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On local edge intersection graphs of paths on bounded degree trees

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Abstract

An undirected graph G is called an EPT graph if it is the edge intersection graph of a family of paths in a tree. We call G a local EPT graph if it is the EPT graph of a collection of paths P which all share a common vertex. In this paper, we characterize the local EPT graphs which can be represented in a host tree with maximum degree h.

1 Introduction and previous results

A graph G is called an **EPT graph** if it is the edge intersection graph of a family of paths in a tree. An **EPT representation** of G is a pair $\langle P, T\rangle$ where P is a family $(P_v)_{v\in V(G)}$ of subpaths of the host tree T satisfying that two vertices v and v' of G are adjacent if and only if P_v and $P_{v'}$ have at least two vertices (one edge) in common.

When the maximum degree of the host tree T is h, the EPT representation of G is called an **(h,2,2)-representation** of G. The class of graphs which admit an $(h, 2, 2)$-representation is denoted by **[h,2,2]**.

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Notice that the class of EPT graphs is the union of the classes [h,2,2] for $h \geq 2$. In [GJ85] it is proved that the recognition of EPT graphs is an NP-complete problem.

The EPT graphs are used in network applications, where the problem of scheduling undirected calls in a tree network is equivalent to the problem of coloring an EPT graph (see [TE96]). The communication network is represented as an undirected interconnection graph, where each edge is associated with a physical link between two nodes. An undirected call is a path in the network. When the network is a tree, this model is clearly an EPT representation. Coloring the EPT graph, such that two adjacent vertices have different colors, implies that paths sharing at least one common edge in the EPT representation have different colors, meaning that undirected calls that share a physical link are scheduled in different times.

In this paper, we examine the local structure of paths passing through a given vertex of a host tree which has maximum degree h, and show these locally EPT graphs are equivalent to the line graphs of certain graphs which have certain properties.

Definition 1.1. *[GJ85] Let $\langle P, T \rangle$ be an EPT representation of a graph G. A **pie of size n** is a star subgraph of T with central vertex q and neighbors $q_1, ..., q_n$ such that each "slice" $q_i q q_{i+1}$ for $1 \leq i \leq n$ is contained in a different member of P; addition is assumed to be module n. (See Figure 1).*

Let $\langle P, T \rangle$ be an EPT representation of a graph G. It was proved (see [GJ85]) that if G contains a chordless cycle of length $n \geq 4$, then $\langle P, T \rangle$ contains a pie of size n.

In a pie all paths share a common central vertex of the tree. Let us pursue this idea further.

Definition 1.2. *[GJ85] We say that $\langle P, T \rangle$ is a **local EPT representation** of G if it is an EPT representation where all the paths of P share*

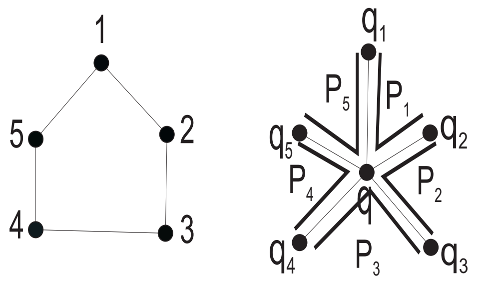

Figure 1: The cycle C_5 and an EPT representation: a pie of size 5.

a common vertex of T. We call G a ***local EPT graph*** *if it has a local EPT representation.*

Let $h \geq 5$, we say that G belongs to the class ***[h,2,2] local*** *if and only if G has a local EPT representation in a host tree T with maximum degree h.*

Definition 1.3. *Let $\langle P, T \rangle$ be a local $(h, 2, 2)$-representation of G, being T a star with central vertex q such that $N_T(q) = \{q_1, q_2, ..., q_h\}$. We say that the edges $qq_i \in E(T)$, with $1 \leq i \leq h$, are the* ***legs*** *of T at q.*

2 Our results

In this Section, we characterize graphs which belongs to the class $[h, 2, 2]$ local.

