Co-disciplinary Physics and Mathematics research and study course (RSC) within three study groups: teachers-in-training, secondary school students and researchers

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Abstract

This work shows the results of a research carried out in secondary school and university developing a co-disciplinary Research and Study Course (RSC) on questions connected to Physics and Mathematics in different groups. We present preliminary results of the RSC developed in three different groups: science education researchers, 25 Mathematics teacher students (training teachers of Mathematics) at University, and 68 secondary school students. The development and scope of the RSC in each group is described by using the RSC components as defined by Chevallard's Anthropological Theory of Didactics (ATD). Furthermore, a short description of the Praxeological Model of Reference is presented. This model was built by the researchers group in order to reply to the generative question, unknown in the beginning, and to anticipate and to conceive the evolution of the possible SRC in each institution. Some general results in each group are discussed and some conclusions concerning the ecology of the pedagogy of research and questioning the world at university and secondary schools are drawn.

Keywords

Co-disciplinary research and study course, Physics and Mathematics teaching, modelling

Résumé

Ce travail présent quelques résultats d'une recherche réalisée dans l'école secondaire et l'université pour développer un parcours d'étude et recherche co-disciplinaire autour des questions liées à la physique et la mathématique, dans trois groups différents. On présent ici des résultats préliminaires obtenues avec chaque groupe: des enseignant-chercheurs, des 25 enseignants de mathématique en formation à l'université et des 68 étudiants de l'école secondaire. Les parcours d'étude et recherche développés dans chaque group sont décrits au moyenne de le schème herbartien et ses composants selon ils ont été définis par la Théorie Anthropologique du didactique du Chevallard. On présent ici une bref description du model praxéologique de référence, qui a été construit par les chercheurs pour répondre à la question génératrice, d'abord inconnue, et pour anticiper des possibles parcours dans chaque institution. Quelques résultats généraux de chaque group sont disputes et quelques conclusions relatives à l'écologie de la pédagogie de l'enquête et du questionnement du monde dans l'enseignement universitaire et secondaire sont tirées.

Mots-Clés

Parcours d'étude et recherche co-disciplinaires, enseignement de la Physique et des Mathématiques, modélisation

Introduction

The present work shows the results obtained from previous research developing a non-traditional pedagogy in secondary schools and university level teaching, known as the *Pedagogy of Research and Questioning the World* (Chevallard, 2012) as proposed by the Anthropological Theory of Didactics (ATD). This pedagogy advocates *an epistemological and didactic revolution* (Chevallard, 2012) of the teaching of mathematics and school disciplines and calls for the dropout of the traditional teaching paradigm, for which knowledge is taught as something important in itself and not because of its usefulness or its potential uses in life and the benefits provided to the ones who are passively "forced" to study it. This event leads to a phenomenon called *monumentalism of knowledge* (Ibid.), in reference to the fact that monuments are visited, admired, worshipped, without any consideration of the reasons of its existence. Thus, we witness paradoxical social events, such as the overvaluation of mathematics whenever a primary student or adolescent is questioned about his school performance, whereas the "futility" of mathematics as taught for day-to-day, practical uses is ignored. Teachers are exposed to the recurring sensible questioning on the part of their students: "What do I need this for?" on the face of which, they cannot produce a convincing answer, given that sharing these views or not, they are immersed in what Chavallard (2001) has metaphorically called "instrumental horror". It is well-known and widely accepted the fact that, not only the school mathematics but most of the items studied at school, have limited usefulness; although to this date, there exists a disregard of the usefulness and of what makes knowledge instrumental, which seems to impose itself to the detriment of those students who will spend long years of their lives to attain knowledge for which only "rare uses" are recognized.

In order to deal with the monumentalization phenomenon, the ATD points at the need to give the *raison d'être* back to the knowledge we teach, i.e., to retrieve the questions which have generated a certain piece of work. Traditional teaching has replaced the study of questions by the study of answers, enforcing a non-motivating encounter with such pieces of work which have an unknown raison d'être and with which we can only have a "magical" connection. Furthermore, it is necessary to re contextualize this knowledge following the current needs arising, so that the mathematics and the science taught at present can regain their functionality. Therefore, the co-disciplinary Research and Study Course (RSC) (Chevallard, 2001) are essential didactic devices to the Research Pedagogy as its objective is the recovery of the meaning and raisons d'être of the praxeologies which are taught throughout the different levels of education, establishing the questions as the starting point of knowledge.

The RSCs can be mono-disciplinary or co-disciplinary, as they can involve mathematics alone or mathematics plus other disciplines. According to Chevallard (2013), the *study gestures* as characteristic of the *research pedagogy* are fully experienced if the questions studied are co-disciplinary in nature, suggesting more complex design, management and implementation. From this previous event it could be said that, although the question is contextualized in a non-mathematical environment, most of the SRCs suggested so far (Barquero, 2009; Fonseca, Pereira & Casas, 2011; Ruíz-Higueras & García, 2011; Parra, Otero & Fanaro, 2012a,b; Costa, Arlego & Otero, 2013; Donvito, Otero & Sureda, 2014), have resulted in mono-disciplinary ones, as in fact, only mathematics is studied. One of the objectives of this research has been to design, develop, implement and evaluate a genuinely co-disciplinary RSC, likely to call for the study of both physics and mathematics, and to describe its functioning in three different *didactic systems* (S). It is also an objective of the present work, broader in scope (Otero et al., 2013) the promotion and analysis of the feasibility of teaching through research in mathematics and physics in the compulsory education at secondary level (CSE) and at university level, as proposed by the ATD.

The starting point is the question Q_0: Why did the Movediza stone in Tandil fall down? Which, in order to be answered – in a provisional and unfinished way- needs the study of praxeology's which arise from Physics and Mathematics jointly. The RSC was developed in three different research groups and closely related: the researchers, two teaching training school groups whose teachers are also researchers, and two groups of students in the secondary level whose teachers are also researchers. The ATD is applied in order to describe in general terms the study procedure in each group, as framed within the Research and Questioning the World Pedagogy (Chevallard, 2001, 2012, 2013). As follows, in a non-exhaustive form, some ATD constructs used in the present work are underlined.

