Schwinger terms in Weyl-invariant and diffeomorphism-invariant 2-d scalar field theory*

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We compute the Schwinger terms in the energy-momentum tensor commutator algebra from the anomalies present in Weyl-invariant and diffeomorphism-invariant effective actions for two dimensional massless scalar fields in a gravitational background. We find that the Schwinger terms are not sensitive to the regularization procedure and that they are independent of the background metric.

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I. INTRODUCTION

The theory of a (quantized) scalar field coupled to gravity has to follow an ad-hoc prescription: the functional integration over the scalar field ϕ involves the evaluation of a determinant of the Laplace operator, which is ambiguous. For massless scalar fields in two-dimensional space-time the standard prescription implements a diffeomorphism invariant regularization that leads to the well known Polyakov action [1] $\Gamma^{\mathrm{P}}[g^{\mu\nu}]$, a functional of the background metric $g_{\mu\nu}$ that is indeed diffeomorphism invariant but has an (equally well known) anomaly with respect to Weyl transformations.

Recently an alternative evaluation of the theory has been given, where a Weyl invariant regularization has been implemented [2–5]. The resulting effective action $\hat{\Gamma}[g^{\mu\nu}]$, while being Weyl invariant, does not remain invariant under general coordinate transformations, but only under those with unit Jacobian.

Gravitational and Weyl anomalies lead to anomalous contributions to the equal-time commutators of the energy-momentum tensor [6,7] (see also [8] for the analogous fact in current algebra). So the question arises whether these two versions of the theory lead to the same anomalous commutators. In this paper we investigate this question and find that, indeed, the anomalous commutators coincide in both versions of the theory and lead to the well known result from Conformal Field Theory [9]. We do this calculation both for flat and curved space-time. In the latter case of general metric the computation is done without any gauge fixing; this is the proper procedure because gauge fixing would be in conflict with the Weyl-invariant regularization, that breaks diffeomorphism invariance. The results, when properly interpreted, lead to the same Schwinger terms as in the flat space-time and, therefore, show that the Schwinger terms do not depend on the curvature.

II. DIFFEOMORPHISM-INVARIANT AND WEYL-INVARIANT REGULARIZATIONS

First we have to fix our conventions. We use the flat Minkowskian metric η_{ab} with signature $(+,-)$. The metric $g_{\mu\nu}(x)$ is related to the zweibein via

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$$g_{\mu\nu}(x) = \eta_{ab} e^a_\mu(x) e^b_\nu(x); \tag{1}$$

we also need the zweibein determinant

$$e(x) := \mathrm{det} e^a_\mu(x) = \sqrt{|\mathrm{det} g_{\mu\nu}(x)|} \tag{2}$$

and the inverse zweibein $E^\mu_a(x)$,

$$E^\mu_a(x) := \eta_{ab} g^{\mu\nu}(x) e^b_\nu(x). \tag{3}$$

For the curvature we use the sign convention $R_{\mu\nu} = -\partial_\alpha \Gamma^\alpha_{\mu\nu} + \ldots$, where $R_{\mu\nu}$ is the Ricci tensor and $\Gamma^\alpha_{\mu\nu}$ is the Christoffel connection.

Weyl transformations act like

$$g_{\mu\nu}(x) \to \exp(2\sigma(x)) g_{\mu\nu}(x), \qquad e^a_\mu(x) \to \exp(\sigma(x)) e^a_\mu(x). \tag{4}$$

When the effective action Γ is not invariant under Weyl transformations, an infinitesimal change $\delta^{\mathrm{W}}_\sigma g_{\mu\nu}(x) = 2\sigma(x) g_{\mu\nu}(x)$ induces a Weyl anomaly $G^{\mathrm{W}}(x)$:

$$\delta^{\mathrm{W}}_\sigma \Gamma := \int d^2x \sigma(x) G^{\mathrm{W}}(x), \tag{5}$$

$$G^{\mathrm{W}}(x) = -2 g^{\mu\nu}(x) \frac{\delta\Gamma}{\delta g^{\mu\nu}(x)} = -e(x) g^{\mu\nu}(x) T_{\mu\nu}(x), \tag{6}$$

where $T_{\mu\nu}$ is the v.e.v. of the energy momentum tensor $\Theta_{\mu\nu}$,

$$T_{\mu\nu}(x) = \langle \Theta_{\mu\nu}(x) \rangle = \frac{2}{e(x)} \frac{\delta\Gamma}{\delta g^{\mu\nu}(x)}. \tag{7}$$