Definition 2.1. *Let G be a connected graph. We say that $v \in V(G)$ is a* ***cut vertex*** *of G if $G - v$ has at least two connected components.*

Theorem 2.1. Let $h \geq 5$. If $G \in [h, 2, 2]$ local and $G \notin [h-1, 2, 2]$ then G has no cut vertices.

Proof: Let $\langle P, T \rangle$ be a local $(h, 2, 2)$-representation of G, being T a star with central vertex q such that $N_T(q) = \{q_1, q_2, ..., q_h\}$.

Suppose, by the contrary, that G has a cutting vertex, say v_1. Then, $G - v_1$ has exactly two connected components C_1 and C_2. Since vertices of C_1 are non-adjacent to vertices of C_2, we have that its corresponding paths use different legs of T at q. Assume that the paths which represent vertices of C_1 use the legs $qq_1, .., qq_n$, with $1 \leq n \leq h-1$, of T at q. And, the paths which represent vertices of C_2 use the legs $qq_{n+1}, .., qq_h$.

We are going to build an $(h-1, 2, 2)$-representation of G, say $\langle P', T' \rangle$.

Case (1): If $n < h-1$.

First we represent the connected component C_1. We define a star with central vertex q' such that $N_{T'}(q') = \{q'_1, ..., q'_n\}$. If $v \in V(C_1)$ with $q_iq_j \in E(P_v)$ in T then $q'_iq'_j \in E(P'_v)$ in T'.

Now, we represent the connected component C_2. We define a star with central vertex q'' such that $N_{T'}(q'') = \{q''_{n+1}, .., q''_h\}$. If $v \in V(C_2)$ with $q_iq_j \in E(P_v)$ in T then $q''_iq''_j \in E(P'_v)$ in T'.

Then, we only have to represent the path P_{v_1} . We put an edge between q' and q'' in T'. Since v_1 is a cutting vertex of G, we have that it is adjacent to at least one vertex of C_1 and at least one vertex of C_2. If $q_i \in V(P_{v_1})$ in T, with $1 \leq i \leq n$, then $q'_i \in V(P'_{v_1})$ in T'. If $q_j \in V(P_{v_1})$ in T, with $n+1 \leq j \leq h$, then $q''_j \in V(P'_{v_1})$ in T'. So, $V(P'_{v_1}) = \{q'_i, q', q'', q''_j\}$.

Case (2): If $n = h-1$, the paths which represent vertices of C_2 use the leg qq_h. We define a star with central vertex q' such that $N_{T'}(q') = \{q'_1, ..., q'_{h-1}\}$. If $v \in V(C_1)$ with $q_iq_j \in E(P_v)$ in T then $q'_iq'_j \in E(P'_v)$ in T'.

Now, we represent the connected component C_2.

Since v_1 is a cutting vertex of G, we have that it is adjacent to at least one vertex of C_1 and at least one vertex of C_2. Hence, $q_i \in V(P_{v_1})$ in T, for some $1 \leq i \leq n$ and $q_h \in V(P_{v_1})$. Suppose, without loss of generality, that $q_1 \in V(P_{v_1})$. Then, we add a vertex q'_h such that $q'_1q'_h \in E(T')$. If $v \in V(C_2)$ with $qq_h \in E(P_v)$ in T then $q'_1q'_h \in E(P'_v)$ in T'. And, $V(P'_{v_1}) = \{q', q'_1, q'_h\}$.

Hence, we have an $(h-1, 2, 2)$-representation of G which contradicts the fact that $G \notin [h-1, 2, 2]$. Therefore, G has no cut vertices.

□

We show that this special subclass of EPT graphs is equivalent to the class of line graphs of certain graphs which have certain properties.