Theoretical Framework

The Research and Study Course require the establishment of a didactic system of the type $S(X;Y;Q)$, i.e., a system which studies and researches a question in the strong sense. The students (X) research and study a question Q following the guidelines from a teacher (y) or group of teachers (Y) with the purpose of providing an $R^{\blacktriangledown}$ answer to Q. The exponent in $R^{\blacktriangledown}$, shows that the answer to Q has been produced under certain conditions and that it is subjected to them, as there is not a universal answer or a universally effective one (Chevallard, 2009c) in order to produce $R^{\blacktriangledown}$, the S didactic system requires instruments, resources, works, i.e., it needs to generate an M didactic medium (Chevallard, 2009a,2009b, 2009c).

$$[S(X;\ Y;\ Q) \rightarrow M] \rightarrow R^{\blacktriangledown} \text{ (Herbartian scheme)}$$

The didactic system produces and organizes the M medium, with which it will produce an $R^{\blacktriangledown}$ answer. The medium M contains all the questions generated from Q and the existing answers accepted by the school culture, "already-made" answers referred to as $R_i^{\Diamond}$ for $i=1,\ldots,n$. A certain book, a teacher´s course, the internet, etc. can be included in the "already-made" answers category. O_j, entities with $j=n+1, \ldots, m$, being work pieces such as theories, experimental constructions, praxeology's, etc. are part of M as well, considered potentially useful in the production of $R^{\Diamond}$ answers obtaining from it something useful in order to generate $R^{\blacktriangledown}$. It is noteworthy that the construction of M is both the students and the teacher´s responsibility, being the latter taken as one information system in the class (media) being that *a media can never be taken their word*. In the ATD, M is written as follows:

$$M = \{R_1^{\Diamond}, R_2^{\Diamond}, R_3^{\Diamond}, \ldots, R_n^{\Diamond}, Q_{n+1}, \ldots, Q_m, O_{m+1}, \ldots, O_p\}$$

An RSC is formalized by means of the Herbatian Scheme I developed (Chevallard, 2007)

$$[S(X,Y,Q) \rightarrow \{R_1^{\diamond}, R_2^{\diamond}, R_3^{\diamond}, \dots, R_n^{\diamond}, Q_{n+1}, \dots, Q_m, O_{m+1}, \dots, O_p\}] \rightarrow R^{\heartsuit}$$

When the RSC is started, the teacher suggests a given question Q, whose study will lead to the encounter or reunion with several mathematical organizations and/or other disciplines, depending on whether it is a mono-disciplinary or co-disciplinary path. Thus, a chain of questions and answers is set as the core of the study process $P=(Q_i ; R_i)$ $1 \leq i \leq n$, being Q_i all the questions deriving from Q and R_i the respective answers to them (Chevallard, 2007).

In the a priori analysis stage, the specific and didactic knowledge which could be involved within an RSC is set up and the Praxeological Reference Model (PRM) is elaborated. The researchers analyze the likely set of questions which the study and the research into Q_0 might encompass; together with the Praxeological infrastructure, mathematics and physics in this case, necessary to answer those questions (Chevallard, 2013). Although in the case of the didactics researchers, the preliminary clarification, elaboration and permanent discussion of PRM are essential, in a more or less explicit way, the PRM is underlying the whole of the teacher, student and researcher´s activity, being always likely and desirable to identify and clarify it, emphasizing the dynamic nature of the PRM.

In the following sections a PRM constructed by the researcher group is developed and presented, in the light of which the actions and the results obtained in each path are interpreted. Then, the components of the developed Herbatian scheme are used (Chevallard, 2007): derived questions, useful ready-made answers to answer the questions, works chosen to be studied and the answer obtained, considering these in each of the three didactic systems: S_R (in which the researchers are students and directors of their own study process $X=Y$), S_{TT} (in which the students are mathematics teachers in training at the University and the researchers are the educators) and S_{CSE} (in which the students are of the compulsory secondary education and the researchers are the teachers as well).

Methodology

This is a qualitative and exploratory study whose aim is to carry out the teaching by research in the Compulsory Secondary Education (CSE) and the University. In this respect, we propose the design, implementation and analysis of a physics and mathematics co-disciplinary RSC, depending on the institution in which it is developed. Co-disciplinary means that in this case, physics does not only trigger the study of mathematics, but rather that both disciplines play a central role, being necessary to study both praxeology's as well. Although it is not the rule, in this case, the researcher and the directors of the study in the RSC, do not know at the beginning the Q_0

answer to the extent that they first have to study the question deeply for their own knowledge, which leads them to the development of a detailed and deep Praxeological analysis in order to produce a feasible answer and then, get into a Praxeological and didactic analysis which considers the various alternatives suitable for the institution where it is performed. Being this question genuinely co-disciplinary with physics and mathematics, the research team took on an RSC within which several discussions arose, until they agreed on a common answer. On the one hand, there was a need to provide support on physics to the colleagues, who were not specialized in the field, in order to build physical models which were nonexistent until now and their corresponding mathematical models. As researchers taking part in the study: physics and mathematics specialist, two mathematics specialists and a physics specialist. At the beginning, they experience the RSC in person being that they do not know the answer beforehand as there are no previous ready-made answers. The team personally experienced the drawbacks when constructing a physical model in order to describe the real situation, first producing an effective model and then, a more complex and phenomenological model, which provides a physics account of certain unjustified parameters displayed in the effective model. Moreover, taking into account the role of the teacher and the didactic, the researchers need the RSC and analyze the scopes and limitations of the proposed RSC, so as to implement it in the CSE and in University.

The RSC was implemented in the state university, in the city of Tandil, Argentina, in a discipline which is part of the didactic studies within the Mathematics Teaching Training Course, in which the researchers are also the teachers. N=12 and N=13 students from the last year (4^{th}), aged 21-33, from the Mathematics Teaching Training Course, took part in it. The lessons were located in the University Library during 10 weeks, with a total of 7 weekly hours, provided in two meetings. Six work groups were organized with approximately 4 members each. The texts used in the study were selected by the students, given the wide availability of books from the library and internet-based searches (in all cases, the students were asked to cite the material consulted in the work). The recordings were obtained at the end of each meeting, from the protocols written by the students, which were in turn digitalized and returned in the subsequent meeting. General audio recordings and field-notes were carried out as well.

