June 18, 1992

Ground state wave functionals for $1+1$-dimensional fermion field theories

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ABSTRACT: We use path-integral methods to derive the ground state wave functions of a number of two-dimensional fermion field theories and related systems in one--dimensional many body physics. We derive the exact wave function for the Thirring/Luttinger and Coset fermion models and apply our results to derive the universal behavior of the wave functions of the Heisenberg antiferromagnets and of the Sutherland model. We find explicit forms for the wave functions in the density and in the Grassmann representations. We show that these wave functions always have the Jastrow factorized form and calculated the exponent. Our results agree with the exponents derived from the Bethe Ansatz for the Sutherland model and the Haldane-Shastri spin chain but apply to all the systems in the same universality class.

I. INTRODUCTION

The quantization of physical systems with a finite number of degrees of freedom is usually carried out in either the Schrödinger picture or in the Heisenberg picture. In Quantum Field Theory (QFT) and in Many-Body Physics the most direct approach involves the use of second quantization and the Heisenberg picture is natural. In this approach, the theory focuses on the construction of the ground state and on the computation of the Green functions. The Schrödinger picture of a QFT, and of a generic many body system, is usually very complicated. The wave functions, whose properties can be guessed by intuition in simple systems with a finite number of degrees of freedom, become very complex objects. In the conventional second quantized picture, the states of the Fock space are represented in terms of eigenstates of the number operators. In the Schrödinger approach, the states are wave *functionals* of the field configurations. These functionals are very difficult to define[1] and they are very rarely used.

In most cases, these theories are discussed in the context of some form of perturbation theory. The unperturbed states are usually very simple and a Feynman diagrammatic approach is certainly simplest in the Heisenberg picture. The situation becomes radically different when it comes to the study of the non-perturbative structure of these theories. To date, there are only two approaches to non-perturbative physics. One are the semiclassical and mean-field approaches. The other is the explicit construction of the exact or approximate eigenstates of the systems. While the semiclassical approach is usually presented in terms of path-integrals, the eigenstate approach uses the first quantized or Schrödinger picture. Although very rarely used in QFT, it is the conventional approach for many-body physics, *e.g.,* the BCS approach to superconductivity and the Laughlin picture of the Fractional Quantum Hall Effect. Although much has been gained from the semiclassical approach[2], much of what is known about non-linear theories in 1+1 dimensions is due to

the Bethe Ansatz approach[3]. However, unlike the BCS and Laughlin wavefunctions, the Bethe Ansatz wavefunctions are so complex that their structure is almost never discussed, except for some very special systems[4,5,6] for which the wave functions become very simple.

In this paper we study the structure of the wave functionals of fermionic theories in 1+1 dimensions using field-theoretic path integral methods and extract their universal content. We focus our attention on a number of field-theoretic model and on their application to the physics of one-dimensional strongly correlated systems and antiferromagnets. In particular, we consider self-interacting theories of massless fermions, including free fermions, the Thirring/Luttinger model and Constrained Fermi systems (Coset models) and apply our results to the study of quantum Heisenberg antiferromagnets and to the Sutherland model. These systems have the interest since they are at a fixed point (*i.e.,* they exhibit conformal invariance), their Green functions exhibit well known universal properties.

We use the methods of references [7] and [8] to show that the ground state wave functionals of of the systems are determined completely by the knowledge of the Green functions. We introduce two representation of the wave functions: the *density* and the *Grassmann* representations. The density representation, which is closely related to the conventional particle number representation of second quantization, is given in terms of the density-density correlation functions. These correlation functions are exactly calculable by path integral methods in all the systems that we discuss which exhibit conformal and Kac-Moody invariance. The wave functionals have universal properties which are determined by the universal properties of the Green functions. The Grassmann representation gives a more detailed picture of the wave function in terms of fermion coherent states.