Under an infinitesimal coordinate transformation (diffeomorphism) $\delta^{\mathrm{D}}_\xi x^\mu = -\xi^\mu(x)$ the metric and zweibein transform like

$$\delta^{\mathrm{D}}_\xi g^{\mu\nu}(x) = -D^\mu \xi^\nu(x) - D^\nu \xi^\mu(x), \qquad \delta^{\mathrm{D}}_\xi e^a_\mu(x) = \xi^\lambda \partial_\lambda e^a_\mu(x) + e^a_\lambda(x) \partial_\mu \xi^\lambda \tag{8}$$

and a diffeomorphism anomaly is given as

$$\delta^{\mathrm{D}}_\xi \Gamma := \int d^2x \xi^\nu(x) G^{\mathrm{D}}_\nu(x), \tag{9}$$

$$G^{\mathrm{D}}_\nu(x) = 2e(x) D^\mu \left(\frac{1}{e(x)} \frac{\delta\Gamma}{\delta g^{\mu\nu}(x)} \right) = e(x) D^\mu T_{\mu\nu}(x). \tag{10}$$

It will be convenient later on to use covariant derivatives acting on the combination $eT_{\mu\nu}$, using the rule $eD_\alpha = (D_\alpha - \Gamma^\lambda_{\alpha\lambda})e$. Thus we rewrite G^{D}_ν as

$$G^{\mathrm{D}}_\nu(x) = (D^\mu - g^{\mu\rho}\Gamma^\lambda_{\rho\lambda})(e(x) T_{\mu\nu}(x)). \tag{11}$$

Further we will frequently use the following variational formulae,

$$\frac{\delta g^{\mu\nu}(x)}{\delta e^a_\alpha(y)} = -\eta_{ac} e^c_\lambda(x) (g^{\mu\alpha}(x) g^{\nu\lambda}(x) + g^{\nu\alpha}(x) g^{\mu\lambda}(x)) \delta^{(2)}(x-y), \tag{12}$$

$$\frac{\delta e(x)}{\delta g^{\mu\nu}(y)} = -\frac{1}{2} e(x) g_{\mu\nu}(x) \delta^{(2)}(x-y), \tag{13}$$

$$\frac{\delta R(x)}{\delta g^{\mu\nu}(y)} = [R_{\mu\nu}(x) + (D_\mu D_\nu - g_{\mu\nu}\Box)_{(x)}] \delta^{(2)}(x-y), \tag{14}$$

where R is the curvature scalar and $R_{\mu\nu}$ is the Ricci tensor.

The classical action of the theory reads

$$S = \int d^2x \frac{e(x)}{2} g^{\mu\nu}(x)\partial_\mu\phi(x)\partial_\nu\phi(x). \tag{15}$$

When a diffeomorphism invariant path integration with respect to ϕ is chosen, one obtains the Polyakov effective action [1]

$$\Gamma^{\mathrm{P}}[g^{\mu\nu}] = -\frac{1}{96\pi}\int d^2xd^2y e(x)R(x)\Box^{-1}(x,y)e(y)R(y), \tag{16}$$

where $\Box^{-1}(x,y)$ is the scalar symmetric Green function of the covariant Laplacian (satisfying $\Box_{(x)}\Box^{-1}(x,y) = e^{-1}(x)\delta^{(2)}(x-y)$). Γ^{P} is diffeomorphism invariant,

$$G^{\mathrm{D}}_\nu(x) = 0, \tag{17}$$

and posseses the well known Weyl anomaly (for a comprehensive review, see for instance [10] and references therein),

$$G^{\mathrm{W}}(x) = -\frac{1}{24\pi}e(x)R(x). \tag{18}$$

The alternative, Weyl invariant evaluation that was discussed in [2–5] relies on the observation that the classical action (15) depends only on the Weyl invariant quantity $\gamma^{\mu\nu}$, where

$$\gamma^{\mu\nu}(x) = e(x)g^{\mu\nu}(x), \qquad \gamma_{\mu\nu}(x) = \frac{1}{e(x)}g_{\mu\nu}(x). \tag{19}$$