Furthermore, two other RSC implementations were performed in two CSE courses in a state school of private administration in the city of Tandil, in the Buenos Aires province, Argentina, in the usual mathematics lessons, with the participation of N=68 students aged 16-17. During 7 weeks ten meetings were carried out with the students, providing a total of 30 hours. The students voluntarily took part in 17 groups of not more than 4 people each. As the classrooms were not equipped with internet access, or library, the texts needed for the study were supplied by the teachers-researchers. Each group of students had at least one netbook, allowing the use of software.

The Research And Study Path And The Praxeological Reference Model (PRM)

The Q_0 question is suggested: Why did the Movediza Stone in Tandil fall down? This enormous basalt stone has remained the city's landmark, providing it with a distinctive feature. Many local people and national celebrities such as presidents, writers, scientists, etc., visited the place to watch closely the natural monument. It was a 248-ton rock, sitting on the top of a 300- meter-high hill, which presented discontinuous oscillations when disturbed in a non-arbitrary spot, as shown in Figure 1. Unexpectedly, on February 28, 1912, the stone fell down the cliff and fractured into three pieces, filling the town with dismay at the loss of their symbol. For over 100 years the event produced all kinds of conjecture, legends, and unlikely scientific explanations for the causes of the fall. Within the groups where the RSC has been performed, there existed a certain curiosity and interest in finding a scientific answer to this question. Once in contact with the available information, the generatrix question evolves into Q'_0: What conjecture is there on the Movediza Stone and which is the most likely to be scientifically dealt with? Taking for granted that the fall can be explained by means of the Mechanical Resonance phenomenon, a number of questions arise which are linked to the physical and mathematical knowledge necessary to understand and answer $Q_{0.}$

Figure 1

Photography of the Movediza Stone (Photo File General of the Argentina Nation, available in: http://bibliocicop.blogspot.com.ar/2012/02/piedra-movediza-100-anos-de-su-caida.html)

This part of the investigation consists in the elaboration of proper physical and mathematical models of the motion in the real system, and obtaining in each model the set of parameters which are compatible with the resonance in the system, as caused by an external torque performed by one or two people, within an appropriate frequency range. In figure 2 a PRM scheme is presented and described in general terms as follows. Further details about physical and mathematical models, as well as the performed calculations and simulations, are beyond the scope of the present paper and will be published in a subsequent work.

If we consider that the real system is an oscillating system, the study can be done within the Mechanic Oscillations, starting from spring or pendulum models which are ideal at the beginning. In this case, non-friction systems are used, in which the only force in action is a restoring force depending in a linear way on the deviation respect to the equilibrium position, and which produces oscillating systems known as harmonic, calculated from the second-order linear differential equation, called by the same name. In the case of the pendulum, the restoring force can be considered depending on the oscillating angle (for small angles).

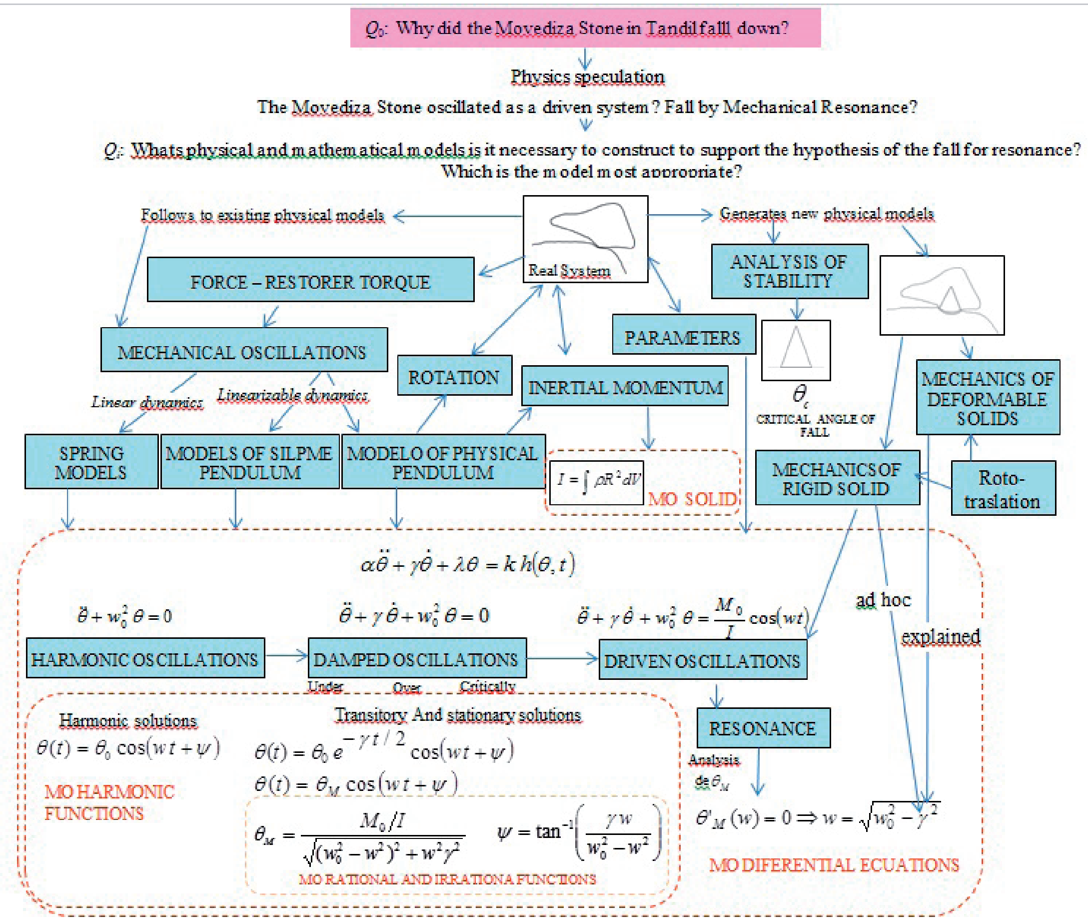
Progressively, the system becomes more complex. If friction-produced damping is considered, it provides a new term to the differential equation connected to the first derivative of the position (speed). Finally, it is possible to study systems that apart from being damped, they oscillate by means of an external force, therefore called driven systems. Whenever this given force is periodic and its frequency coincides with the natural (free of external forces) frequency of the oscillating system, a maximum in the oscillation amplitude is produced, generating the phenomenon known as mechanical Resonance.