The results presented here are directly applicable to a number of interesting models of strongly correlated systems in one space dimension. For instance, this result allows us to show that the wave functions of the Sutherland model have the same universal properties as those of the Thirring/Luttinger model. This is interesting since the Sutherland model

can be viewed as a non-relativistic *regularization* of the Thirring/Luttinger model and it is solvable by the methods of the Bethe Ansatz[4]. The application of the methods of the Bethe Ansatz to models of massless relativistic fermions, such as the Thirring/Luttinger model, is plagued with difficulties. Our results on Coset models are directly applicable to quantum antiferromagnets. It is worthwhile to stress that while the Bethe Ansatz wave functions are usually so complex that there structure is never studied, they too have universal features which can be determined directly with our methods.

The paper is organized as follows. In section II we show how to extract the wave functionals from the path integral formulation of field theory. We begin with a careful definition of the Hilbert space. Two descriptions of the wave functionals are introduced: a) the second quantized or "density" picture and b) the Grassmann or Bargmann picture. In section III we illustrate these ideas in the context of free fermions and free bosons. These results had been derived previously using Schrödinger methods[9,1,10,11]. In section IV we apply these methods to the massless Thirring/Luttinger model. Here we give an explicit expression for the asymptotic behavior of the wave functional in both the density and Grassmann representations. Our results indicate that, while the Grassmann representation is a sum of terms which exhibit left-right factorization, the density representation does not. In addition both pictures yield wave functionals with a Jastrow form but with different exponents. In section V we discuss Coset models and derive the wave functionals for these systems. In section VI we discuss the connection between the Heisenberg and Sutherland models (and their generalizations) with the Thirring/Luttinger model and show that the wave functions have the same long distance behavior. The connection between the Heisenberg model and the Coset field theories is used to derive the long distance form of the wave functions. We show that the asymptotic form at long distances of the wave function for the nearest-neighbor and the $1/|r|^2$ models have the same Jastrow exponent (as required by universality). The conclusions are drawn in section VII.

II. WAVE FUNCTIONALS AND PATH INTEGRALS

Let us begin by introducing a Hilbert space adequate to describe the states of the fermionic systems of interest. Our description is equally applicable to both relativistic and non-relativistic systems in the second quantized formulation. In either case the space of states is Fock space. We will introduce two representations of the wave functions: a) the density representation and b) the Grassmann representation.

Let $|0\rangle$ be a reference state on which we will build the physical Hilbert space. In general we will assume that $|0\rangle$ is a filled fermi sea, although in non-relativistic theories is sometimes useful to consider the case in which $|0\rangle$ is the empty state (*i.e.,* the state with no particles). For the case of case of non-relativistic fermions we will denote with $\hat{c}^\dagger(x)$ and $\hat{c}(x)$ a set of creation and annihilation operators. Then a basis for the N-particle subspace of Fock space has the form $|x_1, \ldots, x_N\rangle = \hat{c}^\dagger(x_1) \ldots \hat{c}^\dagger(x_N)|0\rangle$. These states are eigenstates of the particle density operator $\hat{\rho}(x) = \hat{c}^\dagger(x)\hat{c}(x)$ with eigenvalue $\rho(x) = \sum_{j=1}^{N} \delta(x - x_j)$. We will call this the *density* representation and label these states by the density eigenvalue, *i.e.,* $|x_1, \ldots, x_N\rangle \equiv |\rho(x)\rangle$. For the case of dense systems, we will label the states by their density eigenvalue in the form $|[\rho]\rangle$, where $\rho(x)$, the eigenvalue of $\hat{\rho}(x)$, is a distribution.

Now, the density eigenstates are antisymmetric under particle exchange. However, the density eigenvalue is *even* under exchange. Thus, the density eigenvalue only specifies the presence or absence of a particle at a given point but says nothing about their relative ordering. Nevertheless, we will continue to denote the states by the density eigenvalue and assume that an antisymmetrization (or symmetrization, for the case of bosons) has been adopted. Notice that, in spite of these caveats, the states represented by the density eigenvalue still satisfy the exclusion principle.