Definition 2.2. *Let H be a graph, the **line graph** of H, noted by $L(H)$, has vertices corresponding to the edges of H with two vertices adjacent in $L(H)$ if their corresponding edges of H share an endpoint.*

Definition 2.3. *We say that two vertices $u, v \in V(G)$ are **adjacent dominated vertices** if $uv \in E(G)$ and $N_G(u) \subseteq N_G(v)$ or $N_G(v) \subseteq N_G(u)$.*

Theorem 2.2. If $h \geq 5$, then $G \in [h, 2, 2]$ local, $G \notin [h-1, 2, 2]$ and G has no adjacent dominated vertices if and only if $G = L(H)$ with H a graph such that:

1. $|V(H)| = h$.
2. H has no vertices of degree 1.
3. H is simple.
4. H has no adjacent dominated vertices.
5. H has a cycle C_n, with $4 \leq n \leq h$; and every vertex of $H - C_n$ is in some path between two different vertices of C_n.

Proof: $\Leftarrow$) We know that $G = L(H)$, with H satisfying $(1), .., (5)$.

Let us verify that $G \in [h, 2, 2]$ local: We build a local $(h, 2, 2)$-representation of G as follows. By item (1), we know that $|V(H)| = h$. Let $V(H) = \{q_1, q_2, .., q_h\}$, we define $V(T) = \{q, q_1, .., q_h\}$ and $E(T) = \{qq_i$, for all $1 \leq i \leq h\}$.

For each edge $e_{ij} = q_iq_j \in E(H)$ we define a path P_{ij} in T such that $V(P_{ij}) = \{q_i, q, q_j\}$. Two paths P_{ij}, P_{kl} share an edge in T if and only if $\{i, j\} \cap \{k, l\} \neq \emptyset$, that is, if and only if the corresponding edges e_{ij}, e_{kl} in H share a vertex. Then, we have a local $(h, 2, 2)$-representation of G.

Now, we are going to verify that G has no adjacent dominated vertices: Suppose, by the contrary, that $v_1, v_2 \in V(G)$ are adjacent dominated vertices, that is, $N_G(v_1) \subseteq N_G(v_2)$ and $v_1v_2 \in E(G)$. Then, $v_1, v_2 \in E(H)$ such that they have a common endpoint in H say q_1 (because $v_1v_2 \in E(G)$). We call q_2, q_1 to the endpoints of the edge v_1 of H and q_3, q_1 to the endpoints of the edge v_2 of H. Since H has no vertices of degree 1 (by item (2)) and since every edge of H that has v_1 as an endpoint has v_2 as an endpoint too, we have that q_2 and q_3 are adjacent dominated vertices of H, which is a contradiction.

Let us verify that $G \notin [h-1, 2, 2]$: Suppose, by the contrary, that $G \in [h-1, 2, 2]$. Let $\langle \tilde{P}, \tilde{T} \rangle$ be an $(h-1, 2, 2)$-representation of G. By item (5), we know that H has a cycle C_n, with $4 \leq n \leq h$, and every vertex of $H - C_n$ is in some path between two different vertices of C_n. It is easy to verify that the cycle C_n in H leads to an induced cycle $\tilde{C_n}$ in G. Moreover, since every vertex of $H - C_n$ is in some path between two different vertices of C_n, we have that every edge of $H - C_n$ is in some path between two different vertices of C_n. Hence every vertex of $G - \tilde{C_n}$ is in some path between two different vertices of $\tilde{C_n}$. It is easy to verify that this forces all the paths which represent vertices of G have a vertex of $\tilde{T}$, say q, in common, and this forces the paths to use exactly two legs of $\tilde{T}$ at q.

Then, $\langle \tilde{P}, \tilde{T} \rangle$ is a local $(h-1, 2, 2)$-representation of G, that is, $\tilde{T}$ is a star with central vertex q and legs $\tilde{q}_1, \tilde{q}_2, .., \tilde{q}_{h-1}$.