There is a variety of texts on Physics which provide aid in the study of this phenomena and the associated mathematics, in all the different exhaustive levels. It is also possible to use applets and simulations for each kind of system, especially in the case of springs as pendulums.

On the other hand, the simple devise made by using a pencil and a thread tied around it, whose length can be varied, and from which an small object such as an eraser, or a marble is hanging, allows, by moving the pencil on the edge of the table, to find propioceptibly the proper frequency of the system that matches the movement of the suspension point, observing the abrupt and surprising growth in the oscillation amplitude, when both frequencies coincide, that is to say, the mechanical resonance phenomenon for strings which have different lengths.

By increasing the complexity of the model in relation to the epistemic value of the simplicity which is characteristic of Physics, it is possible to consider a suspended rotating body, instead of a specific mass. This leads to the study of the rotation phenomenon, the torque and the moment of inertia of the rotating body, in relation to its mass distribution.

FIGURE 2

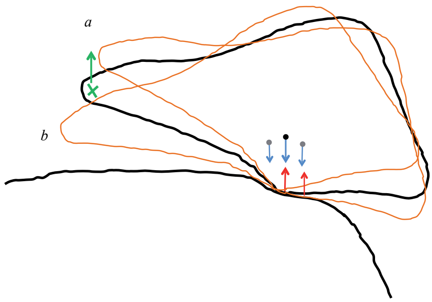


Praxeological Model of Reference (PMR) relative to Q0

Here again, the linear system is for short amplitude oscillations and the damped and driven cases can be also considered, corresponding to the same mathematical model, but in which the parameters have a different physical interpretation. However, as it refers to a suspended oscillating body, this is not a suitable physics physical model for the Movediza stone system.

On the right side of figure 2, the path to be followed is displayed in the case in which a real system is being considered. The morphology of the Movediza stone is well-known from the recent engineering works which have been performed in order to build a fixed replica (Peralta et al., 2008). In this case, an analysis of the stability of the system is carried out. In figure 3, an MS scheme is observed and the place where the system could be disturbed is indicated by a cross. The component of the exerted force parallel to the weight produces a clockwise torque taking the MP to an *a* position, generating a counterclockwise restoring torque resulting from the couple of the weight and the normal force. Therefore, the stone returns to the equilibrium point which is overpassed by the effect of the inertia, oscillating again by an anticlockwise restoring torque until the *b* position. This oscillating and slightly damping movement continues until it comes to a stop.

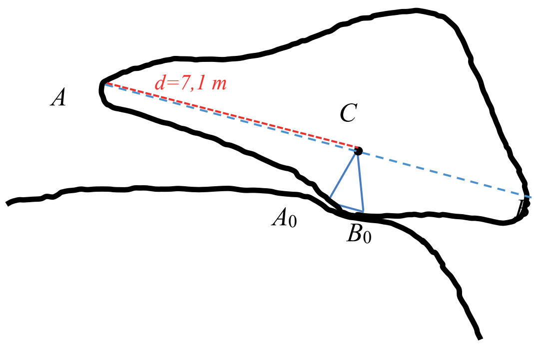
Figure 3



Position of equilibrium the MS and restored torques to the dislocations

In Figure 4, the A and B points are the extreme points of the longest sides of the Stone and C is the gravity centre. The A_0 and B_0 are the extremes of the base, which allow to draw the Isosceles triangle $\overset{\Delta}{A_0CB_0}$.

Figure 4



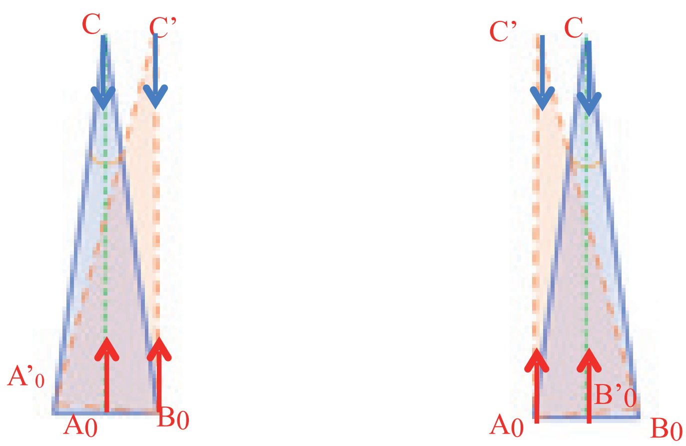
Location of the Movediza Stone (north sight)

The standing point of the MS is A_0B_0=0,4 m, and h=1,81 m is the distance from the gravity centre C to the middle point of the base C_0 . The angle is thus calculated $C_0\hat{C}B_0$

$$C_0\hat{C}B_0 = \tan^{-1}\left(\frac{0,2}{1,81}\right) \Rightarrow C_0\hat{C}B_0 \cong 6,305°$$

Figure 5 illustrates the critical position of the MS in dislocations on the right and left in the rotation plane.

FIGURE 5



Rotation of triangle A_0CB_0 depending on the dislocating of C towards C' and C"

That means that if the oscillating maximum amplitude is less than $6,305° \approx 0,11\ rad$, the MS does not fall down.

However, the base of the Stone was not flat, it is necessary to consider more precise models of the real situation. This leads to the Mechanical FO of rigid solids if we consider a model in which the MS base is curved and it lies on a flat surface, in such a case, it is the so-called roto-translation movement. Calculating the center of mass coordinates and taking into account the energy considerations, there results a mathematical differential equation similar to the one presented in Figure 2, if we consider a likely damping due to friction in the base and a driven system due to a periodical external torque. Clearly, the parameters will not be the same as previous model:

$$\frac{d^2\varphi}{dt^2} + \gamma\frac{d\varphi}{dt} + w_0^2\varphi = \frac{M_0}{I}\cos(wt) \qquad (1)$$

The answer to equation (1) is

$$\varphi(t) = \varphi_M \cos(wt - \psi)$$

Being the amplitude φ_M and the phase ψ

$$\varphi_M = \frac{M_0/I}{\sqrt{(w_0^2 - w^2)^2 + w^2\gamma^2}} \qquad \psi = tg^{-1}\left(\frac{\gamma w}{w_0^2 - w^2}\right)$$

Deriving from this, it is simple to obtain the maximum of φ_M which is found in $w = \sqrt{w_0^2 - \gamma^2}$

$$\frac{d^2\varphi}{dt^2} + \gamma\frac{d\varphi}{dt} + w_0^2\varphi = \frac{M_0}{I}\cos(wt) \qquad (1)$$

The parameters: M_0 (exerted external torque), I (inertia moment), w_0 (oscillation system frequency) and γ (friction coefficient), need to be estimated. Considering that the oscillation plane is the sheet of paper and that this was produced with respect to an axis that is perpendicular to that plane (axis z), the Steiner Theorem can be used to determine the moment of inertia with respect to a parallel axis to z, passing through the standing point or base.