There is an alternative representation in which the antisymmetry of the states is apparent. In this representation, the coordinates which label the states are Grassmann vari-

ables. We will refer to this as the *Grassmann* representation and it is the representation in terms of fermion coherent states[11]. Let $\xi_1, \ldots, \xi_N$ be a set of N Grassmann variables. The coherent state is defined to be $|\xi_1, \ldots, \xi_N\rangle = \exp\{\sum_{j=1}^{N} \xi_j \hat{c}^\dagger(x_j)\}|0\rangle$. These states are overcomplete. Let $|\Psi\rangle$ be an arbitrary state of the Hilbert space. The wave functions are inner products between the states and a set of reference states. In the density representation the wave functions are the usual antisymmetric "orbital" wave functions $\Psi(x_1, \ldots, x_N) = \langle x_1, \ldots, x_N|\Psi\rangle$. In the Grassmann representation, they are Grassmann polynomials $\Psi(\xi_1, \ldots, \xi_N)$. The orbital wave functions are the number coefficients of the Grassmann polynomials.

We will now derive a set of expressions for the wave functions, for both the density and Grassmann representations, in terms of the correlation functions of the specific system. For simplicity, we will discuss only the ground state wave function. These formulas generalize without difficulty for excited states. For the most part they also apply almost without changes to both relativistic and non-relativistic systems. We will discuss first the case of systems of non-relativistic fermions and then generalize our results for relativistic systems.

We begin with the density representation. Following ideas that go back to Landau's work on superfluid helium[12], Dashen and Sharp [13] (DS) have shown that density representation follows naturally from the symmetries of the physical system. If the Hamiltonian has a conserved charge, such as particle number, then the density $\hat{\rho}(x) = \hat{c}^\dagger(x)\hat{c}(x)$ and the current $\hat{j}(x) = \frac{1}{2i}[\hat{c}^\dagger(x)\partial_x\hat{c}(x) - \hat{c}(x)\partial_x\hat{c}^\dagger(x)]$ are the time and space components of a locally conserved vector which obeys the continuity equation, $\partial_t\hat{\rho}(x,t) + \partial_x\hat{j}(x,t) = 0$. From the equal-time canonical anticommutation relations of the field operators, it is easy to see that the density and current operators must satisfy the equal-time commutation relations

$$[\hat{\rho}(x), \hat{j}(x')] = -i\partial_x \left(\delta(x - x')\hat{\rho}(x)\right) \tag{2.1}$$

and

$$[\hat{\rho}(x), \hat{\rho}(x')] = [\hat{j}(x), \hat{j}(x')] = 0 \tag{2.2}$$

Notice that these commutation relations are non-trivial only for *dense* systems, *i.e.,* systems for which the local density operator has non-vanishing expectation values. In other words, this algebra only applies to operators acting on a Hilbert space of states obtained by acting finitely on a reference state which represents a *filled Fermi sea*. We will come back to this point when we discuss relativistic systems.

These commutation relations imply that the states can be either eigenstates of the density *or* eigenstates of the current but not of both. Dashen and Sharp further showed that $\rho(x)$ and $j(x)$ are a satisfactory set of coordinates in the sense that the set of states labelled by $\rho(x)$ *or* by $j(x)$ span the Hilbert space. In the *density* representation, the current operators are represented by

$$\hat{j}(x)|\Psi[\rho]\rangle \equiv -i\rho(x)\partial_x \left(\frac{\delta}{\delta\rho(x)}|\Psi[\rho]\rangle\right) \tag{2.3}$$

(Recall, however, the caveats concerning antisymmetry). Since these states form a complete basis of the Hilbert space, it is possible to resolve the identity operator $\hat{I}$ in the form

$$\hat{I} = \int \mathcal{D}\rho \; |[\rho]\rangle\langle[\rho]| \tag{2.4}$$

up to a normalization constant (which we absorb in the measure).

