\mathbf{PGL}_2(t)$ *for some* t *such that* $q = t^h$*,* $h \in \mathbb{N}$*. If* q *is odd, then there are such subgroups if and only if* h *is even and* $q = t^h$*.* □

Lemma 3.5. *Assume that* q *is either even, or else odd but not a square. Then a unipotent conjugacy class* $\mathcal{O}$ *of* $\mathbf{PSL}_2(q)$ *is sober, hence cthulhu.*

Proof. Let X be a subrack of $\mathcal{O}$; we show that X is either abelian or indecomposable. Let K be the subgroup of $\mathbf{PSL}_2(q)$ generated by X. Since X generates K, it is a union of (unipotent) K-conjugacy classes [5, 1.9]. We may assume that $r_1 \in X$. The order of any element in X is p, so p divides $|K|$; this excludes case (a) in Theorem 3.4 for p odd. Assume that q is even, so that $d = 1$, and K is a dihedral group of order $2(q \pm 1)$. Then X is the rack of involutions of K, which is indecomposable, see Example 2.15.

If K is as in case (b), then $X \subset Q$, hence it is an abelian rack. If $K \simeq \mathbb{A}_4 \simeq \mathbf{PSL}_2(3)$, case (c), then $p = 2$ or 3. If $p = 2$, then X could not generate K, being contained in the normal 2-Sylow subgroup of K. If $p = 3$, then we are reduced to case (f). If $K \simeq \mathbb{S}_4$, case (d), then $p = 2$ or 3; but the 3-cycles in $\mathbb{S}_4$ generate $\mathbb{A}_4$ so 3 is not possible, whereas $p = 2$ is excluded by Theorem 3.4. If $K \simeq \mathbb{A}_5 \simeq \mathbf{PSL}_2(5)$, case (e), then $p = 2, 3$ or 5. If $p = 2$, then X is indecomposable, being the unique class of involutions in $\mathbb{A}_5$. If $p = 3$, then $p^{2m} - 1 \equiv 0 \bmod 5 \iff m$ is even, excluded by hypothesis. If $p = 5$, then we are reduced to case (f).

Assume then that $K \simeq \mathbf{PSL}_2(t)$, $q = t^h$, case (f). For q even, $\mathbf{PGL}_2(t) \simeq \mathbf{PSL}_2(t)$ has just one regular unipotent conjugacy class, so X is indecomposable by [5, 1.9, 1.15]; for $\mathbf{PSL}_2(2) \simeq \mathbb{S}_3$ this is clear. Assume that q is odd. Let $s \in X$; is K-conjugate to r_x for some $x \in \mathbb{F}_t^\times$. But $\mathcal{O}_s^G = \mathcal{O} = \mathcal{O}_{r_1}^G$, hence $x \in (\mathbb{F}_q^\times)^2$. Since m is odd, this only happens when x is a square in $\mathbb{F}_t^\times$, i.e. when $\mathcal{O}_s^K = \mathcal{O}_{r_1}^K$. Hence $X = \mathcal{O}_{r_1}^K$ is indecomposable. Here we have to argue separately for $\mathbf{PSL}_2(3) \simeq \mathbb{A}_4$, but in this case the claim is clear.

Finally, case (g) is excluded when q is odd because q is not a square. □

Lemma 3.6. *A non-trivial unipotent conjugacy class $\mathcal{O}$ in $G = \mathbf{SL}_2(q)$, respectively in $\mathbf{PSL}_2(q)$, is of type D if and only if $q > 9$ is an odd square.*

We excluded $\mathbf{PSL}_2(9) \simeq \mathbb{A}_6$ but in this case $\mathcal{O}_r$ is not of type D by [3, Remark 4.2 (b)].

Proof. By Remark 3.2, we may assume that $\mathcal{O} = \mathcal{O}_r$ with $r = r_1$. Suppose $q \neq 9$ is an odd square. Let $x \in \mathbb{F}_p^\times - (\mathbb{F}_p^\times)^2$; since $q \neq 9$ we may assume that $x \neq 2$. Let $v = r_x$. Since x is not a square in $\mathbb{F}_p^\times$, $\mathcal{O}_r^G = \mathcal{O}_v^G$ but $R := \mathcal{O}_r^{\mathbf{SL}_2(p)} \neq \mathcal{O}_v^{\mathbf{SL}_2(p)} =: S$. Let $s = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ -x & 1 \end{smallmatrix}\right) \in S$. Then $(rs)^2 \neq (sr)^2$ showing that $\mathcal{O}_r^G$ is of type D. Conversely, if q is not an odd square, then $\mathcal{O}_r^G$ is cthulhu by Lemma 3.5, hence not of type D. □

The exceptional isomorphism $\mathbf{PSL}_3(2) \simeq \mathbf{PSL}_2(7)$ motivates the analysis of some semisimple classes in this last group.

Lemma 3.7. *Let $\mathcal{O}$ be the conjugacy class of $x \in \mathbf{PSL}_2(7)$.*

(a) *If* $\operatorname{ord} x = 2$, *then $\mathcal{O}$ is cthulhu.*
(b) *If* $\operatorname{ord} x = 4$, *then $\mathcal{O}$ is sober.*

Hence the class of type $(2, 1)$, *respectively* (3), *in* $\mathbf{PSL}_3(2)$ *is cthulhu, respectively sober.*

Proof. The proper subgroups of $\mathbf{PSL}_2(7)$ are isomorphic either to $\mathbb{D}_3 \simeq \mathbb{S}_3$, $\mathbb{D}_4$, the non-abelian group of order 21, $\mathbb{A}_4$, or $\mathbb{S}_4$, or their subgroups. Let X be a subrack of $\mathcal{O}$ and $K = \langle X \rangle$; we show by inspection that X is either abelian, or indecomposable, or the union of at most 3 subracks that do not fulfill (2.2). Suppose that $\operatorname{ord} x = 2$. First, $\mathcal{O}_{(12)}^{\mathbb{S}_3}$ is indecomposable. Second, $\mathbb{D}_4 = \langle r, s \mid r^4 = s^2 = \mathrm{id}, srs = r^3 \rangle$ has 3 classes of involutions: $\{r^2\}$ which is central, and the abelian racks $\{s, sr^2\}$, $\{sr, sr^3\}$ not commuting with each other; (2.2) does not hold here. The involutions of $\mathbb{A}_4$ generate the 2-Sylow subgroup, so $K = \mathbb{A}_4$ is not possible. If $K = \mathbb{S}_4$, then $\mathcal{O}_{(12)(34)}^{\mathbb{A}_4} = \mathcal{O}_{(12)(34)}^{\mathbb{S}_4}$ does not generate K, $\mathcal{O}_{(12)}^{\mathbb{S}_4}$ is indecomposable by [5, 3.2 (2)], and (2.2) does not hold in $\mathcal{O}_{(12)(34)}^{\mathbb{S}_4} \amalg \mathcal{O}_{(12)}^{\mathbb{S}_4}$. So, $\mathcal{O}$ is cthulhu but not sober. If $\operatorname{ord} x = 4$, then K could be either abelian, $\mathbb{D}_4$ or $\mathbb{S}_4$.

The elements of order 4 in $\mathbb{D}_4$ generate a cyclic subgroup. If $K \simeq \mathbb{S}_4$, then $X = \mathcal{O}_4^{\mathbb{S}_4}$ is indecomposable. $\square$

3.3. *Unipotent classes of type D*

3.3.1. *Odd characteristic*

Here we assume that q is odd.

Lemma 3.8. *Let $u \in G$ be a unipotent element of type $(\lambda_1, \dots, \lambda_k)$. If $\lambda_1 > 2$, then the conjugacy class $\mathcal{O}_u$ is of type D.*

Proof. Assume that u is regular, i.e. $\lambda_1 = n > 2$. By Remark 3.2 we may suppose that $\mathcal{O} = \mathcal{O}_{r_{\mathbf{1}}}$. Let $\zeta \in \mathbb{F}_q^\times$ with $\zeta^3 \neq 1$, t the diagonal matrix $(1, \zeta, \zeta^{-1}, 1, \dots, 1)$ and $\mathbf{b} = (\zeta^{-1}, \zeta^2, 1, \dots, 1)$. Then $tr_{\mathbf{1}}t^{-1} \in R_{\mathbf{b}}$ and $R_{\mathbf{1}} \amalg R_{\mathbf{b}}$ is a decomposable subrack of $\mathcal{O}_{r_{\mathbf{1}}}$. Besides, $r_{\mathbf{1}} \triangleright (r_{\mathbf{b}} \triangleright (r_{\mathbf{1}} \triangleright r_{\mathbf{b}})) \neq r_{\mathbf{b}}$ by (3.7), so that $\mathcal{O}_{r_{\mathbf{1}}}$ is of type D. In the general case, we may assume that u is as in (3.1) with $u_j \in \mathbf{SL}_{\lambda_j}(q)$, $j \in \mathbb{I}_k$. Then Lemma 3.1 applies. $\square$

We next deal with non-trivial unipotent conjugacy classes not covered by the previous lemma, that is those of type $(2, 2, \dots, 1, \dots, 1)$.