Given that we know the distance $d = 7{,}1\ m$ where the external exerted force was placed to set the stone in motion, possible torques can be estimated and also the intensity of the total force needed, as well as the individual force, provided by three, four or five people (as given by the historical chronicles). By using these values, it is possible to study the behavior of the $\varphi_M\ (w)$ function for w_0 in a range of frequencies between 0,7 Hz and 1 Hz, historically recognized as oscillation frequencies in the MS system and calculate in each case the maximum amplitude $\varphi_M\ (w_m)$ that occurs for $w_m = \sqrt{w_0^2 - \gamma^2}$.

The Stone would fall if $\varphi_c \leq \varphi_M(w_m)$, being $\varphi_M(w_m) = \frac{M_0}{w_0 I\gamma}$ the maximum value of the amplitude function, that is to say $\varphi_c \leq \frac{M_0}{w_0 I\gamma}$.

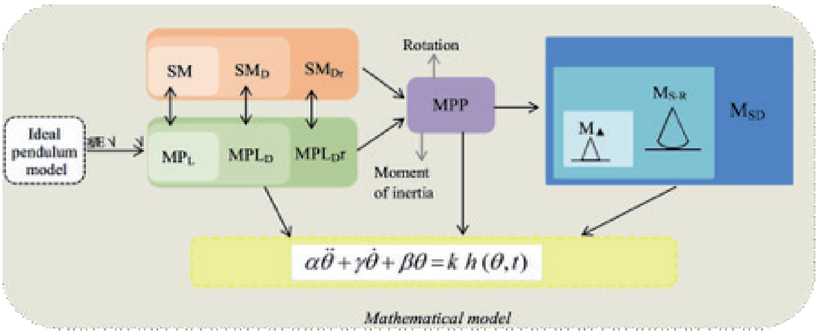
This model presents the problem that we do not know the values of γ which could be "acceptable". If we adopt "ad doc" for this parameter, a magnitude order $\gamma \geq 10^{-2}$, we obtain in the various situations considered with different torques and the interval frequencies previously mentioned, that all the scenarios support the overcoming of the critical angle, that is to say, predict the fall.

The relevance of the Praxeological analysis is highlighted hereby, which evidences the didactic potential of a spreadsheet to simulate the different possibilities and of software in order to show the $\varphi_M\ (w)$ family in a graph with all the different sets of parameters. Each S system will decide how deeply it studies the involved MOs and FOs, in correspondence to dialectics of the getting-out of and into the topic and the grey and clear boxes (Chevallard, 2013).

Later, in search of a more appropriate approximation of the physics model for the damping that is clearly not due to air, we can consider the stone as a deformable solid, where the contact in the support is not a point, but a finite extension. Therefore, the normal force is distributed on such a surface, being higher in the movement direction and generating a rolling resistance, manifested through a torque contrary to the motion due to friction. The rolling resistance can depend on the speed stone, giving a physical interpretation to the damping term. That is to say, the physics behind the damping is the same that makes a tire wheel rolling horizontally on the road come to a stop, but in the case of the stone, the deformities are various orders of magnitude smaller. This model needs of more parameters than the previous one although the predictions coincide, even in the damping values that derive from it, that in this case are no longer ad hoc, but referred to and obtained from the model.

In Figure 6, we present the PMR from the point of view of the modelization it entails. We schematized the successive extensions of the physical model and the mathematical model which gets adapted to all of them, considering the parameters specificity. This makes evident the role given by the ATD to the modelization in the RSCs. In this case, the intrinsic co-disciplinarity in the generatrix question entails a succession of physical models that can be described by the same mathematical model. As it has been shown before, the PMR allows anticipating the maximum scope intended for the RSC, although according to the didactic system arising in the different institutions, the necessary adjustments will be produced in order to preserve its ecologic viability. Likewise, it is important to mention that limiting the PMR, i.e. the Praxeological analysis, equals to limiting the didactic analysis. More specifically, it will be seen that this is one of the main problems that future teachers, and even more traditional teaching, have to face. Therefore, the monumentalization of knowledge and the fact that it is seen as transparent rather than problematic; presents an obstacle to the Praxeological analysis, deeply damaging the didactic analysis.

Figure 6



Analyze of the PMR from the point of view of the modelization

Description Of The Functioning Of Each Didactic System By Means Of The Herbatian Scheme

A general way to describe the RSCs is to consider the components in the Herbatian scheme for each didactic system. As follows, we discuss, on the basis of those components, the RSC experienced in each S. Although the starting point is the analysis and questioning the probable causes of the fall of the stone, finally, each system reformulates the question according to its possibilities (second row on the table) and produces an answer under certain conditions.

The Researchers System S_R

The distinctive feature of this system is that the study is directed towards the search of models which are progressively more sophisticated and closer to the real situation. Initially, there is some questioning and evidence is collected for the resonance conjecture, doubting the historical information by being considered inaccurate and qualitative, given that, although there was the acting of engineers, the dynamics of the system might have never been analyzed. This is translated into the questions that S_R formulates on the existing questions and which it doubts. Firstly, the basic considerations of stability were analyzed and the value of the critical angle of fall (Holmberg, 1912) was more accurately estimated, being approximately of 0,11 rad. In order to consider if the system effectively oscillated, and how it did it, models having different support bases were constructed, and probable restoring and exciting torques were estimated. As this group is responsible for the PMR described above, the Praxeological-didactic analysis went through the left side of Figure 6, given that the non-physics members of the group initially studied the resonance phenomenon without regarding the MS system, and in order to participate in the construction of the successive models provided in the right side of the figure. This allowed a better didactic anticipation for the implementation in the remaining two groups.