We will now follow the methods of references [7,8] to write the wave functions in the density representation. We begin by recalling that the absolute value squared of the ground state wave function in the density representation $|\Psi_0[\rho]|^2$ is given by

$$\begin{aligned} |\Psi_0[\rho]|^2 = \int \mathcal{D}A_0 \; \exp\{-i \int dx_1 \; A_0(x_1) \; \rho(x_1)\} \\ \lim_{A_0(x)\to A_0(x_1)\delta(x_0)} \langle 0|T \exp\{i \int d^2x \; A_0(x) \; \hat{j}_0(x)\}|0\rangle \end{aligned} \tag{2.5}$$

where $\hat{j}_0(x) \equiv \hat{\rho}(x)$. This identity can be proven by inserting the decomposition of the identity, eq (2.4), inside the vacuum expectation value in the integrand of eq (2.5) and

by carrying out the Fourier transforms. Please note that the Fourier transform is a path integral at a fixed time $x_0 = 0$. The operators in this expression are Heisenberg operators of the system in the absence of sources. The vacuum expectation value in the integrand of eq (2.5) can be calculated from the generating functional of density correlation functions. Let us denote by $\mathcal{Z}[A_\mu]$ the generating functional for density and current correlation functions. $\mathcal{Z}[A_\mu]$ is obtained from the path integral of the system by coupling it minimally to an external, classical, gauge potential A_μ,*i.e.,*

$$\mathcal{Z}[A_\mu] = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \ \exp\{iS[\bar{\psi}, \psi, A_\mu]\} \tag{2.6}$$

Clearly, we have

$$|\Psi_0[\rho]|^2 = \int \mathcal{D}A_0 \ \exp\{-i \int dx_1 \ A_0(x_1) \ \rho(x_1)\} \lim_{A_0(x) \to A_0(x_1)\delta(x_0)} \mathcal{Z}[A_0, A_1 = 0] \tag{2.7}$$

Eq (2.7) tells us that $|\Psi_0[\rho]|^2$ is determined by the generating functional of equal-time density correlation functions. In the next sections we will make extensive use of eq (2.7). This formula was derived in reference [7].

It is also possible to find an expression for the wave function itself, not just for the absolute value squared. This is done by using a generalization of the identity eq (2.5). In order to do that we need to be able to get information about the phase of the wave function, not just of its absolute value squared. The commutation relations eq (2.2) imply that the phase information must be related to the current operators. This leads us to consider the full generating function of density and current correlation functions $\mathcal{Z}[A_\mu]$. In particular, we will consider the case in which the space and time components of A_μ are non-zero on *different* time surfaces and look at vacuum expectation values of the form

$$\mathcal{Z}[A_0, A_1] \equiv \langle 0|T \exp\{i \int d^2x \ A_1(x) \ \hat{j}_1(x)\} \ \exp\{i \int d^2x \ A_0(x) \ \hat{j}_0(x)\}|0\rangle \tag{2.8}$$

where we must set $A_0(x) \to A_0(x_1)\delta(x_0 - t)$ first, $A_1(x) \to A_1(x_1)\delta(x_0 - t')$ later and, finally, let $t' \to t$. In what follows we will assume that these operations are done in the

order that we have just specified. This prescription amounts to a choice of order among the operators. By inserting now the resolution of the identity eq (2.4) inside the matrix element in eq (2.8) and after recalling the expression for the current operator in the density representation, eq (2.3), we can write eq (2.8) in the form

$$\begin{aligned}\mathcal{Z}[A_0, A_1] \equiv \\ &\int \mathcal{D}\rho \ \exp\{\int dx_1 \ \partial_x A_1(x_1)\rho(x)\frac{\delta}{\delta\rho(x)}\}\Psi_0^*[\rho] \ \Psi_0[\rho] \ \exp\{i\int dx_1 \ A_0(x_1)\rho(x_1)\} \\ &= \int \mathcal{D}\rho \ \Psi_0^*[\rho + \rho\partial_x A_1] \ \Psi_0[\rho] \ \exp\{i\int dx_1 \ A_0(x_1)\rho(x_1)\}\end{aligned} \tag{2.9}$$