We are going to build a simple graph $\tilde{H}$ with $|V(\tilde{H})| = h-1$ such that $G = L(\tilde{H})$. Let $V(\tilde{H}) = \{\tilde{q}_1, \tilde{q}_2, .., \tilde{q}_{h-1}\}$. If $\tilde{P}_v \in \tilde{\mathcal{P}}$ such that $\{\tilde{q}_i, q, \tilde{q}_j\} \subseteq V(\tilde{P}_v)$ we define $\tilde{e_{ij}} = \tilde{q}_i\tilde{q}_j \in E(\tilde{H})$. Hence, $G = L(\tilde{H})$. Then, since G has no adjacent dominated vertices, we have that $\tilde{H}$ has no multiple edges. And, since all the paths of $\tilde{\mathcal{P}}$ use exactly two legs of $\tilde{T}$ at q we have that $\tilde{H}$ has no loops. Therefore, $\tilde{H}$ is a simple graph. And, by item (3), H is a simple graph too.

Hence, $G = L(H) = L(\tilde{H})$, with H and $\tilde{H}$ simple graphs. But the unique non isomorphic simple connected graphs which have isomorphic line graphs are K_3 and $K_{1,3}$ [BLS99]. Then, $H = K_3$ and $\tilde{H} = K_{1,3}$. So, $|V(H)| = |V(\tilde{H})| = 3$ which is a contradiction. Therefore, $G \notin [h-1, 2, 2]$.

$\Rightarrow$) We know that $G \in [h, 2, 2]$ local, $G \notin [h-1, 2, 2]$ and G has no adjacent dominated vertices. We have to verify that there exists a graph H such that $G = L(H)$ and H satisfies the properties (1), .., (5).

We are going to verify that $|V(H)| = h$: We know that $G \in [h, 2, 2]$ local, with a representation $\langle P, T\rangle$ being T a star with central vertex q such that $N_T(q) = \{q_1, q_2, .., q_h\}$. We are going to build a graph H with $|V(H)| = h$ such that $G = L(H)$. Let $V(H) = \{q_1, q_2, .., q_h\}$.

First, observe that if a path $P_{ii} \in \mathcal{P}$ was such that $\{q_i, q\} \subseteq V(P_{ii})$ and $\{q_j\}$ is not contain in $V(P_{ii})$ for all $i \neq j$, then if P_{ii} is the only path which uses this leg we can obtain an $(h-1, 2, 2)$-representation of G, and if there exists other path using this leg we have that the vertex corresponding to this path is adjacent and dominates the vertex corresponding to the path P_{ii}. In both cases, we have a contradiction.

If a path $P_{ij} \in \mathcal{P}$ was such that $\{q_i, q, q_j\} \subseteq V(P_{ij})$ then $e_{ij} = q_iq_j \in E(H)$. Two paths P_{ij}, P_{kl} share an edge in T if and only if $\{i, j\} \cap \{k, l\} \neq \emptyset$ if and only if the corresponding edges e_{ij}, e_{kl} of H share a vertex. Then, $G = L(H)$ with $|V(H)| = h$.

We are going to verify that H has no vertices of degree 1: Let $V(H) = \{q_1, q_2, .., q_h\}$. Suppose that $q_i \in V(H)$ with $d_H(q_i) = 1$. Let e be the unique edge of H that has q_i as an endpoint. Doing the previous construction we have an $(h, 2, 2)$-representation of G such that T is a star with central vertex q such that $N_T(q) = \{q_1, q_2, .., q_h\}$. Moreover, if the leg qq_i is only contained in the path P_e then we can delete the leg qq_i from T and we have an $(h-1, 2, 2)$-representation of G which contradicts the fact that $G \notin [h-1, 2, 2]$.

We have to verify that H is a simple graph: If e and $\tilde{e}$ were multiple edges in H, then e and $\tilde{e}$ would be true twins in G, which contradicts the fact that G has no adjacent dominated vertices.