Lemma 3.9. *Let $u \in G$ be a unipotent element. Assume that either*

(a) *$n = 4$ and u has type $(2, 2)$ or*
(b) *$n = 3$ and u has type $(2, 1)$.*

Then the conjugacy class $\mathcal{O}_u$ is of type D.

Proof. (a): We may assume $u = \begin{pmatrix} r_1 & 0 \\ 0 & r_1 \end{pmatrix}$, Remark 3.2. Let $\zeta \in \mathbb{F}_q^\times - (\mathbb{F}_q^\times)^2$, $r = u$, $t = \begin{pmatrix} 0 & -\zeta & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\zeta^{-1} \\ 0 & 0 & 1 & 0 \end{pmatrix}$ and $s = t \triangleright u = \begin{pmatrix} 1 & 0 & -\zeta & 0 \\ -\zeta^{-1} & 1 & -1 & \zeta^{-1} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\zeta & 1 \end{pmatrix}$. A direct computation shows that $(rs)^2 \neq (sr)^2$. Moreover, $\langle r, s \rangle$ is strictly contained in the subgroup H of $\mathbf{SL}_4(q)$ of block upper triangular matrices with diagonal blocks in $\mathbf{SL}_2(q)$. Since $\begin{pmatrix} 1 & 0 \\ -\zeta^{-1} & 1 \end{pmatrix}$ is conjugated to $\begin{pmatrix} 1 & \zeta^{-1} \\ 0 & 1 \end{pmatrix}$ in $\mathbf{SL}_2(q)$ and this is not conjugated to r_1 by Lemma 3.3, it turns out that $\mathcal{O}_r^H \neq \mathcal{O}_s^H$, so $\mathcal{O}_r^{\mathbf{SL}_4(q)}$ is of type D.

(b): We may assume that $u = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Take $r = u$ and $s = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \triangleright r = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \in \mathcal{O}_r^{\mathbf{SL}_3(q)}$. Then $(rs)^2 \neq (sr)^2$, since q is odd. Also, $H = \langle r, s \rangle = \left\{ \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ c & b+ac & 1 \end{pmatrix} \mid a, b, c \in \mathbb{F}_p \right\}$ because $[r^k, s^m] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & km & 1 \end{pmatrix}$ and $[r, [r^k, s^m]] = [s, [r^k, s^m]] = 1$. It is not hard to verify that $\mathcal{O}_r^H \neq \mathcal{O}_s^H$. $\square$

If $\mathcal{O}_u^{\mathbf{SL}_n(q)}$ is of type D, then $\mathcal{O}_u^{\mathbf{GL}_n(q)}$ is of type D too. Thus, the previous results apply to nontrivial unipotent conjugacy classes in $\mathbf{GL}_n(q)$ with the prescribed hypothesis. We deal with the remaining cases.

Lemma 3.10. *If $u = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $q > 3$, then $\mathcal{O}_u^{\mathbf{GL}_2(q)}$ is of type D.*

Proof. Consider the subsets of $\mathcal{O}_u$ given by

$$R = \left\{AuA^{-1} : \ A \in \mathbf{GL}_2(q), \det A \notin \left(\mathbb{F}_q^\times\right)^2\right\},$$
$$S = \left\{AuA^{-1} : \ A \in \mathbf{GL}_2(q), \det A \in \left(\mathbb{F}_q^\times\right)^2\right\}.$$

Let $A, B \in \mathbf{GL}_2(q)$ and set $r = AuA^{-1}$ and $s = BuB^{-1} \in \mathcal{O}_u$. Then

$$r \triangleright s = \left(AuA^{-1}\right)\left(BuB^{-1}\right)\left(AuA^{-1}\right)^{-1} = \left(AuA^{-1}B\right)u\left(B^{-1}Au^{-1}A^{-1}\right)$$

and $\det(AuA^{-1}B) = \det B$. Hence R, S are subracks of $\mathcal{O}_u$, $R \triangleright S \subseteq S$ and $S \triangleright R \subseteq S$. Moreover, $R \cap S \neq \emptyset$ if and only if there exists $B \in C_{\mathbf{GL}_2(q)}(u)$ with $\det B$ not a square. But $C_{\mathbf{GL}_2(q)}(u) = \{\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) : \ a, b \in \mathbb{F}_q\}$, so that the two racks are disjoint and $Y = R \amalg S$ is a decomposable subrack of $\mathcal{O}_u$. Now let $d \in \mathbb{F}_q$ be not a square and $r_d = d^{-1}\left(\begin{smallmatrix} d-1 & 1 \\ -1 & d+1 \end{smallmatrix}\right) = A_d u A_d^{-1} \in R$ with $A_d = \left(\begin{smallmatrix} 1 & 0 \\ 1 & d \end{smallmatrix}\right)$. If we set $s = u$, a direct computation in $\mathbf{GL}_2(q)$ shows that

$$s \triangleright (r_d \triangleright s) = d^{-2}\begin{pmatrix} d^2 + d - 2 & d^2 - 4d + 4 \\ -1 & d^2 - d + 2 \end{pmatrix} = t.$$

Hence, $r_d \triangleright (s \triangleright (r_d \triangleright s)) = s$ if and only if $t = r_d^{-1} \triangleright s$ if and only if

$$\begin{pmatrix} d^2 - d - 1 & (d+1)^2 \\ -1 & d^2 + d + 1 \end{pmatrix} = \begin{pmatrix} d^2 + d - 2 & (d-2)^2 \\ -1 & d^2 - d + 2 \end{pmatrix},$$

if and only if $(d+1)^2 = (d-2)^2$ and $2d = 1$ in $\mathbb{F}_q$. Thus, $r_d \triangleright (s \triangleright (r_d \triangleright s)) \neq s$ if $2d \neq 1$. If $q \neq 3$, such a d always exists, showing that $\mathcal{O}_u$ is of type D. □

3.3.2. Even characteristic

Here we assume that q is even.

Lemma 3.11. *Let $u \in G$ be a unipotent element of type $\lambda = (\lambda_1, \dots, \lambda_k)$; assume that $\lambda_i \geq \lambda_{i+1} \geq 3$ for some $1 \leq i \leq k-1$. Then the conjugacy class $\mathcal{O} := \mathcal{O}_u$ is of type D.*

Proof. By Lemma 3.1, it is enough to look at the following specific unipotent class: If $\lambda = (\lambda_1, \lambda_2)$ with $\lambda_1 \geq \lambda_2 \geq 3$, then $\mathcal{O}$ is of type D.

Let $x_i = r_{(1,\dots,1)} \in \mathbb{F}^{\lambda_i\times\lambda_i}$, $i = 1, 2$. By Remark 3.2 we may assume that $u = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}$. Notice that $x_1^{-1} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 0 & 1 & \dots & 1 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & 1 \end{pmatrix} \in \mathcal{O}_{x_1}$ by Lemma 3.3. Let

$$R_1 = \left\{ \begin{pmatrix} x_1 & Z \\ 0 & x_2 \end{pmatrix} \;\middle|\; Z = (z_{ij}) \in \mathbb{F}_2^{\lambda_1\times\lambda_2} \right\}, \qquad R = R_1 \cap \mathcal{O};$$
$$S_1 = \left\{ \begin{pmatrix} x_1^{-1} & Z \\ 0 & x_2 \end{pmatrix} \;\middle|\; Z = (z_{ij}) \in \mathbb{F}_2^{\lambda_1\times\lambda_2} \right\}, \qquad S = S_1 \cap \mathcal{O}.$$

Since $Y_1 = R_1 \amalg S_1$ is a decomposable subrack, so is $Y = R \amalg S \subset \mathcal{O}$. Let

$$z_1 = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \end{pmatrix}, \qquad z_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 1 & 1 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \end{pmatrix},$$
$$z_3 = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 1 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & 1 & \dots & 0 \end{pmatrix}.$$

If $P = \begin{pmatrix} \mathrm{id}_{\lambda_1} & e_{\lambda_1,1} \\ 0 & \mathrm{id}_{\lambda_2} \end{pmatrix} = P^{-1} \in G$, then $v = P \begin{pmatrix} x_1^{-1} & 0 \\ 0 & x_2 \end{pmatrix} P = \begin{pmatrix} x_1^{-1} & z_1 \\ 0 & x_2 \end{pmatrix} \in S$,

$$(uv)^2 = \begin{pmatrix} \mathrm{id} & x_1 z_1 \\ 0 & x_2^2 \end{pmatrix}^2 = \begin{pmatrix} \mathrm{id} & x_1 z_1(\mathrm{id} + x_2^2) \\ 0 & x_2^4 \end{pmatrix} = \begin{pmatrix} \mathrm{id} & z_2 \\ 0 & x_2^4 \end{pmatrix}, \quad \text{and}$$
$$(vu)^2 = \begin{pmatrix} \mathrm{id} & z_1 x_2 \\ 0 & x_2^2 \end{pmatrix}^2 = \begin{pmatrix} \mathrm{id} & z_1 x_2(\mathrm{id} + x_2^2) \\ 0 & x_2^4 \end{pmatrix} = \begin{pmatrix} \mathrm{id} & z_3 \\ 0 & x_2^4 \end{pmatrix} \quad \text{when } \lambda_2 > 3; \quad \text{but}$$

when $\lambda_2 = 3$, $(uv)^2 = \begin{pmatrix} \mathrm{id} & e_{2,3} \\ 0 & x_2^4 \end{pmatrix}$, $(vu)^2 = \begin{pmatrix} \mathrm{id} & e_{1,3}+e_{2,3} \\ 0 & x_2^4 \end{pmatrix}$. Thus $\mathcal{O}$ is of type D. □

Lemma 3.12. *Let $u \in G$ of type $(\lambda_1, \dots, \lambda_k)$ and assume that either*

(a) $\lambda_1 = 4$.
(b) $\lambda_1 = 3$, $\lambda_2 = 1$.
(c) $\lambda_1 = \lambda_2 = 2$.