Hence, after a Fourier transform we can write

$$\Psi_0^*[\rho + \rho\partial_x A_1] \ \Psi_0[\rho] = \int \mathcal{D}A_0 \exp\{-i\int dx_1 \ \rho(x_1)A_0(x_1)\}\mathcal{Z}[A_0, A_1] \tag{2.10}$$

where $\mathcal{Z}[A_0, A_1]$ is the expression calculated with the prescription of eq (2.8). We can now choose $A_1(x)$ to be such that, for arbitrary $\rho(x)$, $\rho(x) + \rho(x)\partial_x A_1(x) = \bar{\rho}$, where $\bar{\rho}$ is the average density. We can further pick the arbitrary choice of setting $\Psi_0[\bar{\rho}] = 1$ (*i.e.,* a fixed real number). With these prescriptions we can now write an expression for the full wavefunction:

$$\Psi_0[\rho] = \int \mathcal{D}A_0 \exp\{-i\int dx_1 \ \rho(x_1)A_0(x_1)\}\mathcal{Z}[A_0, A_1] \tag{2.11}$$

provided that $A_1(x)$ satisfies the condition $\rho(x)+\rho(x)\partial_x A_1(x) = \bar{\rho}$ and that $\mathcal{Z}$ is calculated in the limit (and with the order) prescribed above.

The density representation can be applied almost without changes to relativistic fermions. Let us consider a relativistic Fermi field $\psi(x)$. For the most part, we will not write the two Dirac components explicitly in what follows. The field ψ obeys canonical equal anti-commutation relations. The $U(1)$ charge current $J_\mu = \bar{\psi}\gamma_\mu\psi$, normal-ordered relative to the filled Dirac sea, obeys the equal-time $U(1)$ Kac-Moody algebra

$$[J_0(x), J_0(x')] = [J_1(x), J_1(x')] = 0 \qquad [J_0(x), J_1(x')] = -\frac{i}{\pi}\partial_x\delta(x - x') \tag{2.12}$$

These commutation relations, which are the relativistic version of the non-relativistic commutation relations of eq (2.2), determine the choice of representation for the wave functions. Here too, we will adopt the density representation, *i.e.,* we will pick the wave functions to be eigenstates of the density operator $J_0(x)$. However, in general, the density eigenvalue alone does not specify the state completely. For instance, for free fermions it is also necessary to specify the densities of left and right movers separately. In the case of interacting theories, because of the presence of the chiral anomaly, it is not generally possible to conserve both the right and left moving densities separately. One consequence of the anomaly is the existence of anomalous dimensions of various operators. In what follows we will assume that the regularization that we are using is such that the $U(1)$ charge current is conserved even in the presence of interactions.

We can now use the same approach that we developed above for non-relativistic systems, and apply it to relativistic models, such as the Thirring/Luttinger model. We will consider the Hilbert space built on the state $|0\rangle$, the *filled Dirac sea*. The $U(1)$ charge density is a diagonal operator in the basis of states $\{|x_1, x_1'; \ldots; x_n, x_n'\rangle\}$ which denotes a set of n particles (electrons) and n antiparticles (holes), with the eigenvalue $\rho(x) = \sum_{j=1}^{n}(\delta(x - x_j) - \delta(x - x_j'))$. The probability $|\Psi_0[\rho]|^2$ that a given $U(1)$ charge density profile $\rho(x)$ is observed in the ground state is still given exactly by the same equation that we derived for non-relativistic systems, eq (2.7). If we set the amplitude for the state with uniform (zero) density to unity, the *amplitude* $\Psi_0[\rho]$ is given by eq (2.11) but with $A_1(x)$ given by

$$A_1(x) = -\pi \int_{-\infty}^{x} dx' \rho(x') \tag{2.13}$$

This change is consequence of the fact that the Kac-Moody algebra of eq (2.12) is a little simpler that its non-relativistic version eq (2.2).