If H had a loop e that has q_i as its endpoint, then q_i must be the endpoint of another edge, say $\tilde{e}$. Then, e,$\tilde{e} \in V(G)$ such that $e\tilde{e} \in E(G)$ and $N_G(e) \subseteq N_G(\tilde{e})$, which contradicts the fact that G has no adjacent dominated vertices.

We have to verify that H has no adjacent dominated vertices: Suppose that q_1 and q_2 are adjacent dominated vertices of H, such that $N_H(q_1) \subseteq N_H(q_2)$. Then, since H has no vertices of degree 1 and H is a simple graph, we have that there exists $q_3 \in V(H)$ such that $q_1q_3 \in E(H)$ and $q_2q_3 \in E(H)$. Let $e \in E(H)$ and $\tilde{e} \in E(H)$ such that $e = q_1q_3$ and $\tilde{e} = q_2q_3$, we have that $e,\tilde{e} \in V(G)$ are adjacent dominated vertices such that $N_G(e) \subseteq N_G(\tilde{e})$, which contradicts the fact that G has no adjacent dominated vertices.

We have to verify that H has a cycle $\tilde{C_n}$, with $4 \leq n \leq h$: Since $G \notin [h-1, 2, 2]$, with $h \geq 5$, we have that $G \notin (EPT \cap Chordal) = [3, 2, 2]$ (see [JM05]). Hence, $G \notin Chordal$, that is, G has an induced cycle C_n, with $n \geq 4$. So, it must be that H has a cycle C_n as a subgraph, with $n \geq 4$.

On the other hand, if H had a C_n, with $n \geq h+1$, as a subgraph, then G would have an induced cycle C_n, with $n \geq h+1$, as a subgraph, which contradicts the fact that $G \in [h, 2, 2]$.

Finally, we are going to verify that every vertex of $H - C_n$ is in some path between two different vertices of C_n: We know that H has a cycle C_n, with $4 \leq n \leq h$. Suppose, by the contrary, that there exists $x_1 \in V(H) - V(C_n)$ such that x_1 is not in a path between different vertices of C_n. Since $|V(H)| = h$, we have that $|C_n| \leq h$. Since H is a connected graph there exists a path between x_1 and some vertex of the cycle C_n, say v_1. We choose the shortest path, say P, which is an induced path. Let $G = L(H)$, it is clear that C_n leads to an induced cycle in G, and P leads to an induced path in G. Moreover, if e_1 was the edge of P that had v_1 as an extreme vertex in H, then e_1 would be a cut vertex of G. This contradicts Theorem 2.1. $\square$

3 Conclusion

We examine the local structure of paths passing through a given vertex of a host tree which has maximum degree h, that is local EPT graphs which can be represented in a host tree with maximum degree h. We show these locally EPT graphs are equivalent to the line graphs of certain graphs which have certain properties.

Conjecture 3.1. Let $h \geq 5$. If $G \in [h, 2, 2]$, $G \notin [h-1, 2, 2]$ but $G - v \in [h-1, 2, 2]$, for all $v \in V(G)$, then $G \in [h, 2, 2]$ local.

References

[BLS99] Andreas Brandstädt, Van Bang Le, and Jeremy P. Spinrad, *Graph classes: a survey*, SIAM Monographs on Discrete Mathematics and Applications, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1999. MR 1686154

[GJ85] Martin Charles Golumbic and Robert E. Jamison, *The edge intersection graphs of paths in a tree*, J. Combin. Theory Ser. B **38** (1985), no. 1, 8–22. MR 782622

[JM05] Robert E. Jamison and Henry Martyn Mulder, *Constant tolerance intersection graphs of subtrees of a tree*, Discrete Math. **290** (2005), no. 1, 27–46. MR 2117355

[TE96] K. Jansen T. Erlebach, *Scheduling of virtual connections in fast networks*, Proc. of the 4th Parallel Systems and Algorithms Workshop. PASA' 1996, Germany, 10 - 12 April 1996, World Scientific, 1996, pp. 13–32.

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