As the breaking of the classical Weyl invariance in Polyakov's path integration may be traced back to a diffeomorphism-invariant and Weyl non-invariant normalization for the path integral measure,

$$\int \mathcal{D}\phi \exp(i\int d^2x e(x)\phi^2(x)) = 1, \tag{20}$$

the Weyl invariant evaluation can be achieved by choosing instead

$$\int \mathcal{D}\phi \exp(i\int d^2x \phi^2(x)) = 1. \tag{21}$$

This leads to a Weyl-invariant effective action $\hat\Gamma[g^{\mu\nu}]$ which depends on $g^{\mu\nu}(x)$ only through the combination $\gamma^{\mu\nu}$. By construction the two effective actions Γ^{P} and $\hat\Gamma$ coincide for metrics with unit determinant, therefore

$$\hat\Gamma[g^{\mu\nu}] \equiv \Gamma^{\mathrm{P}}[\gamma^{\mu\nu}] = -\frac{1}{96\pi}\int d^2xd^2y\hat R(x)\Box^{-1}(x,y)\hat R(y), \tag{22}$$

where $\hat R(x)$ is the curvature scalar evaluated from $\gamma^{\mu\nu}$ (notice that $\hat R(x)$ is not a true scalar).

$\hat\Gamma$ is Weyl-invariant, but it acquires an anomaly under coordinate transformations with Jacobian not equal to unity. This anomaly may actually be easily computed from the Weyl anomaly of the Polyakov action. The v.e.v. of the energy-momentum tensor computed from $\hat\Gamma$ is

$$\begin{aligned}\hat T_{\mu\nu}(x) &= \frac{2}{e(x)}\frac{\delta\hat\Gamma}{\delta g^{\mu\nu}(x)} = 2\frac{\delta\hat\Gamma}{\delta\gamma^{\mu\nu}(x)} - \gamma_{\mu\nu}\gamma^{\alpha\beta}\frac{\delta\hat\Gamma}{\delta\gamma^{\alpha\beta}(x)}\\ &= T^{\mathrm{P}}_{\mu\nu}(\gamma) - \frac{1}{2}\gamma_{\mu\nu}\gamma^{\alpha\beta}T^{\mathrm{P}}_{\alpha\beta}(\gamma).\end{aligned} \tag{23}$$

Here $T^{\mathrm{P}}_{\mu\nu}(\gamma)$ is the energy-momentum tensor $T^{\mathrm{P}}_{\mu\nu}$, as computed from the Polyakov action, evaluated at $g^{\mu\nu} = \gamma^{\mu\nu}$. Obviously, there is no Weyl anomaly, $g^{\mu\nu}\hat T_{\mu\nu} = 0$.

In order to evaluate the diffeomorphism anomaly we need the identity $D_\mu(g^{\mu\nu}\hat T_{\nu\alpha}) = \frac{1}{e}\hat D_\mu(\gamma^{\mu\nu}\hat T_{\nu\alpha})$, which may be easily proven by using the tracelessness and symmetry of $\hat T_{\mu\nu}$ (here $\hat D_\mu$ is the covariant derivative for the metric $\gamma^{\mu\nu}$). We then find for the diffeomorphism anomaly

$$
\begin{aligned}
\hat{G}^{\mathrm{D}}_\alpha &= eD_\mu(g^{\mu\nu}\hat{T}_{\nu\alpha}) \\
&= \hat{D}_\mu\left(\gamma^{\mu\nu}T^{\mathrm{P}}_{\nu\alpha}(\gamma) - \frac{1}{2}\gamma^{\mu\nu}\gamma_{\nu\alpha}\gamma^{\beta\delta}T^{\mathrm{P}}_{\beta\delta}(\gamma)\right) \\
&= -\frac{1}{2}\hat{D}_\alpha(\gamma^{\beta\delta}T^{\mathrm{P}}_{\beta\delta}(\gamma)) \\
&= -\frac{1}{48\pi}\partial_\alpha\hat{R}.
\end{aligned} \tag{24}
$$

Here we have used the vanishing of the diffeomorphism anomaly for Γ^{P} and the fact that $\hat{D}_\alpha$ reduces to the ordinary derivative on scalars. The anomaly is a pure divergence because only the symmetry with respect to transformations with non-unit Jacobian is broken (see [3]).

