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On the Existence of Critical Clique-Helly Graphs

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Abstract

A graph is *clique-Helly* if any family of mutually intersecting cliques has non-empty intersection. Dourado, Protti and Szwarcfiter conjectured that every clique-Helly graph contains a vertex whose removal maintains it a clique-Helly graph. We will present a counterexample to this conjecture.

Keywords: Helly property, Clique-Helly graphs, clique graphs.

1 Introduction

A set family $\mathcal{F}$ satisfies the *Helly property* if the intersection of all the members of any pairwise intersecting subfamily of $\mathcal{F}$ is non-empty. This property, originated in the famous work of Eduard Helly on convex sets in the Euclidean

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space, has been widely studied in diverse areas of theoretical and applied mathematics such as extremal hypergraph theory, logic, optimization, theoretical computer science, computational biology, data bases, image processing and, clearly, graphs theory. A few surveys have been written on the Helly property, see for instance [1,2,4,5].

From the computational and algorithmic point of view, the relevance of the Helly property has been highlighted in the survey [3]. In the section *Proposed Problems* of that work, the authors posed the following open question:

Conjecture 1.1 (Dourado, Protti and Szwarcfiter) *Every clique-Helly graph (the family of maximal cliques of the graph satisfies the Helly property) contains a vertex whose removal maintains it a clique-Helly graph.*

In this work, we prove that the conjecture is false: in Section 3 we will exhibit a clique-Helly graph G such that $G - v$ (the graph obtained from G by removing vertex v) is not clique-Helly for every vertex v of G.

2 Definitions and preliminary results

Given a finite and simple graph G, we let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively.

The *open* and *closed neighborhoods* of a vertex $v \in V(G)$ are denoted by $N_G(v)$ and $N_G[v]$, respectively. The *degree* of v is the cardinality of $N_G(v)$.

If $S \subseteq V(G)$ then the subgraphs of G induced by S and by $V(G)\backslash S$ are denoted by $G[S]$ and $G - S$, respectively. When S contains a unique vertex v, we write $G - v$ for $G - \{v\}$.

The *complete graph* on n vertices is denoted by K_n. A *complete set* of G is a subset of $V(G)$ inducing a complete subgraph. A *clique* is a maximal (with respect to the inclusion relation) complete set. We let $\mathcal{C}(G)$ be the family of cliques of G. When $\mathcal{C}(G)$ satisfies the Helly property, we say that G is a *clique-Helly graph*. The *clique graph* $K(G)$ of G is the intersection graph of $\mathcal{C}(G)$: the vertices of $K(G)$ are the cliques of G and two different cliques of G are adjacent in $K(G)$ if and only if they have non-empty intersection.

A *chordless cycle* in G is a sequence of at least three distinct vertices $v_1, v_2, \ldots, v_k$ of G such that two of them are adjacent in G if and only if they are consecutive in the sequence or they are v_1 and v_k. The positive integer k is the *length* of the cycle. The chordless cycle of length k is denoted by C_k. The *girth* $g(G)$ of G is the length of a shortest chordless cycle in G (if G has no cycle, then $g(G) = \infty$). The *local girth* of G *at a vertex* $v \in V(G)$ is the girth of the subgraph induced by the open neighborhood of v in G, i.e