Then the conjugacy class $\mathcal{O}_u$ is of type D. Furthermore, the class in $\mathbf{PSL}_4(2)$ of type $(2, 1, 1)$ is cthulhu.

Proof. Assume that $n = 4$, $q = 2$: Since $\mathbf{PSL}_4(2) \simeq \mathbb{A}_8$, we apply [3]. There are two classes of involutions in $\mathbb{A}_8$, of types $(1^4, 2^2)$ or (2^4); with centralizers of orders 96 and 192, respectively. The former is of type D [3, Table 2], and the latter is cthulhu because its proper subracks generated by two elements are abelian racks and dihedral racks with 3 and 4 elements [3, 4.2 (f)]. Now the class in $\mathbf{PSL}_4(2)$ of type $(2, 1, 1)$, respectively type $(2, 2)$, has centralizer of order 192 so it is cthulhu, respectively of order 96 and so is of type D. Also, there are two classes of elements of order 4 in $\mathbb{A}_8$, of types $(1^2, 2, 4)$ or (4^2), both of type D [3, Table 1 and Step 9]. Hence the classes in $\mathbf{PSL}_4(2)$ of types $(3, 1)$ and (4) are of type D. Now the claim for the classes $(2, 2)$, $(3, 1)$ and (4), for q even, follows as $\mathbf{SL}_n(q) < \mathbf{SL}_n(q^j)$ for any $j \in \mathbb{N}$; here Remark 3.2 is needed. Finally Lemma 3.1 applies. $\square$

We now present a negative result.

Proposition 3.13. *The unipotent classes of type* $(2, 1, 1, 1, 1, \dots)$ *in* $\mathbf{SL}_n(q)$ *for* q *even and* $n \geq 2$ *are not of type D.*

Proof. Let $\mathcal{O}$ be a class of type $(2, 1, 1, 1, 1, \dots)$ in $G = \mathbf{SL}_n(q)$. Let $\mathbb{U}^F$, respectively $\mathbb{T}^F$, the subgroup of unipotent upper-triangular, respectively diagonal matrices, in G. Without loss of generality we may assume that it is represented by $r = \mathrm{id}_n + e_{1,n}$, which lies in $Z(\mathbb{U}^F)$. We will show that if $s \in \mathcal{O}$ satisfies $[s, r] \neq 1$ and $\mathcal{O}_r^{\langle r,s\rangle} \neq \mathcal{O}_s^{\langle r,s\rangle}$, then $(rs)^2 = (sr)^2$. Let $s \in \mathcal{O}$ satisfy $[r, s] \neq 1$, and let $g \in G$ such that $s = grg^{-1}$. By [21, 24.1] g can be decomposed as $g = un_w tv$ where n_w is a monomial matrix with coefficients in $\mathbb{F}_2$; $u, v \in \mathbb{U}^F$ and $t \in \mathbb{T}^F$. Then $s = un_w trt^{-1}n_w^{-1}u^{-1}$. We have:

$$
\begin{aligned}
trt^{-1} &= \mathrm{id}_n + \xi e_{1,n}, && \text{for some } \xi \in \mathbb{F}_q^\times, \\
\sigma := n_w trt^{-1} n_w^{-1} &= \mathrm{id}_n + \xi e_{i,j}, && \text{for some } i \neq j.
\end{aligned}
$$

Further, $[r, s] \neq 1$ iff $u^{-1}[r, s]u = [r, \sigma] \neq 1$ and this happens only if either $i = n$ or $j = 1$, or both. Assume first $(i, j) = (n, 1)$. Since $r \in Z(\mathbb{U}^F)$ we have $K := \langle r, s\rangle \simeq u^{-1}\langle r, s\rangle u = \langle r, \sigma\rangle \simeq \langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ \xi & 1 \end{smallmatrix}\right)\rangle \leq \mathbf{SL}_2(q)$. By Lemma 3.5, $\mathcal{O}_r^K = \mathcal{O}_s^K$. Assume now $i = n$ and $j \neq 1, n$. Then

$$
\begin{aligned}
u^{-1}(rs)^2 u = (r\sigma)^2 &= \big((\mathrm{id}_n + e_{1,n})(\mathrm{id}_n + \xi e_{n,j})\big)^2 = \mathrm{id}_n + \xi e_{1,j}, \\
u^{-1}(sr)^2 u = (\sigma r)^2 &= \big((\mathrm{id}_n + \xi e_{n,j})(\mathrm{id}_n + e_{1,n})\big)^2 = \mathrm{id}_n + \xi e_{1,j}.
\end{aligned}
$$

The case $j = 1$, $n \neq 1, n$ can be treated similarly. $\square$

Lemma 3.14. *The unipotent classes of type* (3) *in* $\mathbf{PSL}_3(2^{2m})$ *are of type D for every* $m \geq 1$.

Proof. Let $\zeta \in \mathbb{F}_4^\times - \mathbb{F}_2$, $r = \begin{pmatrix} 1&1&0\\0&1&1\\0&0&1\end{pmatrix}$, $s = \begin{pmatrix} \zeta^2&0&\zeta^2\\ \zeta&1&\zeta^2\\ \zeta&\zeta^2&\zeta^2\end{pmatrix}$. Then $(rs)^2 \neq (sr)^2$, $\mathcal{O} = \mathcal{O}_r^{\mathbf{SL}_3(4)} = \mathcal{O}_s^{\mathbf{SL}_3(4)}$ and $\mathcal{O}_r^H \neq \mathcal{O}_s^H$, where $H = \langle r, s\rangle < \mathbf{SL}_3(4)$. Indeed, $|H| = 108$ and it can be presented as the group generated by two elements r, s satisfying the relations $r^4 = s^4 = 1$, $(rs)^3 = 1$, $(r \triangleright (s^{-1} \triangleright (r \triangleright s)))s^{-1} = 1$. Thus, also $(sr)^3 = 1$. In particular, $sr^{-1}s \in C_H(r)$, $rs^{-1}r \in C_H(s)$ and

$$\mathcal{O}_r^H = \left\{ \left(r^i s^j\right) \triangleright r \;\middle|\; 0 \le i \le 3,\ 0 \le j \le 2 \text{ and } i = 0 \text{ if } j = 0 \right\} \quad \text{and}$$
$$\mathcal{O}_s^H = \left\{ \left(s^i r^j\right) \triangleright s \;\middle|\; 0 \le i \le 3,\ 0 \le j \le 2 \text{ and } i = 0 \text{ if } j = 0 \right\},$$

with $|\mathcal{O}_r^H| = 9 = |\mathcal{O}_s^H|$. A direct computation shows that $s \notin \mathcal{O}_r^H$. The lemma follows, as $\mathbf{SL}_3(4) < \mathbf{SL}_3(2^{2m})$. □

3.4. Unipotent conjugacy classes of type F

Here assume that q is even and investigate when a unipotent class is of type F; recall that not all classes are of type D, see Proposition 3.13.

Lemma 3.15. *Let $u \in G$ of type $(\lambda_1, \dots, \lambda_k)$ and assume that either*

(a) $\lambda_1 \ge 5$.
(b) $\lambda_1 = 3$, $\lambda_2 = 2$.
(c) $\lambda_1 = 3$ *and* $q \ge 8$, *or*
(d) $\lambda_1 = 2$ *and* $\lambda_j = 1$ *for at least* 3 *different* j.

Then the conjugacy class $\mathcal{O}_u$ is of type F.

Proof. By Lemma 3.1, it is enough to look at some specific unipotent classes; when these are regular we may assume that $\mathcal{O} = \mathcal{O}_{r_1}$ by Remark 3.2.

Case 1. If $n > 4$, then a regular unipotent class $\mathcal{O}$ in G is of type F.