III. SCHWINGER TERMS

In this section we want to relate the anomalies of the previous section to the equal-time commutators (ETCs) of the energy-momentum tensor, both in flat and curved space-time. Here we will follow a method that was developed in [11] and used there for the calculation of ETCs in the flat space-time limit. We want to find the Schwinger terms in the general case of a non flat space-time, too, which makes things slightly more complicated. We choose the hypersurface $x^0 = 0$ as a quantization surface. For ETCs we write

$$
\delta(x^0 - y^0)[e(x)\Theta^\mu_a(x), e(y)\Theta^\nu_b(y)] = \Theta^{\mu\nu}_{ab}(x,y) + S^{\mu\nu}_{ab}(x,y), \tag{25}
$$

where we have used the zweibein formalism in order to conform with [11] (i.e. μ, ν are space-time indices whereas a, b are Lorentz indices). In eq. (25) $\Theta^{\mu\nu}_{ab}$ is the canonical part, depending again on the regularized energy-momentum operators $\Theta^\mu_a(x)$, whereas $S^{\mu\nu}_{ab}$ are c-numbers (the Schwinger terms). In the flat case regularization means just normal ordering, and therefore the v.e.v. of eq. (25) arises only from $S^{\mu\nu}_{ab}$ in the r.h.s. In the general case this is no longer true [12] but our knowledge of the flat case will still enable us to identify the individual pieces.

In the flat case it is well known that the canonical part is proportional to the first spatial derivative of the delta function, e.g. $\Theta^{00}_{01}(x,y) \sim i(\Theta^0_0(x) + \Theta^0_0(y))\delta(x^0 - y^0)\delta'(x^1 - y^1)$, whereas the Schwinger term is proportional to a triple spatial derivative, $S^{00}_{01}(x,y) \sim c\delta(x^0 - y^0)\delta'''(x^1 - y^1)$ (c is a constant).

In the general case both the expression for the classical energy-momentum tensor (see (15)) and the regularization will introduce a dependence on the metric and its derivatives in eq. (25). However, we will assume that the number of derivatives on the delta function remains unchanged, i.e. we will continue to identify the δ''' piece of the v.e.v. of eq. (25) with the Schwinger term. By treating the deviation from the flat space-time action (15) as interaction, $S_{\mathrm{I}} = S[g^{\mu\nu}] - S[\eta^{\mu\nu}]$, $\Gamma = -i\ln <0|T^*\exp iS_{\mathrm{I}}|0> = -i\ln\mathcal{Z} = -i\ln <out|in>$ we find for the two point function

$$
\begin{aligned}
-i\frac{\delta^2\Gamma}{\delta e^a_\mu(x)e^b_\nu(y)} &= <out|T^*(e(x)\Theta^\mu_a(x)e(y)\Theta^\nu_b(y))|in> \\
&\quad - <out|e(x)\Theta^\mu_a(x)|in><out|e(y)\Theta^\nu_b(y)|in> \\
&\quad + <out|\frac{1}{i}\frac{\delta((e(x)\Theta^\mu_a(x))}{\delta e^b_\nu(y)}|in> \\
&:= T^{\mu\nu}_{ab}(x,y) + \Omega^{\mu\nu}_{ab}(x,y),
\end{aligned} \tag{26}
$$

where $T^{\mu\nu}_{ab}(x,y)$ is the connected, time-ordered two-point function

$$
\begin{aligned}
T^{\mu\nu}_{ab}(x,y) = &<out|T(e(x)\Theta^\mu_a(x)e(y)\Theta^\nu_b(y))|in> \\
&- <out|e(x)\Theta^\mu_a(x)|in><out|e(y)\Theta^\nu_b(y)|in>
\end{aligned} \tag{27}
$$

and $\Omega^{\mu\nu}_{ab}$ contains the remaining pieces and is local (i.e. proportional to $\delta(x-y)$ and derivatives thereof).

Now we want to relate this two-point function to functional derivatives of the anomalies in eqs. (6,10). Defining these functional derivatives as

$$
I^\alpha_{ab}(x,y) := -iE^\mu_a\frac{\delta G^{\mathrm{D}}_\mu(x)}{\delta e^b_\alpha(y)}, \tag{28}
$$

$$
\Pi^\alpha_b(x,y) := -i\frac{\delta G^{\mathrm{W}}(x)}{\delta e^b_\alpha(y)}, \tag{29}
$$

we find the relations

$$
\begin{aligned}
-I^{\alpha}_{ab}(x,y)+A^{\alpha}_{ab}(x,y) &= (D_\rho-\Gamma^{\lambda}_{\rho\lambda})_{(x)}(T^{\rho\alpha}_{ab}(x,y)+\Omega^{\rho\alpha}_{ab}(x,y)) \\
&= S^{0\alpha}_{ab}(x,y)+(D_\rho-\Gamma^{\lambda}_{\rho\lambda})_{(x)}\Omega^{\rho\alpha}_{ab}(x,y)
\end{aligned}
\tag{30}
$$

and

$$
\Pi^{\alpha}_{b}(x,y)+B^{\alpha}_{b}(x,y)=e^{a}_{\mu}(x)\Omega^{\mu\alpha}_{ab}(x,y). \tag{31}
$$