We end this section with a few comments on how the Grassmann representation is constructed using these methods. Except for the presence of various indices (Dirac com-

ponents, *etc.*), the method is the same for both relativistic and non-relativistic theories. The Grassmann representation of the wave functions can be constructed in a manner very much analogous to the construction of the density representation. The main difference is that now we want to project the ground state onto fermion coherent state $|[\chi]\rangle$. It is straightforward to show that this can be achieved by considering the generating function of *fermion* correlation functions $\mathcal{Z}[\bar{\eta},\eta]$, *i.e.,*

$$\mathcal{Z}[\bar{\eta},\eta] = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\{iS(\bar{\psi},\psi) + i\int d^2x[\bar{\eta}\psi + \bar{\psi}\eta]\} \tag{2.14}$$

The probability for the state $|[\chi]\rangle$ to occur in the ground state is

$$|\Psi[\bar{\chi},\chi]|^2 = \int \mathcal{D}\bar{\eta}\mathcal{D}\eta\mathcal{Z}[\bar{\eta},\eta]\ \exp\{-i\int dx[\bar{\eta}\chi + \bar{\chi}\eta]\} \tag{2.15}$$

with the same equal-time limits used in the density representation.

III. WAVE FUNCTIONALS FOR FREE THEORIES IN TWO DIMENSIONS

As a warm-up exercise, we will consider first the cases of a free relativistic scalar field (a theory of bosons) and free relativistic fermions, both in $1+1$-dimensions. The non--relativistic case can also be considered without difficulty (in any case, non-relativistic fermions at finite density are, in one space dimension, equivalent to relativistic fermions). For simplicity we will work in Euclidean space *i.e.,* $ix_0 \to \tau$. Note that besides the usual modifications in the partition functions $iS_M \to S_E$, the density-current correlation functions (2.1), (2.12) acquires an extra factor of i.

a. Free bosons

The (Euclidean) action for a free scalar field is:

$$S[\phi] = \frac{1}{2}\int d^2x \partial_\mu\phi\partial_\mu\phi \tag{3.1}$$

so that the partition function $Z[J]$ is given by:

$$Z[J] = \int D\phi \exp(-S[\phi] + \int d^2x J.\phi) \tag{3.2}$$

or,

$$Z[J] = Z[0]\exp[-\frac{1}{2}\int d^2x d^2y J(x)G(x,y)J(y)] \tag{3.3}$$

with $G(x,y)$ the 2-dimensional Laplacian Green function,

$$G(x,y) = \frac{1}{2\pi}\log|x-y|. \tag{3.4}$$

As explained in the precedent section, one should take for the source J:

$$J = J(x_1)\delta(x_0). \tag{3.5}$$

Then, calling

$$\hat{G}(x_1,y_1) = \lim_{x_0\to y_0} G(x,y) \tag{3.6}$$

we easily find, applying formula (2.7):

$$|\Psi[\varphi]|^2 = \int DJ \exp[-\frac{1}{2}\int dx_1 dy_1 J(x_1)\hat{G}(x_1,y_1)J(y_1) + i\int dx_1 J(x_1)\varphi(x_1)] \tag{3.7}$$

or,

$$|\Psi[\varphi]|^2 = \exp[-2\int dx_1 dy_1 \varphi(x_1)\hat{G}^{-1}(x_1,y_1)\varphi(y_1)]. \tag{3.8}$$

We have now to evaluate $\hat{G}^{-1}(x_1,y_1)$ satisfying:

$$\int dz_1 \hat{G}^{-1}(x_1,z_1)\hat{G}(z_1,y_1) = \delta(x_1-y_1). \tag{3.9}$$

Writing:

$$\hat{G}^{-1}(x_1, y_1) = \int \frac{dp}{2\pi} g^{-1}(p) \exp[ip(x_1 - y_1)] \tag{3.10}$$

one gets:

$$g^{-1}(p) = -2|p| \tag{3.11}$$

or

$$\hat{G}^{-1}(z) = -\frac{2}{\pi}\frac{1}{z^2} \tag{3.12}$$

so that the ground-state wave-functional for two dimensional free-bosons is given by:

$$|\Psi[\varphi]|^2 = \exp[-\frac{4}{\pi}\int dx_1 dy_1 \varphi(x) \frac{1}{(x-y)^2} \varphi(y)] \tag{3.13}$$

which is the correct answer.

b. Free fermions

Let us now compare this result with that obtained for the ground-state wave functional of a system of two-dimensional free fermions. The (Euclidean) action for fermions coupled to Grassmann valued external sources $\bar{\eta}, \eta$ is given by:

$$S[\bar{\eta}, \eta] = \int d^2x (\bar{\psi} i \not{\partial} \psi - \bar{\eta}\psi - \bar{\psi}\eta). \tag{3.14}$$

(With our conventions $\gamma_\mu \gamma_5 = i\epsilon_{\mu\nu}\gamma_\nu$ and $\gamma_5 = i\gamma_0\gamma_1$).