TABLE 1

Components of the Herbatian Scheme for S_R

	S_R(Y,Y,Q) X=Y: 4 researchers
	Q: What physical and mathematical models is it necessary to construct to support the hypothesis of the fall down by resonance? Which is the most appropriate model?
$R_1^{\Diamond}, \ldots, R_n^{\Diamond}$ Ready-made answers	The stone oscillated as a driven system. (Conjecture of Holmberg: overthrow accidental for "sum of impulses"). Movement of the stone (Analysis of stability of the equilibrium →critical angle, dynamic qualitative analysis). Morphology of the Stone of 1870, before the fall down. Morphology based on a reconstruction, for the installation of a replica fixes in 2007. Physics of oscillations. Damped and driven systems. Resonance. Physical pendulum. Moment of inertia, restoring and external torque. Rigid solid. Roto-traslation. Deformable solids. Physical models and mathematical models.
$Q_{n+1}, \ldots, Q_m$ Derivative questions	Is it possible that the fall down was a product of a resonant torque, carried out by 2 or more persons? How was it the morphology of the Stone? Which was its equilibrium position? Did it really oscillate? Had it a restoring torque? How was it? Which was the critical angle, or the maximum amplitude admissible? Which were the natural frequency oscillations of the system? How was its base of support? How can be modelled? How to construct progressively more sophisticated models of MS? Which are the mathematical models of the physical models mentioned? Which is the external torque? What characteristic has it? How many persons performed it? Where? How to estimate the moments of inertia and respect to what axes? What type of damping had the Stone? How to estimate it and to calculate it? How to find the unknown parameters? How to restrict them? How to teach Mathematics and Physics from Q_0? What would be the potential courses in the secondary school level and at the University? What physical and mathematical organizations would meet again at each level? What software is necessary to enter to M at every level?
$O_{m+1}, \ldots, O_p$ Work	$MO_{\text{Differential and integral calculus}}$ $MO_{\text{Differential equations}}$ $FO_{\text{Mechanical oscillations}}$ $FO_{\text{Mechanics of the rigid and deformable solid}}$
$R^{\heartsuit}$	Physical and Mathematical model of the oscillating system PM like a type of solid deformable in roto-traslation. Possible initializations of the Model allow supporting the hypothesis of the fall down for Resonance by means of an external appropriate torque, with parameters and real possible values.

Starting by the $R^{\lozenge}_{n}$ and the mathematical and physical knowledge mentioned in Table 1 and in the description of the PMR, an effective model of the system was elaborated, considering that the base is a portion of a solid in roto-translation and carrying out considerations on the energy. Within this model, several possibilities are calculated and analyzed, by means of the family of functions that the oscillation system amplitude function generates if the parameters vary in ranges that are compatible with the real situation. Previously, the external torques performed by two to five people were estimated and it was decided to adopt a range of possible natural frequencies that contemplate the value indicated in the historical chronicles. This allows the calculation of the damping coefficient of the system according to the parameters, but its origin is not provided. Then, a physical model was made for the damping considering the stone as a deformable solid, where the contact in the standing point is not a point but rather, a finite extension. Therefore, the normal force is distributed alongside this surface, being higher in the direction of the motion and generating a rolling resistance, which manifests itself by means of a torque contrary to the motion due to friction.

These successive models are not only an adaptation of the already-made answers, but rather they suppose a certain "creative leap" leading to a more sophisticated $R^{\heartsuit}$. This is not an ordinary event in the RSCs known so far and would be a consequence of the problematicity of Q_0. Once spread, the answer will become an "already-made" answer, non-existent until now. This recreates the actual work of any researcher facing a problem and expands the RSC dimension. Once the question is analyzed from the point of view of physics and mathematics, anticipations on the functioning of the RSC in the CSE and the University were made.

The teachers-in-training system S_{TT}

The RSC has been implemented in two courses. During the first implementation, in the S_{TT1} the mathematical activity aims at answering how and why the stone fell down the cliff. The TT busily searched for an "already-made" mathematical and physical model, which allows them to solve a differential equation in a specific way. For example, some TT expected that the model let them answer: what time did the stone fall down? This shows the difficulties they present in order to understand what a mathematical model can provide and what it cannot. Initially in both implementations, the organizations showing on the left side of Figure 2 and Figure 6 were studied, generating several questions, as the main preoccupation of the TT is the study of the FO oscillations, because it is a thoroughly new organization to them. In any of the two groups, the underlying mathematics seemed to present difficulties at the beginning, given that in parallel they are studying the Differential Equations course. The most important differences in the S_{TT} are summarized in Table 2. They are indicated in the table using a (+) for what is added in the S_{TT2} compared to S_{TT1}; and by means of a (-) for what is not applicable; i.e., only the differences are described.

TABLE 2

Components of the Herbatian Scheme for S_{TT}

	S_{TT1} (X_{TT1}, Y, Q), X_{TT1}: **students of 4th year of the mathematics teaching training course.**	S_{TT2} (X_{TT2}, Y, Q), X_{TT2}: **students of 4th year of the mathematics teaching training course.**
	Q: How did the stone fall down? Why?	
$R_1^{\Diamond}, \ldots, R_n^{\Diamond}$ Ready-made answers	Conjectures on the fall: Politics, Geologic, of Holmberg. Morphology based on a reconstruction, for installing a fixed replica in 2007. Oscillations. SHM. Spring and pendulum models. Damped and driven system. Resonance. Physical Pendulum. Rotation. Moment of inertia. Equation of motion. Resolution of the differential equation of the physical pendulum.	(+) Equilibrium conditions. Stability of the system. Verification of the differential equation solution of the physical pendulum. Roto-traslation (-) Resolution of the differential equation of the physical pendulum.
$Q_{n+1},\ldots,Q_m$ Derivative questions	What conjectures are there? And which is the most likely to be scientifically dealt with? What is an oscillation and what types are there? Which are the equations that describe the motion? Which are the solutions? Which of the known physical models is compatible with MS? What is the underlying mathematical model? (find the solutions of the equations) How was the morphology of the stone? Which is the moment of inertia? How to calculate the moment of inertia of an irregular solid? What regular solid is a good approximation of MS to calculate I? How to calculate at least one natural frequency? What initial conditions are there? How equation is solved for initial values of the stone? How to restrict the parameters of the model? What kind of damping had MS? What happens if the damping is not considered? Which is the driving torque? What information is obtained from the solution of the differential equation? What exact moment did the stone fall down?	(+) The stone was in equilibrium? What kind of equilibrium had it? What force was required to take out the equilibrium to the stone? Why the stone "jumped" to fall? What was the natural frequency? What was the critical angle of stability? (-) What happens if the damping is not considered? How to calculate the moment of inertia of an irregular solid? What regular solid is a good approximation of MS to calculate I?