Here $A^{\alpha}_{ab}(x,y)$ and $B^{\alpha}_{b}(x,y)$ stem from variations of the anomalies (6,10) that do not vary the one-point function $e(x)T_{\mu\nu}(x)$ (e.g. $B^{\alpha}_{b}(x)(x,y)=-\left(\frac{\delta g^{\mu\nu}(x)}{\delta e^{b}_{\alpha}(y)}\right)e(x)T_{\mu\nu}(x)$). They produce δ functions and first derivatives thereof and vanish in the flat limit. They are unimportant in the sequel. Further, we have assumed in eqs. (30,31) that the anomalies of the Heisenberg operators Θ^{a}_{μ} are themselves c-numbers. Under this assumption the anomalies do not contribute to the connected two-point function, e.g. $< T\left((D^{\mu}\Theta_{\mu\nu}(x))\Theta_{\alpha\beta}(y)\right) >_c = 0$. (Here we slightly differ in the conventions from [11]. They treat the operator $\Theta^{\mu}_{a}(x)$ as an interaction picture operator and, therefore, obtain additional commutators $[\Theta^{0}_{a}(x), L_{\mathrm{I}}(x^0)]$ in their relations.)

As we use the zweibein formalism, we need the corresponding equation for the Lorentz anomaly, even though the latter vanishes in both regularizations of our theory. Under infinitesimal Lorentz transformations the zweibein changes as

$$
\delta^{\mathrm{L}}_{\alpha}e^{a}_{\mu}=-\alpha^{a}_{b}e^{b}_{\mu}, \tag{32}
$$

inducing a variation of the effective action

$$
\delta^{\mathrm{L}}_{\alpha}\Gamma := \int d^2x\,\alpha^{ab}G^{\mathrm{L}}_{ab}(x) \tag{33}
$$

where

$$
G^{\mathrm{L}}_{ab}(x)=-\frac{1}{2}(\eta_{ac}e^{c}_{\mu}\frac{\delta}{\delta e^{b}_{\mu}}-\eta_{bc}e^{c}_{\mu}\frac{\delta}{\delta e^{a}_{\mu}})\Gamma. \tag{34}
$$

Then, defining

$$
L^{\alpha}_{cab}(x,y):=-i\frac{\delta G^{\mathrm{L}}_{ab}(x)}{\delta e^{c}_{\alpha}(y)} \tag{35}
$$

we find a further set of equations

$$
L^{\alpha}_{cab}(x,y)+C^{\alpha}_{cab}(x,y)=\eta_{cd}e^{d}_{\mu}\Omega^{\mu\alpha}_{ab}(x,y)-\eta_{ad}e^{d}_{\mu}\Omega^{\mu\alpha}_{cb}(x,y) \tag{36}
$$

(where C^{α}_{cab} is irrelevant, analogous to the above A and B).

Next we need the explicit expressions for the functional derivatives of the anomalies (L^{α}_{cab} being zero in both cases of interest). For the Polyakov action Γ^{P} we have $G^{\mathrm{D}}=0$ and

$$
\begin{aligned}
\frac{\delta G^{\mathrm{W}}(x)}{\delta e^{b}_{\alpha}(y)} &= -\frac{1}{24\pi}\int d^2z\,\frac{\delta(e(x)R(x))}{\delta g^{\mu\nu}(z)}\frac{\delta g^{\mu\nu}(z)}{\delta e^{b}_{\alpha}(y)} \\
&= \frac{1}{24\pi}\eta_{bc}e^{c}_{\lambda}(y)(g^{\mu\alpha}g^{\nu\lambda}+g^{\mu\lambda}g^{\nu\alpha})_{(y)}(D_\mu D_\nu-g_{\mu\nu}\Box)_{(x)}\delta^{(2)}(x-y),
\end{aligned}
\tag{37}
$$

whereas for the Weyl-invariantly regularized effective action $\hat{\Gamma}$ we find $\hat{G}^{\mathrm{W}}=0$ and