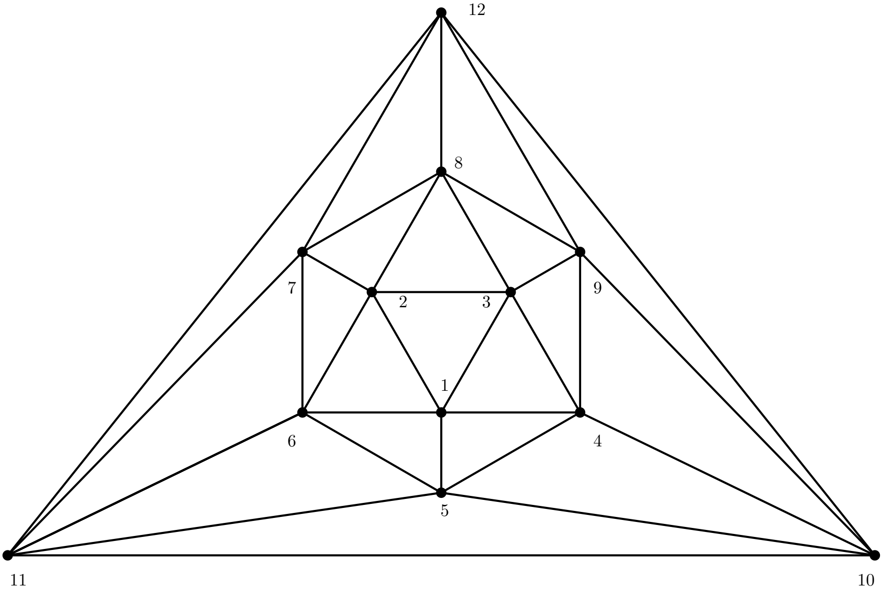

Fig. 1. The icosahedron.

$lg_v(G) = g(G[N(v)])$. The minimum of the local girths at the different vertices of G is denoted by $lg(G)$ and named the *local girth* of G, i.e.

$$lg(G) = \min\{lg_v(G) : v \in V(G)\}.$$

Theorem 2.1 ([6]) *If the local girth of the graph G is greater than 6 (i.e. $lg(G) \geq 7$) then $K(G)$ is clique-Helly.*

Definition 2.2 A graph G is *critical clique-Helly* if G is clique-Helly and $G - v$ is not clique-Helly for every $v \in V(G)$.

Note that in terms of the previous definition, the conjecture of Dourado, Protti and Szwarcfiter postulates that there are no critical clique-Helly graphs. In what follows, a counterexample to that conjecture will be obtained as the clique graph of the tensor product of the icosahedron and the complete graph with three vertices K_3 (also called a *triangle*).

The *icosahedron* I is the graph with vertex set $\{1, 2, \ldots, 12\}$ depicted in Fig. 1. The following properties of I can be easily checked.

Proposition 2.3 (i) *Every vertex of I has degree 5.*

(ii) *The open neighborhood of each vertex of I induces a C_5.*

(iii) *The cliques of I are precisely its faces which are all triangles.*

(iv) *Every vertex of I is in exactly 5 cliques.*

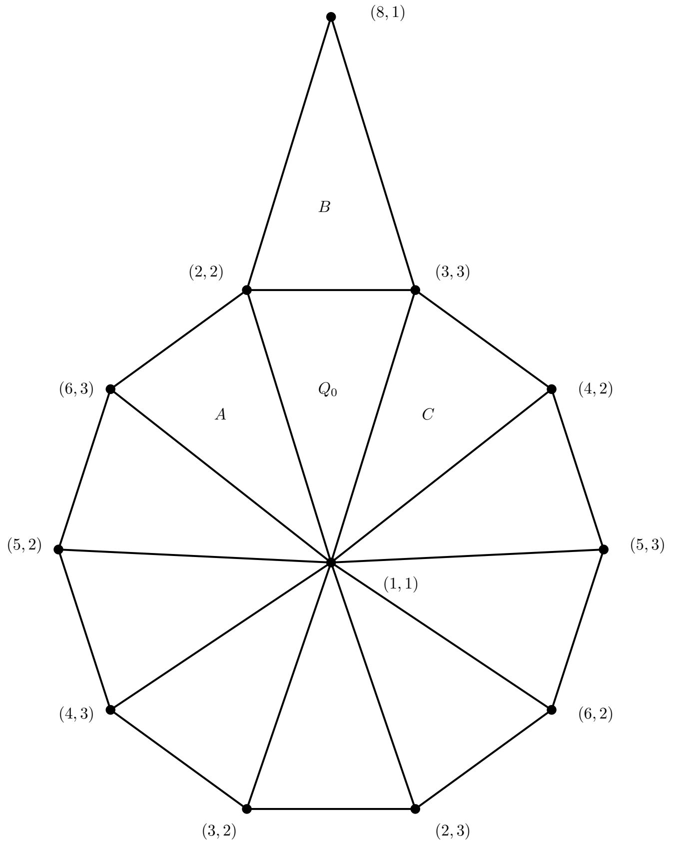

Fig. 2. A partial drawing of $I \times K_3$.