Let $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{u} = (u_1, u_2, u_3) \in \mathbb{F}_q^3$ and set

$$x_{\mathbf{a}}(\mathbf{u}) = \begin{pmatrix} 1 & 1 & 0 & 0 & \dots & a_1 & u_1 & u_3 \\ 0 & 1 & 1 & 0 & \dots & 0 & a_2 & u_2 \\ 0 & 0 & 1 & 1 & \dots & 0 & 0 & a_3 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & \dots & \dots & & \dots & 0 & 1 & 1 \\ 0 & \dots & \dots & \dots & \dots & \dots & 0 & 1 \end{pmatrix}.$$

Let $X_{\mathbf{a}} = \{x_{\mathbf{a}}(\mathbf{u}) : \mathbf{u} = (u_1, u_2, u_3) \in \mathbb{F}_q^3\} \subset \mathcal{O}$. Then $x_{\mathbf{a}}(\mathbf{u})x_{\mathbf{b}}(\mathbf{v}) = x_{\mathbf{b}}(\mathbf{v} + \mathbf{w})x_{\mathbf{a}}(\mathbf{u})$, where

$$\mathbf{w} = (a_1 + a_2 + b_1 + b_2, a_2 + a_3 + b_2 + b_3, a_1 + a_2 + b_1 + b_2 + u_1 + u_2 + v_1 + v_2).$$

Thus $X_{\mathbf{a}} \triangleright X_{\mathbf{b}} = X_{\mathbf{b}}$, for every $\mathbf{a}, \mathbf{b} \in \mathbb{F}_q^3$. Let

$$A = \{(1, 1, 1), (1, 1, 0), (1, 0, 1), (0, 1, 1)\}.$$

If $\mathbf{a} \neq \mathbf{b} \in A$, then $x_{\mathbf{a}}(\mathbf{u}) \triangleright x_{\mathbf{b}}(\mathbf{v}) \neq x_{\mathbf{b}}(\mathbf{v})$ for any $\mathbf{u}$, $\mathbf{v}$, and $\mathcal{O}$ is of type F.

Case 2. If $\mathcal{O}$ is unipotent of type $(3, 2)$, then $\mathcal{O}$ is of type F.

Let $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{u} = (u_1, u_2, u_3) \in \mathbb{F}_q^3$ and set $x_{\mathbf{a}}(\mathbf{u}) = \begin{pmatrix} 1 & 1 & a_1 & u_1 & u_3 \\ 0 & 1 & 1 & a_2 & u_2 \\ 0 & 0 & 1 & 0 & a_3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$. Let $X_{\mathbf{a}} = \{x_{\mathbf{a}}(\mathbf{u}) : \mathbf{u} = (u_1, u_3, u_3) \in \mathbb{F}_q^3\}$. It can be shown that $X_{\mathbf{a}} \subset \mathcal{O}$ if and only if $\mathbf{a} \in I = \{(a_1, a_2, a_3) \in \mathbb{F}_q^3 : a_2 = a_3\}$. Let $\mathbf{a}, \mathbf{b} \in I$. Now

$$x_{\mathbf{a}}(\mathbf{u})x_{\mathbf{b}}(\mathbf{v}) = x_{\mathbf{b}}(\mathbf{v} + \mathbf{w})x_{\mathbf{a}}(\mathbf{u}), \tag{3.15}$$

where $\mathbf{w} = (a_2 + b_2, 0, a_2 + b_2 + a_1 b_2 + a_2 b_1 + u_1 + u_2 + v_1 + v_2)$. By (3.15), $X_{\mathbf{a}} \triangleright X_{\mathbf{b}} = X_{\mathbf{b}}$, for every $\mathbf{a}, \mathbf{b} \in I$. Let

$$A = \{\mathbf{a}_1 = (1, 0, 0),\ \mathbf{a}_2 = (1, 1, 1),\ \mathbf{a}_3 = (0, 1, 1),\ \mathbf{a}_4 = (0, 0, 0)\} \subset I;$$
$$r_1 = x_{\mathbf{a}_1}(1, 0, 0), \qquad r_j = x_{\mathbf{a}_j}(0, 0, 0), \quad 2 \le j \le 4.$$

Then $r_j \in R_j := X_{\mathbf{a}_j}$ and $r_i \triangleright r_j \neq r_j$ for $i \neq j \in \mathbb{I}_4$, so $\mathcal{O}$ is of type F.

Case 3. If $n = 3$ and $q \ge 8$, then a regular unipotent class $\mathcal{O}$ is of type F.

Let $A = \{\mathbf{a} := (a^2, a^{-1}) \in (\mathbb{F}_q^\times)^2 : a \in \mathbb{F}_q^\times\}$. Then $(R_{\mathbf{a}})_{\mathbf{a} \in A}$ is a family of mutually disjoint subracks of $\mathcal{O}$, by Lemma 3.3. Now $\theta^1_{\mathbf{a},\mathbf{b}} = 0 \iff a^3 = b^3 \iff a = b$. Hence $r_{\mathbf{a}} \triangleright r_{\mathbf{b}} \neq r_{\mathbf{b}}$ for $a \neq b$, by (3.6). As $|\mathbb{F}_q^\times| \ge 4$, $\mathcal{O}$ is of type F.

Case 4. If $u \in G$ is unipotent of type $(2, 1, 1, 1)$, then $\mathcal{O} = \mathcal{O}_u$ is of type F.

We may assume that $u = \begin{pmatrix} r_1 & 0 \\ 0 & \mathrm{id}_3 \end{pmatrix}$. Let $(\mathbf{e}_j)_{j \in \mathbb{I}_4}$ be the canonical basis of $\mathbb{F}_q^4$ and $R_j = R_{\mathbf{e}_j} \cap \mathcal{O}$; then $R_j \triangleright R_k \subseteq R_k$ for $k, j \in \mathbb{I}_4$. Let $r_1 = r_{\mathbf{e}_1}$, $r_2 = r_{\mathbf{e}_2}$,

$$r_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \mathrm{id}_2 & e_{2,1} \\ 0 & \mathrm{id}_3 \end{pmatrix} r_{\mathbf{e}_3} \begin{pmatrix} \mathrm{id}_2 & e_{2,1} \\ 0 & \mathrm{id}_3 \end{pmatrix},$$

$$r_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \mathrm{id}_3 & e_{2,1} + e_{3,1} \\ 0 & \mathrm{id}_2 \end{pmatrix} r_{\mathbf{e}_4} \begin{pmatrix} \mathrm{id}_3 & e_{2,1} + e_{3,1} \\ 0 & \mathrm{id}_2 \end{pmatrix}.$$

Then $r_j \in R_j$ and $r_j \triangleright r_k \neq r_k$, $j \neq k \in \mathbb{I}_4$. Thus $\mathcal{O}$ is of type F. □

By Proposition 3.13, the classes of type $(2,1)$ in $\mathbf{SL}_3(q)$, q even, are not of type D. Now we show that they are not of type F, hence are chtulhu.

Proposition 3.16. *The unipotent classes of type* $(2,1)$ *in* $G = \mathbf{SL}_3(q)$ *for* q *even are not of type F.*

Proof. Let $(r_a)_{a\in\mathbb{I}_4}$ in G such that $\mathcal{O}_{r_a}^{\langle r_a : a\in\mathbb{I}_4\rangle} \neq \mathcal{O}_{r_b}^{\langle r_a : a\in\mathbb{I}_4\rangle}$ and $r_a \neq r_b$, for all $a \neq b$ in $\mathbb{I}_4$. Without loss of generality we may assume $r_1 = \mathrm{id}_3 + e_{1,3}$. Then, arguing as in the proof of Proposition 3.13 we have:

- $r_a = u_a n_a t_a \triangleright r_1$, for $a \in \{2,3,4\}$, where $u_a \in \mathbb{U}^F$, $t_a \in \mathbb{T}^F$ and n_a is monomial in $\mathbf{SL}_3(2)$;
- $\sigma_a := n_a t_a \triangleright r = \mathrm{id}_3 + \xi_a e_{i_a,j_a}$ for some $\xi_a \in \mathbb{F}_q^\times$ and $(i_a, j_a) \in \{(2,1),(3,2)\}$.

Thus, there are $a \neq b$ in $\{2,3,4\}$ such that $(i_a, j_a) = (i_b, j_b)$. We claim that if $(i_a, j_a) = (i_b, j_b) = (2,1)$ then $|r_a r_b|$ is either 2 or odd. Hence by Remark 2.7, either $r_a r_b = r_b r_a$ or $\mathcal{O}_{r_a}^{\langle r_a, r_b\rangle} = \mathcal{O}_{r_b}^{\langle r_a, r_b\rangle}$, a contradiction to our assumption. Since matrices in $\mathrm{id}_3 + \mathbb{F}_q e_{2,3}$ commute with σ_a and σ_b, there is no loss of generality in taking $u_a, u_b \in \mathrm{id}_3 + \mathbb{F}_q e_{1,2} + \mathbb{F}_q e_{1,3}$. Further,

$$|r_a r_b| = \left|u_a \sigma_a u_a^{-1} u_b \sigma_b u_b^{-1}\right| = \left|\sigma_a\big((u_a^{-1} u_b) \triangleright \sigma_b\big)\right|$$

so to prove the claim we may take $r_a = \sigma_a$, $r_b = (u_a^{-1} u_b) \triangleright \sigma_b$. Then, for $u_a^{-1} u_b = \mathrm{id}_3 + x e_{1,2} + y e_{1,3}$ we have

$$r_b = \begin{pmatrix} 1 + \xi_b x & \xi_b x^2 & \xi_b xy \\ \xi_b & 1 + \xi_b x & \xi_b y \\ 0 & 0 & 1 \end{pmatrix}, \qquad r_a r_b = \begin{pmatrix} A & \mathbf{c} \\ 0 & 1 \end{pmatrix},$$

$$\text{where } A = \begin{pmatrix} 1 + \xi_b x & \xi_b x^2 \\ \xi_a + \xi_a \xi_b x + \xi_b & \xi_a \xi_b x^2 + 1 + \xi_b x \end{pmatrix}, \ \mathbf{c} = \begin{pmatrix} \xi_b xy \\ \xi_a \xi_b xy + \xi_b y \end{pmatrix}.$$