$$
\begin{aligned}
\frac{\delta\hat{G}^{\mathrm{D}}_{\lambda}(x)}{\delta e^{b}_{\alpha}(y)} &= -\frac{1}{48\pi}\partial^{x}_{\lambda}\int d^2z\,d^2z'\,\frac{\delta\hat{R}(x)}{\delta\gamma^{\rho\sigma}(z)}\frac{\delta\gamma^{\rho\sigma}(z)}{\delta g^{\beta\delta}(z')}\frac{\delta g^{\beta\delta}(z')}{\delta e^{b}_{\alpha}(y)} \\
&= \frac{e(y)}{48\pi}(\delta^{\rho}_{\mu}\delta^{\sigma}_{\nu}-\frac{1}{2}g_{\mu\nu}g^{\rho\sigma})_{(y)}\eta_{bc}e^{c}_{\epsilon}(y)(g^{\beta\alpha}g^{\delta\epsilon}+g^{\delta\alpha}g^{\beta\epsilon})_{(y)} \\
&\quad\times\partial^{x}_{\lambda}(\hat{D}_\rho\hat{D}_\sigma-\gamma_{\rho\sigma}\Box)_{(x)}\delta^{(2)}(x-y).
\end{aligned}
\tag{38}
$$

Now the procedure of [11] for evaluating the Schwinger terms $S^{0\alpha}_{ab}$ consists in expanding all the local functions of eqs. (30,31,36) into derivatives of δ functions, e.g.

$$I^{\alpha}_{ab}(x,y) = \sum_{n,k} I^{\alpha(k,n-k)}_{ab}(x)\partial_0^k\partial_1^{n-k}\delta^{(2)}(x-y). \tag{39}$$

The index $k = 0, \cdots, n$ counts the number of time derivatives, while $n-k$ counts space derivatives. In particular, $S^{0\alpha}_{ab}(x,y)$ has only spatial derivatives of δ functions,

$$S^{0\alpha}_{ab}(x,y) = \sum_{n} S^{0\alpha(n)}_{ab}\partial_1^n\delta^{(2)}(x-y). \tag{40}$$

Thus, one obtains a system of linear equations for the unknown coefficient functions $S^{0\alpha(n)}_{ab}$ and $\Omega^{\mu\alpha(k,n-k)}_{ab}$.

First let us briefly review the flat space-time computation that was done in [11] (they used it for chiral fermions, too, where diffeomorphism and Weyl anomalies are present). In this case all derivatives only act on the δ functions. Therefore the explicit expression analogous to (38) for I^{α}_{ab} contains only terms with three derivatives, and the corresponding expression (37) for Π^{α}_{a} only terms with two derivatives. Further, the covariant derivative in eq. (30) turns into an ordinary derivative. As a consequence, the resulting system of equations may be solved separately for each fixed number of derivatives (n derivatives for I, S and $n-1$ derivatives for Π, Ω); for each fixed n the number of unknowns $S^{0\alpha(n)}_{ab}$ and $\Omega^{\mu\alpha(k,n-k)}_{ab}$ equals the number of equations. As only $\Pi^{\alpha(k,2-k)}_{a}$ and $I^{\alpha(k,3-k)}_{ab}$ are non-zero, one finds a non-zero result only for $S^{0\alpha(3)}_{ab}$, $\Omega^{\mu\alpha(k,2-k)}_{ab}$ (even in the non-flat case, we will only consider the coefficient of the triple derivative of the Schwinger term, therefore we drop the superscript (3)). Eliminating the Ωs, one arrives at the flat space result

$$S^{0\alpha}_{0b} = -I^{\alpha(0,3)}_{0b} - I^{\alpha(1,2)}_{1b} - I^{\alpha(2,1)}_{0b} - I^{\alpha(3,0)}_{1b} - \Pi^{\alpha(1,1)}_{b}, \tag{41}$$

$$S^{0\alpha}_{1b} = -I^{\alpha(0,3)}_{1b} - I^{\alpha(1,2)}_{0b} - I^{\alpha(2,1)}_{1b} - I^{\alpha(3,0)}_{0b} - \Pi^{\alpha(0,2)}_{b} - \Pi^{\alpha(2,0)}_{b}. \tag{42}$$

These equations we have to evaluate for the two versions Γ^{P} and $\hat{\Gamma}$ of our theory in the flat limit. In the first case only Π^{α}_{b} are non-zero, in the second case only $\hat{I}^{\alpha}_{ab}$. Both versions lead to the same Schwinger terms,

$$S^{00}_{00} = S^{00}_{11} = 0, \tag{43}$$

$$S^{00}_{01} = S^{00}_{10} = \frac{i}{12\pi}. \tag{44}$$

For the Weyl anomaly this result was in fact already computed in [11] (we differ in signs because of different metric and curvature conventions). For the diffeomorphism anomaly we find the same result, showing that the Schwinger terms are not sensitive to the regularization prescription.