Components of the Herbatian Scheme for S_{TT}

	S_{TT1} (X_{TT1}, Y, Q), X_{TT1}: students of 4th year of the mathematics teaching training course.	**S_{TT2} (X_{TT2}, Y, Q), X_{TT2}: students of 4th year of the mathematics teaching training course.**
	Q: How did the stone fall down? Why?	
$O_{m+1},\ldots,O_p$ Work	MO Vectors MO Regular solids MO differential and integral calculus FO Oscillations FO Equilibrium conditions, torque, moment of inertia	(-) MO Regular solids (+) FO Rigid solid
R♥	The Stone fell down by resonance. Mathematical Model. "Solutions of the differential equations have too many parameters and this makes impossible to calculate "the exact moment when the stone fell down".	The Stone fell down by resonance. Mathematical Model: • Analysis, calculation and estimation of the parameters of Differential Equations. • Calculating the critical angle of oscillation • Agreed answer with the problem of the fall down of the Stone

In both implementations, the lessons were delivered in the University library, which seems to maximize the path, given that the TT analyzed and discussed in situ which physics and mathematics books would be the most suitable for the study, and at the same time, searched information on the internet about the MS and about the physics of the problem. In the S_{TT1} the study was dominated by the need to find a physics model suitable for the situation. In both implementations, the teachers-in-training decided on the pendulum model initially, whose mathematical model is similar to the one the problem might represent, although physically inadequate. However, the path performed in the first implementation was different from the second. In the first case, the TTs do not question the physics model, and provide plenty of time to the study of the moments of inertia and how to calculate them if we consider a regular or amorphous solid, which would result in an appropriate model for the irregular shape of the stone. A number of questions were therefore generated, for which the answers were provided. The students who had understood the resonance phenomenon for springs, ideal pendulums and physical pendulums, calculated the characteristic frequency of the system, making use of the moment of inertia previously obtained. Thus, only a parameter resulted undetermined: the damping. The physical pendulum model becomes an obstacle, given that the damping they considered was due to air-friction, which is not applicable to the stone as it is a supported body, not hanging.

At this point, the torque (concept about which there were different trials to analyze and estimate it) and the solution of the equation remained unstudied. Until this moment, the solution for the differential equation did not seem present any obstacles. They considered they were facing an initial value problem. Once they had obtained the parameters, which they considered fixed the solution seemed simple. However, they had problems in order to arrive at a final solution, even though this can be found in textbooks of Physics (though without its deduction), that is why it was discarded and they decided to do the calculation on their own. This event complicated the quantification they aimed at obtaining as the physical interpretation. Some groups in this cohort removed the term of damping, as a way to lower degrees of freedom; thus, the stone would have been in perpetual motion. This did not create any contradiction to them. Other ones adopted a damping value due to air-friction, which also led to wrong results.

In summary, instead of adopting and adapting the solution that was presented in books of physics, the TTs in the cohort dismissed it and did not interpret the answer in the texts concerning the Stone. The posteriori analysis shows that the TTs do not understand the usefulness of the mathematical model, neither the role the parameters could play, which were considered as fixed and universal. In consequence, they failed to establish different sets of parameters and did not generate the feasible families of functions and values, whose compatibility with the physical situation could have been analyzed. Instead, they unnecessarily complicated the solution of the differential equation and had to draw upon the teachers of the differential calculus discipline, obtaining as a result the notion that it was "a particular case" and "beyond the syllabus". They also drew upon some physicists from the institution, and got hints indicating that the solution was not immediate and needed of a long time to be found. Considering that the context of the study is a faculty of Sciences -institution provided with an important team of researchers on physics and mathematics- the strangeness and discomfort produced by the questioning on the part of some of the students, is an indicator of the magnitude of the drawbacks that the teaching by questioning has to face in the University, as it is opposed to the mainstream pedagogy in this institution.

In the second cohort, the research directors had already perceived that the fundamental problem seemed to be in the models and in the modelization, thus devoting 8 sessions to the development of two intra-mathematical RSCs (Chappaz & Michon, 2003; Ruíz, Bosch & Gascón, 2007), that the TTs could experience in their own flesh, therefore emphasizing the role of the modelization and the use of such devices as spreadsheets and plotters. Besides, within our RSC, the research directors acted in different ways, as soon as the pendulum and spring models aroused which developed as described in Figure 6. One group studied the AMS for the simple pendulum, the spring, and the physical pendulum, another group studied the spring model in all its possibilities and the third one did not develop further than the AMS in simple pendulum and spring. The time of summarizing corresponding to that class allowed the production of a complete answer for the three models and its possibilities, from which the TTs from S_{TT2} arrived at the conclusion that the same mathematical model represented (9) nine different physical systems. A big amount of time was devoted to pondering on the differences and similarities between the mathematical and physical models and their connection with the real system we aimed at modeling. Then, the answers to the equations presented in the books were checked out.

In both cohorts, as a fixed route that is inevitably set by the books, the TTs came across the FO of the physical pendulum. However, in the second cohort some students presented strong objections to the possibility of using it in the case of the stone, as it is not likely to refer to a supported body as if it was an "inverted" physical pendulum. This drew the discussion once more towards the real system and the standing point, so that the path went through the models which refer specifically to the system and that are not in books. Firstly, the equilibrium was analyzed and the critical angle was calculated, and then, the model of the base of the MS was sophisticated. For the study of the rigid and deformable solid physical model, two little texts were introduced into the didactic medium. There was also a development in the calculation and estimation of the parameters of the DE solution, and in the elaboration of an answer that allows the explanation, by means of a model, of the characteristics of the fall.

The Secondary School system S_{CSE}

In the secondary school, firstly there was an attempt to understand the guesses on the fall by resonance and the connections with mathematics. Initially, the study referred to the oscillations and the systems of the pendulum and spring type, the AMS and the harmonic functions that describe it, being all this unknown to the students.