Now, $(r_a r_b)^k = \left(\begin{smallmatrix} A^k & (A^{k-1} + \dots + \mathrm{id}_2)\mathbf{c} \\ 0 & 1 \end{smallmatrix}\right)$. Besides, $A \in \mathbf{SL}_2(q)$ so it is either semisimple or unipotent, the latter occurring if and only if $Tr(A) = 0$, if and only if $x = 0$. In this case, $r_a r_b = r_b r_a$. Otherwise A is semisimple, hence $|A| = h$ is odd and $A^{h-1} + \dots + \mathrm{id}_2 = 0$, so $|r_a r_b| = h$; the claim follows. □

Table 2
Collapsing unipotent classes in $\mathbf{PSL}_n(q)$.

n	q	Type $(\lambda_1, \dots, \lambda_k)$	Criterium
2	odd square > 9	(2)	D, 3.6
> 2	odd	$\lambda_1 \geq 3$	D, 3.8
		$(2, 2, \dots)$	D, 3.9 (a)
		$(2, 1 \dots)$	D, 3.9 (b)
	even	$\lambda_1 \geq 5$	F, 3.15 (a)
		$\lambda_1 = 4$	D, 3.12 (a)
		$(3, 3, \dots)$	D, 3.11
		$(3, 2, \dots)$	F, 3.15 (b)
		$(3, 1, \dots)$	D, 3.12 (b)
		$(2, 2, \dots)$	D, 3.12 (c)
		$(2, 1, 1, 1, \dots)$	F, 3.15 (d)
	even ≥ 8	$\lambda_1 = 3$	F, 3.15 (c)
	4		D, 3.14

3.5. Collapsing unipotent classes in $\boldsymbol{G} = \mathbf{PSL}_n(q)$

We summarize the results in Section 3.3 and 3.4 showing the unipotent classes in $\boldsymbol{G}$ that collapse in Table 2. Recall that we assume $q \neq 2, 3, 4, 5, 9$, when $n = 2$. The information in Table 2 is minimal; many orbits collapse by different reasons, but we omit to discuss this in detail.

Now we deal with the Nichols algebras of irreducible Yetter–Drinfeld modules associated to the remaining classes in Table 1. We recall the useful *little triangle* Lemma. Let G be a finite group. A conjugacy class $\mathcal{O}$ in G contains a *little triangle* if there are different elements $(\sigma_i)_{i \in \mathbb{I}_3}$ such that

- $\sigma_1^h = \sigma_2 \sigma_3$ for an odd integer h;
- $\sigma_i \sigma_j = \sigma_j \sigma_i$, $i, j \in \mathbb{I}_3$;
- there are $g_2, g_3 \in G$ such that $\sigma_i = g_i \sigma_1 g_i^{-1}$ and $g_3 g_2, g_2 g_3 \in C_G(\sigma_1)$.

Lemma 3.17. *(See [16, Lemma 2.3].) If $\mathcal{O}$ in G contains a little triangle, then* $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$, *for every* $\rho \in \operatorname{Irr} C_G(\sigma_1)$. □

Clearly, if $\mathcal{O}_x^G$ of $x \in G$ contains a little triangle and $\psi : G \to H$ is a group homomorphism, then $\mathcal{O}_{\psi(x)}^H$ contains a little triangle. In particular,

- If $x \in G < H$ and $\mathcal{O}_x^G$ of G contains a little triangle, then $\mathcal{O}_x^H$ also contains a little triangle.
- If T is an (outer) automorphism and $\mathcal{O}$ in G contains a little triangle, then so does $T(\mathcal{O})$.

Lemma 3.18. *Let $\mathcal{O}$ be the conjugacy class of $x \in \boldsymbol{G}$. In the cases listed below,* $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$*, for every* $\rho \in \operatorname{Irr} C_{\boldsymbol{G}}(x)$.

(a) $\boldsymbol{G} = \mathbf{PSL}_2(q)$, *$x$ of type* (2).
(b) $\boldsymbol{G} = \mathbf{PSL}_3(q)$ *with q even, x of type* $(2, 1)$.
(c) $\boldsymbol{G} = \mathbf{PSL}_4(q)$ *with q even, x of type* $(2, 1, 1)$.

Proof. (a): [15, 3.1] for q even, [16, 4.1, 4.3] for q odd. (b) and (c): $\mathbf{PSL}_3(2) \simeq \mathbf{PSL}_2(7)$ contains a copy of $\mathbb{A}_4$, so the class of involutions contains a little triangle [16, 4.3]. Now the previous remarks apply. □

4. Non-semisimple classes in $\mathbf{PSL}_n(q)$

4.1. Preliminaries

In this section we apply the results in Section 3 on unipotent classes to non-semisimple classes in $\boldsymbol{G} = \mathbf{PSL}_n(q)$. Let $x \in \boldsymbol{G}$ and pick $\mathtt{x} \in \mathbf{SL}_n(q)$ such that $\pi(\mathtt{x}) = x$; if $\mathtt{x} = \mathtt{x}_s\mathtt{x}_u$ is the Chevalley–Jordan decomposition of $\mathtt{x}$, then $x_s = \pi(\mathtt{x}_s)$ and $x_u = \pi(\mathtt{x}_u)$ form the Chevalley–Jordan decomposition of x. Now $\mathtt{x}_u$ belongs to $\mathcal{K} := C_{\mathbf{SL}_n(q)}(\mathtt{x}_s)$, thus $x_u \in K := \pi(\mathcal{K})$ and there are morphisms of racks $\mathcal{O}^{\mathcal{K}}_{\mathtt{x}_u} \simeq \mathcal{O}^{K}_{x_u} \hookrightarrow \mathcal{O}^{\boldsymbol{G}}_{x}$. Hence, in many cases it will be enough to deal with $\mathcal{O}^{\mathcal{K}}_{\mathtt{x}_u}$ and to start with we describe $\mathcal{K} = C_{\mathbf{GL}_n(q)}(\mathtt{x}_s) \cap \mathbf{SL}_n(q)$. Up to conjugation by a matrix in $\mathbf{SL}_n(q)$, we may assume that

$$\mathtt{x}_s = \begin{pmatrix} \mathtt{S}_1 & 0 & \dots & 0 \\ 0 & \mathtt{S}_2 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & \mathtt{S}_k \end{pmatrix}, \tag{4.1}$$

with $\mathtt{S}_i \in \mathbf{GL}_{\lambda_i}(q)$ irreducible, that is, its characteristic polynomial $\chi_{\mathtt{S}_i}$ is irreducible in $\mathbb{F}_q[X]$. Furthermore, $\prod_{i \in \mathbb{I}_k} \det \mathtt{S}_i = 1$. Now, if $\sigma \in \mathbb{S}_k$, then there is $T \in \mathbf{SL}_n(q)$ such that $T\mathtt{x}_sT^{-1} = \begin{pmatrix} \mathtt{S}_{\sigma(1)} & 0 & \dots & 0 \\ 0 & \mathtt{S}_{\sigma(2)} & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & \mathtt{S}_{\sigma(k)} \end{pmatrix}$.

If $\mathtt{S} \in \mathbf{GL}_\Lambda(q)$ is irreducible, then the subalgebra $\mathcal{C}_{\mathtt{S}}$ of matrices commuting with $\mathtt{S}$ is a division ring by Schur Lemma; being finite, is isomorphic to $\mathbb{F}_{q^\mu}$ for some $\mu \in \mathbb{N}$. We claim that $\mu = \Lambda$. Indeed, the characteristic and minimal polynomials of $\mathtt{S}$ coincide and have degree Λ, so standard arguments for finite fields imply the claim.

Remark 4.1. Let $\mathtt{S}$, $\mathtt{R} \in \mathbf{GL}_\Lambda(q)$ be semisimple and conjugate in $\mathbf{GL}_\Lambda(\Bbbk)$. Then there exists $T \in \mathbf{SL}_\Lambda(q)$ such that $T\mathtt{S}T^{-1} = \mathtt{R}$; that is, $\mathtt{S}$ and $\mathtt{R}$ are conjugate under $\mathbf{SL}_\Lambda(q)$.

Indeed, $\mathtt{R}$ and $\mathtt{S}$ are conjugate in $\mathbf{GL}_\Lambda(q)$ by [19, 8.5], [23, I.3.5]. Also we may assume that $\mathtt{S}$ is irreducible. Let $T_0 \in \mathbf{GL}_\Lambda(q)$ such that $T_0\mathtt{S}T_0^{-1} = \mathtt{R}$. Since $\det : \mathcal{C}_{\mathtt{S}}^{\times} \to \mathbb{F}_q^{\times}$

equals the norm $N : \mathbb{F}_{q^\Lambda}^\times \to \mathbb{F}_q^\times$ which is surjective, we may pick $T_1 \in \mathcal{C}_{\mathtt{S}}$ such that $\det T_1 = \det T_0^{-1}$. Then $T = T_0T_1$ does the job.