Next we want to discuss the case of general metric. In this case one has covariant derivatives in eqs. (30,37,38), and therefore the system of equations (30,31,36) mixes different number of derivatives. However, I^{α}_{ab} and Π^{α}_{b} still contain at most three and two derivatives, respectively, acting on δ functions. If one also assumes that $\Omega^{\alpha\mu}_{ab}$ contains at most two derivatives (which is a very reasonable assumption, as all diagrams contributing to $< T(e(x)\Theta^{a}_{\mu}(x)e(y)\Theta^{b}_{\nu}(y)) >$ are at most quadratically divergent), it still holds that the subsystem of equations containing the maximal number of derivatives (three for I, S and two for Π, Ω) may be solved separately.

This system of equations is a little bit more complicated and leads again to the same solution for both the Weyl anomaly of Γ^{P} or the diffeomorphism anomaly of $\hat{\Gamma}$. The coefficients of $\partial_1^3\delta^{(2)}(x-y)$ in the Schwinger terms read

$$\begin{aligned} S^{00}_{00} = S^{00}_{11} &= -\frac{i}{6\pi}\frac{e^0_1 e^1_1}{(g_{11})^2}, \\ S^{00}_{10} = S^{00}_{01} &= \frac{i}{12\pi}\frac{(e^0_1)^2 + (e^1_1)^2}{(g_{11})^2}, \end{aligned} \tag{45}$$

and (defining $\kappa = \frac{i}{12\pi e (g_{11})^3}$)

$$\begin{aligned} S^{01}_{00} &= \kappa(-e^0_0 e^0_1 g_{01} g_{11} - e^0_0 e^1_1 e g_{11} + (e^0_1)^2((g_{01})^2 + e^2) + 2 e^0_1 e^1_1 e g_{01}), \\ S^{01}_{01} &= \kappa(e^0_1 e^1_0 g_{01} g_{11} + e^1_0 e^1_1 e g_{11} - e^0_1 e^1_1((g_{01})^2 + e^2) - 2(e^1_1)^2 e g_{01}), \\ S^{01}_{10} &= \kappa(e^0_0 e^1_1 g_{01} g_{11} + e^0_0 e^0_1 e g_{11} - e^0_1 e^1_1((g_{01})^2 + e^2) - 2(e^0_1)^2 e g_{01}), \\ S^{01}_{11} &= \kappa(-e^1_0 e^1_1 g_{01} g_{11} - e^0_1 e^1_0 e g_{11} + (e^1_1)^2((g_{01})^2 + e^2) + 2 e^0_1 e^1_1 e g_{01}). \end{aligned} \tag{46}$$

Although some components look rather ugly, this result is precisely what one expects, as we want to discuss now.

Let us transform $S^{\mu\alpha}_{ab}$ to pure space-time indices via

$$S^{\mu'\alpha'}_{ab} = E^\nu_a E^\beta_b g^{\mu'\mu} g^{\alpha'\alpha} S_{\mu\nu\alpha\beta}. \tag{47}$$

Notice that we cannot invert this relation because we do not know all the components of $S^{\mu\alpha}_{ab}$. However, due to the symmetries $S_{\mu\nu\alpha\beta} = S_{\nu\mu\alpha\beta} = S_{\alpha\beta\mu\nu}$, $S_{\mu\nu\alpha\beta}$ actually consists of six independent components. The expressions (45,46) for $S^{\mu'\alpha'}_{ab}$ lead to five independent equations for $S_{\mu\nu\alpha\beta}$. Therefore we are able to express all components of $S_{\mu\nu\alpha\beta}$ in terms of one unknown function Λ, where the form of Λ is restricted by the requirement that all $S_{\mu\nu\alpha\beta}$ tend to their well known Minkowski space version in the flat limit. We obtain