The *tensor product* $I \times K_3$ is the graph with $V(I \times K_3) = V(I) \times \{1, 2, 3\}$ and $E(I \times K_3)$ defined as follows: two vertices (i, j) and (i', j') are adjacent in $I \times K_3$ if and only if i is adjacent to i' in I and $j \neq j'$. Clearly, $I \times K_3$ is a graph on 36 vertices. Fig. 2 shows an induced subgraph of $I \times K_3$ including the neighborhood of the vertex $(1, 1)$.

Lemma 2.4 (i) *Every vertex of $I \times K_3$ has degree 10.*

(ii) *The open neighborhood of each vertex of $I \times K_3$ induces a C_{10}.*

(iii) *The cliques of $I \times K_3$ are triangles $\{(i, 1), (j, 2), (k, 3)\}$ for any triangle $\{i, j, k\}$ of I.*

(iv) *Every vertex of $I \times K_3$ is in exactly ten cliques; and any other clique of $I \times K_3$ (i.e. any clique which does not contain the given vertex) intersects at most three of those ten cliques.*

Proof. (i) Consider the vertex $(1,1)$ of $I \times K_3$. Since $N_I(1) = \{2,3,4,5,6\}$ (see Fig. 1), we have that

$$N_{I\times K_3}((1,1)) = \{(i,j) : i \in \{2,3,4,5,6\} \text{ and } j \in \{2,3\}\}.$$

The regularity of I extends the proof to any other vertex of $I \times K_3$.
(ii) Again consider the vertex $(1,1)$ of $I \times K_3$ and its ten neighbors. It is easy to check that the adjacencies between them are exactly the ones depicted in Fig. 2; thus $N_{I\times K_3}((1,1))$ induces a C_{10} in $I \times K_3$. The regularity and symmetry of I extends the proof to any other vertex of $I \times K_3$.
(iii) It is a clear consequence of the previous two assertions.
(iv) One more time, without loss of generality, consider the vertex $(1,1)$ of $I \times K_3$. That $(1,1)$ is in exactly ten cliques follows from (i) and (ii), see Fig. 2. On the other hand, if Q is a clique which does not contain the vertex $(1,1)$ then Q contains at most two consecutive vertices of the cycle induced by the neighbors of $(1,1)$ which implies that Q intersects at most three of the ten cliques containing $(1,1)$. □

3 The main theorem

Theorem 3.1 *The graph $K(I \times K_3)$ is critical clique-Helly.*

Proof. By the assertion (ii) of Lemma 2.4, the local girth of $I \times K_3$ equals 10. Therefore, by Theorem 2.1, $K(I \times K_3)$ is clique-Helly.

Let Q_0 be any vertex of $K(I \times K_3)$, i.e. Q_0 is a clique of $I \times K_3$. Without loss of generality assume that $Q_0 = \{(1,1),(2,2),(3,3)\}$ (see Fig. 2). We will prove that $K(I \times K_3) - Q_0$ is not clique-Helly.

For $i \in \{1,2,3\}$, let D_i be the set of vertices of $K(I \times K_3) - Q_0$ corresponding to the cliques of $I \times K_3$ containing the vertex (i,i), that is

$$D_i = \{Q \in \mathcal{C}(I \times K_3) : (i,i) \in Q\} \setminus \{Q_0\}.$$

By the assertion (iv) of Lemma 2.4, D_i is a clique of $K(I \times K_3) - Q_0$ for $i \in \{1,2,3\}$. We claim that these three cliques are pairwise intersecting but the intersection of all three of them is empty. Indeed, the vertices of $K(I \times K_3) - Q_0$ corresponding to the cliques $\{(1,1),(2,2),(6,3)\}$, $\{(2,2),(3,3),(8,1)\}$ and $\{(1,1),(3,3),(4,2)\}$ of $I \times K_3$ (named A, B and C, respectively, in Fig. 2) belong to $D_1 \cap D_2$, $D_2 \cap D_3$ and $D_1 \cap D_3$, respectively. Finally, assume in order to obtain a contradiction that a vertex Q of $K(I \times K_3) - Q_0$ belongs to

$D_1 \cap D_2 \cap D_3$. Then, by definition of these sets, Q is a clique of $I \times K_3$ such that $(i, i) \in Q$ for $i \in \{1, 2, 3\}$. Thus, by the assertion (iii) of Lemma 2.4, $Q = \{(1, 1), (2, 2), (3, 3)\} = Q_0$ which contradicts the fact that Q is a vertex of $K(I \times K_3) - Q_0$. $\square$

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