Assume that $\mathtt{S}$ is irreducible but not in $\mathbb{F}_q$; then $\chi_{\mathtt{S}}(\mathtt{S}^q) = (\chi_{\mathtt{S}}(\mathtt{S}))^q = 0$, so $\mathtt{S}$ and $\mathtt{S}^q$ are conjugate under $\mathbf{SL}_\Lambda(q)$, but $\mathtt{S} \neq \mathtt{S}^q$. Indeed, it can be shown that $\mathtt{S}^{q^i}$, $i \in \mathbb{I}_\Lambda$, are all the roots of $\chi_{\mathtt{S}}$ in $\mathbb{F}_q[\mathtt{S}] \simeq \mathbb{F}_{q^\Lambda}$.

Remark 4.2. Let $\pi : \mathbf{GL}_\Lambda(q) \to \mathbf{PGL}_\Lambda(q)$ and let $\mathtt{S} \in \mathbf{GL}_\Lambda(q)$ irreducible with $\Lambda > 1$; hence $\mathtt{S} \neq \mathtt{S}^q$. Then $\pi(\mathtt{S}) = \pi(\mathtt{S}^q)$ if and only if $\chi_{\mathtt{S}}$ belongs to

$$\begin{aligned}\mathfrak{I}(q) = \big\{F \in \mathbb{F}_q[X] \text{ irreducible} : F|X^{q-1} - c,\\ \text{for some } c \in \mathbb{F}_q^\times,\ c \neq 1,\ c^{\deg F} = 1\big\}.\end{aligned} \tag{4.2}$$

4.2. Centralizers

By the previous considerations, we may regroup the blocks so that there exist integers $h_1, \dots, h_\ell$ such that $\mathtt{S}_i$ and $\mathtt{S}_j$ are conjugate under $\mathbf{SL}_{\lambda_i}(q)$ if and only if there exists a (unique) $t \in \mathbb{I}_\ell$ such that $i, j \in \mathbb{J}_t$, where

$$\mathbb{J}_t = \{i \in \mathbb{N} : h_1 + \ldots + h_{t-1} + 1 \leq i \leq h_1 + \ldots + h_t\}. \tag{4.3}$$

So, we set $\Lambda_t = \lambda_i$, if $i \in \mathbb{J}_t$, $t \in \mathbb{I}_\ell$. In other words, h_1 is the number of blocks $\mathtt{S}_i$ that are isomorphic to $\mathtt{S}_1$, all of size Λ_1; h_2 is the number of blocks $\mathtt{S}_i$ that are isomorphic to $\mathtt{S}_{h_1+1}$, all of size Λ_2, and so on.

Proposition 4.3. *(See [12].)* $C_{\mathbf{GL}_n(q)}(\mathtt{x}_s) \simeq \mathbf{GL}_{h_1}(q^{\Lambda_1}) \times \ldots \times \mathbf{GL}_{h_\ell}(q^{\Lambda_\ell})$.

Proof. Let $\mathtt{S} \in \mathbf{GL}_N(q)$, $\mathtt{R} \in \mathbf{GL}_P(q)$ be irreducible. Let $Z = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in M_{N+P}(q)$, where A is of size $N \times N$. Then Z commutes with $\left(\begin{smallmatrix} \mathtt{S} & 0 \\ 0 & \mathtt{R} \end{smallmatrix}\right)$ iff

$$A\mathtt{S} = \mathtt{S}A, \qquad B\mathtt{R} = \mathtt{S}B, \qquad C\mathtt{S} = \mathtt{R}C, \qquad D\mathtt{R} = \mathtt{R}D.$$

So that $A \in \mathcal{C}_{\mathtt{S}} \simeq \mathbb{F}_{q^\Lambda}$, $D \in \mathcal{C}_{\mathtt{R}} \simeq \mathbb{F}_{q^\nu}$. If $\mathtt{S}$ and $\mathtt{R}$ are not conjugated, then $B = 0$, $C = 0$ by Schur Lemma. Otherwise, $N = P$; we may assume $\mathtt{S} = \mathtt{R}$, hence $A, B, C, D \in \mathcal{C}_{\mathtt{S}} \simeq \mathbb{F}_{q^\Lambda}$. The claim follows from this. For, assume that $\mathtt{x}_s$ is of the form (4.1). Let $Z = (Z_{ij}) \in \mathbf{GL}_n(q)$, where $Z_{ij} \in \mathbb{F}_q^{\lambda_i \times \lambda_j}$, $i, j \in \mathbb{I}_k$. Then $Z \in C_{\mathbf{GL}_n(q)}(\mathtt{x}_s)$ iff $Z_{ij} = 0$ unless $i, j \in \mathbb{J}_t$ for some t, in which case $Z_{ij} \in \mathbb{F}_{q^{\Lambda_t}}$. Thus every $Z \in C_{\mathbf{GL}_n(q)}(\mathtt{x}_s)$ is a matrix of blocks

$$Z = \begin{pmatrix} W_1 & 0 & \dots & 0 \\ 0 & W_2 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & W_\ell \end{pmatrix} \tag{4.4}$$

where in turn W_t is a matrix of h_t^2 blocks, each of size $\Lambda_t \times \Lambda_t$ and belonging to $\mathcal{C}_{\mathtt{S}_i} \simeq \mathbb{F}_{q^{\Lambda_t}}$, if $i \in \mathbb{J}_t$, $t \in \mathbb{I}_\ell$. Thus W_t can be thought of as a matrix $\widetilde{W_t} \in M_{h_t}(q^{\Lambda_t})$, and the map $\psi_t : W_t \mapsto \widetilde{W_t}$ is an isomorphism of monoids. Also, $\det Z \neq 0$ iff $\det W_t \neq 0$ in $\mathbf{GL}_{h_t\Lambda_t}(q)$ for all $t \in \mathbb{I}_\ell$, iff $\det \widetilde{W_t} \neq 0$ in $\mathbf{GL}_{h_t}(q^{\Lambda_t})$ for all $t \in \mathbb{I}_\ell$. Thus ψ_t gives rise to an isomorphism from the group G_t of matrices (4.4) with all $W_r = \mathrm{id}$, except for $r = t$, to $\mathbf{GL}_{h_t}(q^{\Lambda_t})$. □

Let $\Psi : C_{\mathbf{GL}_n(q)}(\mathtt{x}_s) \to \mathbf{GL}_{h_1}(q^{\Lambda_1}) \times \ldots \times \mathbf{GL}_{h_\ell}(q^{\Lambda_\ell})$ be the isomorphism given by Proposition 4.3. Then

$$\mathcal{K} \simeq \left\{X \in \mathbf{GL}_{h_1}\left(q^{\Lambda_1}\right) \times \ldots \times \mathbf{GL}_{h_\ell}\left(q^{\Lambda_\ell}\right) : \det \circ \Psi^{-1}(X) = 1\right\}. \tag{4.5}$$

In particular, if $\mathbf{SL}_{h_t}(q^{\Lambda_t}) \neq \mathbf{SL}_2(2), \mathbf{SL}_2(3)$, then it is perfect, hence

$$\mathbf{SL}_{h_t}\left(q^{\Lambda_t}\right) \hookrightarrow \mathcal{K}, \quad 1 \leq t \leq \ell. \tag{4.6}$$

If $\mathbf{SL}_{h_t}(q^{\Lambda_t}) = \mathbf{SL}_2(2)$ or $\mathbf{SL}_2(3)$, then (4.6) also holds, being $\Lambda_t = 1$.

4.3. End of the proof of Theorem 1.3

Proposition 4.4. *Let $x \in \boldsymbol{G}$ be neither semisimple nor unipotent. Then $\mathcal{O}_x^{\boldsymbol{G}}$ collapses.*

Proof. Let $\mathtt{x} \in \mathbf{SL}_n(q)$ with $x = \pi(\mathtt{x})$, and let $\mathtt{x} = \mathtt{x}_s\mathtt{x}_u$ be its Chevalley–Jordan decomposition. By our assumption $\mathtt{x}_s$ is not central and $\mathtt{x}_u \neq e$.

We assume that $\mathtt{x}_s$ is in the form (4.1); then there are natural numbers $h_1, \ldots, h_\ell$, $\Lambda_1, \ldots, \Lambda_\ell$ such that the structure of $\mathcal{K}$ is given by (4.5). Then $\mathtt{x}_u = (u_1, \ldots, u_\ell)$ with $u_t \in \mathbf{GL}_{h_t}(q^{\Lambda_t})$ unipotent, $t \in \mathbb{I}_\ell$. For simplicity, we write also $\mathtt{x}_s = (S_1, \ldots, S_\ell)$. Up to a further reordering, there exists $M \in \mathbb{I}_\ell$ such that $u_t \neq \mathrm{id}$ iff $t \leq M$, and $h_1 \geq \ldots \geq h_M > 1$; since $\mathtt{x}_u \neq e$, $M > 0$. Recall that $\mathcal{O}_{u_t}^{\mathbf{SL}_{h_t}(q^{\Lambda_t})}$ is a subrack of $\mathcal{O}_{\mathtt{x}_u}^{\mathcal{K}}$ for all t by (4.6).