$$\begin{aligned}
S_{0000} &= \frac{4ie^3 g_{00} g_{01}}{12\pi (g_{11})^3} - \frac{8ie^3 (g_{01})^3}{12\pi (g_{11})^4} + \frac{(g_{01})^5}{(g_{11})^4}\Lambda, \\
S_{0001} &= \frac{ie^3 g_{00}}{12\pi (g_{11})^2} - \frac{4ie^3 (g_{01})^2}{12\pi (g_{11})^3} + \frac{(g_{01})^4}{(g_{11})^3}\Lambda, \\
S_{0101} = S_{0011} &= -\frac{2ie^3 g_{01}}{12\pi (g_{11})^2} + \frac{(g_{01})^3}{(g_{11})^2}\Lambda, \\
S_{0111} &= -\frac{ie^3}{12\pi g_{11}} + \frac{(g_{01})^2}{g_{11}}\Lambda, \\
S_{1111} &= g_{01}\Lambda,
\end{aligned} \tag{48}$$

where Λ may be non-zero (but finite) in the flat limit.

For a proper interpretation of this result we need some basic facts about canonical quantization in curved space-time. We chose the hypersurface $x^0 = \text{const}$ as a quantization surface. The direction of the (arbitrarily chosen) time coordinate is not an intrinsic property of this surface, and, therefore, time components of tensors are not invariant under coordinate transformations that do not change the coordinates on the hypersurface. Instead one has to choose the projection of the time components onto the timelike vector l^μ orthogonal to the surface, e.g. ($T_{\mu\nu}$ is a general tensor, i is the space index)

$$T_{\mu\nu} \to T_{ij}, \quad l^\mu T_{\mu j}, \quad l^\nu T_{i\nu}, \quad l^\mu l^\nu T_{\mu\nu} \tag{49}$$

(see e.g. [13]). The vector l^μ is given by

$$l^\mu = e g^{0\mu}. \tag{50}$$

Here we chose the normalization $l^\mu l_\mu = -g_{11}$, which is the proper normalization in order to obtain the correct commutator algebra on the quantization surface, see e.g. [6,14] (this normalization corresponds to the requirement that l^μ is a vector, not a vector density: for a general tangent vector b^μ_1 to the hypersurface, the orthogonal covector l_μ is $l_\mu = \bar\epsilon_{\mu\nu} b^\nu_1$, where $\bar\epsilon_{\mu\nu} = e^a_\mu e^b_\nu \epsilon_{ab} = e\epsilon_{\mu\nu}$ is a tensor. For our specific choice $b^\mu_1 = \delta^\mu_1$ one finds precisely (50) for l_μ). Further we should remember that $S^{\mu\nu}_{ab}$ was defined as the commutator of $[e(x)\Theta^\mu_a(x), e(y)\Theta^\nu_b(y)]$ (see eq. (25)), i.e. to obtain the commutators of the Θ^μ_a themselves we still have to divide by e^2. Doing so, and performing the projections, we recover precisely the central extension of the Virasoro algebra [15]

$$l^\mu l^\nu l^\alpha l^\beta S_{\mu\nu\alpha\beta} = l^\mu l^\nu S_{\mu\nu 11} = l^\mu l^\alpha S_{\mu 1\alpha 1} = 0 \tag{51}$$

$$e^{-2}(x) l^\mu l^\nu l^\alpha S_{\mu\nu\alpha 1} = e^{-2}(x) l^\mu S_{\mu 111} = \frac{i}{12\pi} \tag{52}$$

and the arbitrary function Λ cancels out in all expressions (51, 52). The pure space component $S_{1111} = g_{01}\Lambda$, which is not related to any symmetry generator, remains undetermined by our procedure.

IV. CONCLUSIONS

We have analyzed the anomalous Schwinger terms in the equal-time energy-momentum tensor algebra in two different regularizations of 2-d scalar field theory in a curved background.

The usual computations make use of the conformal gauge, which is of course appropriate for the diffeomorphism-invariant regularization. Once the metric is set to its conformally flat form, all the machinery of Conformal Field Theory can be applied essentially as in flat space-time [16]. In contrast, the gauge fixing can not be performed in the Weyl-invariant version of the theory. In order to compare both regularizations one then needs a more general framework, in which no gauge fixing is made at any step.

In this framework we have achieved a two-fold result. On the one hand, we have shown that the energy-momentum operators continue to obey the Virasoro algebra in the case of a general metric, *without using any gauge fixing* for the computation. On the other hand, we have proven that both versions of the theory, eq. (16) and eq. (22), obey the same commutation relations, regardless of the symmetries broken by the regularization procedures.

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