The study of the harmonic functions meant a quite prolonged "getting out of the subject" and the later getting in. The Geogebra software was used to analyze the families of functions and to represent and interpret the motion equations of concrete oscillating systems. Then, they studied the damped and driven systems of the previously mentioned models in order to establish analogies with the system of interest. The study of damped systems, both driven and resonant and their differences was performed from energetic and vector-related considerations, using representations Energy time and of the modules of the position, speed, acceleration and force vectors in the different sections of the motion. Considering the institutional restrictions, such as the unavailability of internet access and of student-friendly books, texts elaborated by the teachers were introduced into the medium M. Some simple but amazing experiments were also performed in the classroom, which enabled the propioceptive experiencing of the resonance phenomenon and its connection with the available information about the fall of the Movediza Stone.

TABLE 3

Components of the Herbatian Scheme for S_{SCE}

$S(X_{CSE}, Y, Q)$, X_{CSE}: math students of 5th year at the secondary school Y: teachers - researchers Q: Which is the real cause of the fall of the stone?	
$R_1^{\Diamond}, \ldots, R_n^{\Diamond}$ Ready-made answers	Conjectures on the fall: Politics, Geologic, of Holmberg. Oscillations. SHM. Harmonic solutions. Damped and driven systems. Resonance. Experimentation y software
$Q_{n+1}, \ldots, Q_m$ Derivative questions	What conjectures is there? Which are more viable? What is an oscillating system? Which characteristic has the Simple Harmonic Movement? (SHM)? Which equations of motion are known? How they are interpreted and how are graphed? What and which characteristics have the harmonic functions? What are the damped and driven oscillations? How does the energy change in time in the different oscillating systems? What is the mechanic resonance? How would it apply to MS?
$O_{m+1}, \ldots, O_p$ Work	$MO_{\text{Harmonic Functions}}$ MO_{Vectors} $MO_{\text{Exponential Functions}}$ $FO_{\text{Oscillations}}$
$R^{\heartsuit}$	*"The Movediza stone fell down because it entered in resonance. For this it was only necessary that two or more people push in the right place with the right frequency y and thus the energy of the system and the amplitude increased so much that provoked the fall down."*

In S_{CSE}, the answer that most satisfied the students was the understanding of the fact that the stone oscillated whenever it was disturbed in an appropriate way by the people (not spontaneously and constantly as they had assumed at first), and that, if such a disturbance met certain periodicity and placement requirements *"in the right place, with the right frequency"*, depending on the system, the oscillation amplitude could have been increased to the point of causing the fall.

Discussion

In the present study we have described in general terms, by making use of the components of an SRC alone, the distinctive characteristics that such a device displays when it is performed in three different didactic systems, from a genuinely co-disciplinary question. Moreover, we have provided a detailed description of the Praxeological Reference Model and its wide scope and didactic implications. Due to space constraints, an exhaustive development of the results in each system has been avoided, which will be presented in a separate work.

Besides the several limitations arising in both the Secondary School and the University, the groups experienced a teaching by research, within its means. Also, they studied physics and mathematics thoroughly and showed a good disposition to deal with questions they had never considered before. This displayed disposition on the part of the research group was present in the secondary school as well. There is a visible initial reluctant attitude on the part of the trainee teachers: Why should physics be studied if we are teachers of mathematics? Within this system, it was gradually understood that the idea was to experience a genuinely co-disciplinary RSC, in order to analyze it and comprehend the meaning of the teaching by research by means of an RSC. Even though the TT has studied the ATD and other didactic theories, they do it in a traditional way comparable to the traditional training they get.

The different approaches concerning the generatrix question in each S have been described. The most evident event, in our opinion, is the ability to generate and use both physics and mathematical models. It was not expected that the TTs developed the models which were constructed in I, but that having studied physics in that way and providing that their underlying closeness to mathematics played a positive role, they used the mathematical answers presented in physics books in a pertinent and exoteric manner. This fact did not occur in the first group and improved in the second one from the decision to make a previous incursion into mono-disciplinary RSC particularly suitable for evidencing the role of the modelization.

The TT's attitude is interpreted from the fact that although they have experienced four years of "hard" university studies, the functionality of the science they aim at teaching had never been visible. This would be, in our view, the most relevant drawback so as to let the TTs at least understand what the teaching by research seeks to do, no matter what didactic theory is being used. However, it is important to notice that the sporadic incursions in the modelization activity do not seem enough to allow the TTs develop such school practices. Neither seems likely that the teachers-to-be nor the already-graduated teachers, can develop complex didactic devices like the RSC individually. This would be an equivalent to asking a doctor to "make" his own medicine. That is to say, that the didactic technology needs to be developed further still, so as to provide the teaching professional with devices and means to perform trials, adjustments and efficient changes which can constitute the framework of their daily professional practices.

In the secondary school, the modelization is also alien to the mainstream pedagogy. Nevertheless, the CSE environment seems to be more open to the study of new questions, and even to the incursion into modelization activities. However, in this first implementation, we were halfway. Taking into account the models of the physics system we have constructed, it is necessary to enlarge our path, including in between a set of information and experiences that allow the students at that level to develop some modelization activities. These are aspects that are being improved concerning the device development from the experimentation.

Finally, it is important to question: What is the reason why both TTs and students in the CSE got into a genuinely co-disciplinary study? It has been considered that it could be due to the co-disciplinary formation of the research team, which alongside directs the teaching practice. Without this gathering of specialists in physics and mathematics, the co-disciplinary would not have been possible; neither the question would have developed likewise. It can also be mentioned the freedom and flexibility in the use of spaces, time, books, computing resources, provided by the University, not always available in the secondary school. As far as the teachers of CSE cannot work in teams this type of endeavors do not seem likely.

Nevertheless, even when arising from contextualized questions in a discipline (but not from problems in itself) which needs of mathematics, it is possible to set up an RSC, in which in the end only mathematical questions are studied and researched, with the objective of bringing about, whenever it is likely, a teaching practice based on questions in both the University and the secondary school. It is unlikely that a teacher whose training has been answers-based can teach by means of questions. The questioning of this pedagogical model in all the teaching institutions results as urgent as it is important, without falling into the illusion that that change will be sufficient.

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