By the unipotent part of Theorem 1.3, we may assume that $h_t \leq 4$ and u_t appears in Table 1 or it is of type (2) and q is in $\{2, 3, 4, 5, 9\}$, for all $t \in \mathbb{I}_M$. Let X be a unipotent orbit either of type (3) with $q^\Lambda = 2$; or else of type (2) with q^Λ even or 9 or odd not a square; or else of type (2,1) or (2,1,1) with q^Λ even. By inspection, we see that

(a) There exist $x_1, x_2 \in X$ such that $(x_1x_2)^2 \neq (x_2x_1)^2$.
(b) There exist $y_1, y_2 \in X$ such that $y_1y_2 = y_2y_1$, except when X is of type (2) with $q^\Lambda = 2$ or 3.

Case 1. $M = 1 = \ell$. Then $\mathcal{O}_x^{\boldsymbol{G}}$ is of type D.

In this case $x_u = u_1$. In addition, $\Lambda = \Lambda_1 > 1$ since $\mathtt{x}_s$ is not central; so $q^\Lambda \neq 2$. Hence type (3) and (2) with $q^\Lambda = 2$ are excluded. Let $\mathtt{S} = \mathtt{S}_1$. Assume that $\chi_{\mathtt{S}} \notin \mathfrak{I}(q)$. By Remarks 4.1 and 4.2,

$$(\mathcal{O}^{\mathcal{K}}_{\mathtt{x}_u})^{(2)} \overset{2.10}{=} \mathcal{O}^{\mathcal{K}}_{\mathtt{x}_u} \amalg \mathcal{O}^{\mathcal{K}}_{\mathtt{x}_u} \simeq \pi(\mathtt{S})\mathcal{O}^{K}_{x_u} \amalg \pi(\mathtt{S}^q)\mathcal{O}^{K}_{x_u} \hookrightarrow \mathcal{O}^{\boldsymbol{G}}_{x}.$$

By (a), $\mathcal{O}^{\boldsymbol{G}}_{x}$ is of type D. Now, if $\chi_{\mathtt{S}} \in \mathfrak{I}(q)$, then $\mathtt{S}$ is conjugated to $\mathtt{S}^q = c\mathtt{S}$ for some $c \in \mathbb{F}_q^\times - 1$. Pick $Y \in \mathbf{SL}_\Lambda(q)$ such that $Y\mathtt{S}Y^{-1} = c\mathtt{S}$. If $\mathtt{x}_u$ is of type (2), then take

$$r = \begin{pmatrix} \mathtt{S} & \mathtt{S} \\ 0 & \mathtt{S} \end{pmatrix}, \qquad s = \begin{pmatrix} \mathrm{id} & 0 \\ 0 & Y \end{pmatrix} \triangleright r = \begin{pmatrix} \mathtt{S} & \mathtt{S}Y^{-1} \\ 0 & c\mathtt{S} \end{pmatrix};$$

$$R_1 = \left\{ \begin{pmatrix} \mathtt{S} & * \\ 0 & \mathtt{S} \end{pmatrix} \in \mathbb{F}_q^{2\Lambda\times 2\Lambda} \right\}, \qquad R = \pi(R_1) \cap \mathcal{O} \ni \pi(r);$$

$$\mathtt{S}_1 = \left\{ \begin{pmatrix} \mathtt{S} & * \\ 0 & c\mathtt{S} \end{pmatrix} \in \mathbb{F}_q^{2\Lambda\times 2\Lambda} \right\}, \qquad \mathtt{S} = \pi(\mathtt{S}_1) \cap \mathcal{O} \ni \pi(s).$$

Then r and s are conjugated in $\mathbf{SL}_{2\Lambda}(q^\Lambda)$ and $R \amalg S \hookrightarrow \mathcal{O}^{\boldsymbol{G}}_{x}$ is decomposable. Also $(rs)^2 = (sr)^2$ means that

$$\begin{pmatrix} \mathtt{S}^4 & (c+c^2)\mathtt{S}^4 + \mathtt{S}^4Y^{-1} + c\mathtt{S}^2Y^{-1}\mathtt{S}^2 \\ 0 & c^2\mathtt{S}^4 \end{pmatrix} = \begin{pmatrix} \mathtt{S}^4 & (c+1)\mathtt{S}^4 + \mathtt{S}^3Y^{-1}\mathtt{S} + c\mathtt{S}Y^{-1}\mathtt{S}^3 \\ 0 & c^2\mathtt{S}^4 \end{pmatrix}$$

$$\iff \quad \mathtt{S}^4Y^{-1} + c^2\mathtt{S}^4 + c\mathtt{S}^2Y^{-1}\mathtt{S}^2 = \mathtt{S}^3Y^{-1}\mathtt{S} + \mathtt{S}^4 + c\mathtt{S}Y^{-1}\mathtt{S}^3$$

$$\iff \quad c^2(c^2-1)\,\mathrm{id} = (1-c^2)Y$$

$$\iff \quad c^2 = 1,$$

where we have used $\mathtt{S}Y^{-1} = cY^{-1}\mathtt{S}$ and that Y is not a scalar matrix. Thus, if q is even, then $c = 1$, a contradiction; and if q is odd and $c \neq 1$, then $\operatorname{ord} c = 2$, hence Λ is even and q^Λ is a square. Hence $\mathcal{O}^{\boldsymbol{G}}_{x}$ is of type D, except when $q^\Lambda = 9$. If $q^\Lambda = 9$, then $q = 3$ and $\Lambda = 2$. Let $\mathtt{S} = \left(\begin{smallmatrix} 0 & 1 \\ 2 & 0 \end{smallmatrix}\right)$, $\mathtt{R} = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right) \in \mathbf{SL}_2(3)$; they are conjugated in $\mathbf{SL}_2(3)$ and $\mathtt{SR} = -\mathtt{RS}$, so that $\pi(\mathtt{S})\pi(\mathtt{R}) = \pi(\mathtt{R})\pi(\mathtt{S})$. It is enough to deal with $\mathcal{O} = \mathcal{O}^{\boldsymbol{G}}_{x}$ where

$$\mathtt{x} = r = \begin{pmatrix} \mathtt{S} & \mathtt{S} \\ 0 & \mathtt{S} \end{pmatrix}, \qquad s = \begin{pmatrix} 0 & \mathrm{id}_2 \\ 2\,\mathrm{id}_2 & 0 \end{pmatrix} \triangleright \begin{pmatrix} \mathtt{R} & \mathtt{R} \\ 0 & \mathtt{R} \end{pmatrix} = \begin{pmatrix} \mathtt{R} & 0 \\ 2\mathtt{R} & \mathtt{R} \end{pmatrix};$$

$$R_1 = \left\{ \begin{pmatrix} a\mathtt{S} & b\mathtt{S} \\ d\mathtt{S} & c\mathtt{S} \end{pmatrix} \in \mathbf{SL}_4(9) : a,b,c,d \in \mathbb{F}_3 \right\}, \qquad R = \pi(R_1) \cap \mathcal{O} \ni \pi(r);$$

$$S_1 = \left\{ \begin{pmatrix} a\mathtt{R} & b\mathtt{R} \\ d\mathtt{R} & c\mathtt{R} \end{pmatrix} \in \mathbf{SL}_4(9) : a,b,c,d \in \mathbb{F}_3 \right\}, \qquad S = \pi(S_1) \cap \mathcal{O} \ni \pi(s).$$

Then r and s are conjugated in $\mathbf{SL}_4(9)$, $(rs)^2 \neq (sr)^2$, $R \amalg S \hookrightarrow \mathcal{O}^{\boldsymbol{G}}_{x}$ and $\mathcal{O}$ is of type D. The types $(2,1)$ and $(2,1,1)$ are treated as above, with

$$r = \begin{pmatrix} \mathtt{S} & \mathtt{S} & 0 \\ 0 & \mathtt{S} & 0 \\ 0 & 0 & \mathtt{S} \end{pmatrix}, \quad \text{respectively } r = \begin{pmatrix} \mathtt{S} & \mathtt{S} & 0 & 0 \\ 0 & \mathtt{S} & 0 & 0 \\ 0 & 0 & \mathtt{S} & 0 \\ 0 & 0 & 0 & \mathtt{S} \end{pmatrix},$$

$$s = \begin{pmatrix} \mathrm{id} & 0 & 0 \\ 0 & Y & 0 \\ 0 & 0 & \mathrm{id} \end{pmatrix} \triangleright r = \begin{pmatrix} \mathtt{S} & \mathtt{S}Y^{-1} & 0 \\ 0 & c\mathtt{S} & 0 \\ 0 & 0 & \mathtt{S} \end{pmatrix}, \quad \text{respectively } s = \begin{pmatrix} \mathtt{S} & \mathtt{S}Y^{-1} & 0 & 0 \\ 0 & c\mathtt{S} & 0 & 0 \\ 0 & 0 & \mathtt{S} & 0 \\ 0 & 0 & 0 & \mathtt{S} \end{pmatrix}.$$

Case 2. $M = 1 < \ell$. Then $\mathcal{O}_x^{\boldsymbol{G}}$ is of type D.

In this case $x_u = (u_1, 1, ..., 1)$. Assume that $\Lambda_i > 1$ for some i. Then $\mathtt{y}_s = (\mathtt{S}_1, \dots, \mathtt{S}_i^q, \dots, \mathtt{S}_\ell)$ (all $\mathtt{S}_h$ equal to $\mathtt{S}_i$ raised to the q), is conjugated to $\mathtt{x}_s$; clearly $\pi(\mathtt{x}_s) \neq \pi(\mathtt{y}_s)$. By Remarks 4.1 and 4.2

$$\left(\mathcal{O}_{\mathtt{x}_u}^{\mathcal{K}}\right)^{(2)} \overset{2.10}{=} \mathcal{O}_{\mathtt{x}_u}^{\mathcal{K}} \amalg \mathcal{O}_{\mathtt{x}_u}^{\mathcal{K}} \simeq \pi(\mathtt{x}_s)\mathcal{O}_{x_u}^{K} \amalg \pi(\mathtt{y}_s)\mathcal{O}_{x_u}^{K} \hookrightarrow \mathcal{O}_x^{\boldsymbol{G}}.$$

By (a), $\mathcal{O}_x^{\boldsymbol{G}}$ is of type D. Assume then that $\Lambda_i = 1$ for all $i \in \mathbb{I}_\ell$. Since $\ell > 1$ the case u_1 of type (3) with $q = 2$ is excluded, so u_1 is of type (2), $(2, 1)$ or $(2, 1, 1)$. We consider first the case when $\ell = 2$ and u_1 is of type (2). Let

$$r = \begin{pmatrix} \mathtt{S}_1 & \mathtt{S}_1 & 0 \\ 0 & \mathtt{S}_1 & 0 \\ 0 & 0 & \mathtt{S}_3 \end{pmatrix}, \qquad s = \begin{pmatrix} \mathtt{S}_3 & 0 & 0 \\ 0 & \mathtt{S}_1 & \mathtt{S}_1 \\ 0 & 0 & \mathtt{S}_1 \end{pmatrix};$$

$$R_1 = \left\{ \begin{pmatrix} \mathtt{S}_1 & * & * \\ 0 & \mathtt{S}_1 & * \\ 0 & 0 & \mathtt{S}_3 \end{pmatrix} \in \mathbb{F}_q^{3\times 3} \right\}, \qquad R = \pi(R_1) \cap \mathcal{O} \ni \pi(r);$$

$$\mathtt{S}_1 = \left\{ \begin{pmatrix} \mathtt{S}_3 & * & * \\ 0 & \mathtt{S}_1 & * \\ 0 & 0 & \mathtt{S}_1 \end{pmatrix} \in \mathbb{F}_q^{3\times 3} \right\}, \qquad \mathtt{S} = \pi(\mathtt{S}_1) \cap \mathcal{O} \ni \pi(s).$$

Then r and s are conjugated in $\mathbf{SL}_3(q)$, $R \amalg S \hookrightarrow \mathcal{O}_x^{\boldsymbol{G}}$ is decomposable,

$$(rs)^2 = \begin{pmatrix} \mathtt{S}_1^2\mathtt{S}_3^2 & \mathtt{S}_1^3(\mathtt{S}_1 + \mathtt{S}_3) & \mathtt{S}_1^3(\mathtt{S}_1 + 2\mathtt{S}_3) \\ 0 & \mathtt{S}_1^4 & \mathtt{S}_1^3(\mathtt{S}_1 + \mathtt{S}_3) \\ 0 & 0 & \mathtt{S}_1^2\mathtt{S}_3^2 \end{pmatrix}$$

$$(sr)^2 = \begin{pmatrix} \mathtt{S}_1^2\mathtt{S}_3^2 & \mathtt{S}_1^2\mathtt{S}_3(\mathtt{S}_1 + \mathtt{S}_3) & \mathtt{S}_1^2\mathtt{S}_3^2 \\ 0 & \mathtt{S}_1^4 & \mathtt{S}_1^2\mathtt{S}_3(\mathtt{S}_1 + \mathtt{S}_3) \\ 0 & 0 & \mathtt{S}_1^2\mathtt{S}_3^2 \end{pmatrix}.$$

Hence $(\pi(r)\pi(s))^2 \neq (\pi(s)\pi(r))^2$ and thus $\mathcal{O}_x^{\boldsymbol{G}}$ is of type D. The other cases are dealt with in a similar way.

Case 3. $M > 1$, and $q^{\Lambda_t} \neq 3$ for some $t \in \mathbb{I}_M$. Then $\mathcal{O}_x^{\boldsymbol{G}}$ is of type D.

Assume q is odd. Since $M > 1$ there is $k \in \mathbb{I}_M - \{t\}$ such that $u_k \neq 1$. We set $X = \mathcal{O}_{u_k}^{\mathbf{SL}_{h_k}(q^{\Lambda_k})}$, $Y = \mathcal{O}_{u_t}^{\mathbf{SL}_{h_t}(q^{\Lambda_t})}$. By Lemma 2.10, (a) and (b) $X \times Y$, and $\mathcal{O}_{\mathtt{x}_u}^{\mathcal{K}}$, are of type D.

If q is even, then the same argument applies except when u_t is of type (2) with $q^{\Lambda_t} = 2$ for all $t \in \mathbb{I}_M$. But here $\mathtt{S}_t \in \mathbb{F}_2^{\times}$, i.e., $\mathtt{S}_t = 1$ so $M = 1$, a contradiction.

Case 4. $M > 1$, and $q^{\Lambda_t} = 3$ for all $t \in \mathbb{I}_M$. Then $\mathcal{O}_x^{\boldsymbol{G}}$ is of type D.

According to our reduction we need only to consider the case in which $h_t = 2$ and u_t is of type (2) for every $t \in \mathbb{I}_M$. Then $\mathtt{S}_t \in \mathcal{C}_{S_t} \simeq \mathbb{F}_{q^{\Lambda_t}} = \mathbb{F}_3$, so $\mathtt{S}_1 = \mathtt{S}_2 = 1$, $\mathtt{S}_3 = \mathtt{S}_4 = 2$ and $M = 2$. The rack of unipotent conjugacy classes in $\mathbf{SL}_2(3)$ is the union of two conjugacy classes $\mathcal{O}_1 \ni r_1$ and $\mathcal{O}_2$, both isomorphic to the tetrahedral rack. If $M < \ell$, then $\mathcal{K} \supseteq \{(g_1, g_2, g_3) \in \mathbf{GL}_2(3) \times \mathbf{GL}_2(3) \times \mathbb{F}_3^{\times} : \det g_1 \det g_2 = g_3^{-1}\}$; thus $\mathcal{O}_{(r_1,r_1)}^{\mathcal{K}} = \coprod_{i,j\in\mathbb{I}_2} \mathcal{O}_i \times \mathcal{O}_j$ is of type D, being $\mathcal{O}_1 \times \mathcal{O}_1 \amalg \mathcal{O}_1 \times \mathcal{O}_2$ of type D.

If $M = \ell = 2$, then $\mathcal{O}_{(r_1,r_1)}^{\mathcal{K}}$ is cthulhu, so we consider $\mathcal{O} = \mathcal{O}_x^{\boldsymbol{G}}$. There are two different conjugacy classes with the same x_s, namely those with representative $r = \begin{pmatrix} 1&1&0&0\\0&1&0&0\\0&0&2&2\\0&0&0&2 \end{pmatrix}$, respectively $\begin{pmatrix} 1&2&0&0\\0&1&0&0\\0&0&2&2\\0&0&0&2 \end{pmatrix}$, but they are isomorphic as racks being conjugated in $\mathbf{PGL}_3(4)$. Now let $s = \begin{pmatrix} 1&0&0&0\\0&0&1&0\\0&0&0&1\\0&1&0&0 \end{pmatrix} \triangleright r = \begin{pmatrix} 1&0&0&1\\0&2&2&0\\0&0&2&0\\0&0&0&1 \end{pmatrix}$; hence $(\pi(r)\pi(s))^2 \neq (\pi(s)\pi(r))^2$. Let

$$R_1 = \left\{ \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 2 & * \\ 0 & 0 & 0 & 2 \end{pmatrix} \in \mathbb{F}_3^{4\times 4} \right\}, \qquad R = \pi(R_1) \cap \mathcal{O} \ni \pi(r);$$

$$S_1 = \left\{ \begin{pmatrix} 1 & * & * & * \\ 0 & 2 & * & * \\ 0 & 0 & 2 & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathbb{F}_3^{4\times 4} \right\}, \qquad S = \pi(S_1) \cap \mathcal{O} \ni \pi(s).$$

Then $R \amalg S \hookrightarrow \mathcal{O}$ is a decomposable subrack, and $\mathcal{O}$ is of type D. $\square$

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