The partition function $Z[\bar{\eta}, \eta]$ is then given by:

$$Z[\bar{\eta}, \eta] = \int D\bar{\psi} D\psi \exp(S[\bar{\eta}, \eta]) \tag{3.15}$$

or

$$Z[\bar{\eta}, \eta] = \exp(-\int d^2x d^2y \bar{\eta}(x) G_F(x, y) \eta(y)) \tag{3.16}$$

where the fermionic Green function G_F given by:

$$G_F(x - y) = -\frac{i}{2\pi}\frac{\gamma_\mu(x_\mu - y_\mu)}{(x-y)^2} \tag{3.17}$$

As before, we take $\eta(x) = \eta(x_1)\delta(x_0)$ (and the same for $\bar{\eta}$) and then we need to compute:

$$\hat{G}_F(x_1, y_1) = \lim_{y_0 \to x_0} G_F(x - y) = -\frac{i}{2\pi} \frac{\gamma_1}{(x_1 - y_1)} \tag{3.18}$$

Then, applying formula (2.15), we have:

$$|\Psi[\bar{\chi}, \chi]|^2 = \int D\bar{\eta} D\eta Z[\bar{\eta}, \eta] \exp(-i \int d^2x (\bar{\chi}\eta + \bar{\eta}\chi)) \tag{3.19}$$

or:

$$\begin{aligned} |\Psi[\bar{\chi}, \chi]|^2 = \int D\bar{\eta} D\eta \exp(- \int dx_1 dy_1 \bar{\eta}(x_1) \hat{G}_F(x_1, y_1) \eta(y_1) \\ - i \int dx_1 (\bar{\chi}\eta + \bar{\eta}\chi)). \end{aligned} \tag{3.20}$$

The quadratic path-integral in (3.20) can be easily evaluated. The answer is:

$$|\Psi[\bar{\chi}, \chi]|^2 = \exp(\int dx_1 dy_1 \bar{\chi}(x_1) \hat{G}_F^{-1}(x_1, y_1) \chi(y_1)) \tag{3.21}$$

where $\hat{G}_F^{-1}(x_1, y_1)$ satisfies:

$$\int dz_1 \hat{G}_F^{-1}(x_1, z_1) G_F(z_1, y_1) = \delta(x_1 - y_1). \tag{3.22}$$

Fourier transforming (3.22) we get,

$$\hat{G}_F^{-1}(z) = \int \frac{dp}{2\pi} g(p) \exp(ipz) \tag{3.23}$$

with

$$g(p) = -2\gamma_1 \frac{p}{|p|}. \tag{3.24}$$

We then finally have:

$$|\Psi[\bar{\chi}, \chi]|^2 = \exp(-\frac{2i}{\pi} \int dx_1 dy_1 \bar{\chi}(x_1) \frac{\gamma_1}{x_1 - y_1} \chi(y_1)). \tag{3.25}$$

This wave functional can be written in a more familiar way if work in a subspace of the Hilbert space with a fixed number of particles. Indeed expanding the exponential in the Grassmann variables we get

$$|\Psi[\bar{\chi}, \chi]|^2 = \sum_{n=0}^{\infty} \left(\frac{-2i}{\pi}\right)^n \frac{1}{n!^2} \int \left(\prod_{i=1}^{n} dx_i dy_i\right) \prod_{j=1}^{n} \left(\bar{\chi}_{\alpha_j}(x_j) \chi_{\beta_j}(y_j)\right) \; F_0[\{x_j, y_j\}]_{\{\alpha_k \beta_k\}} \tag{3.26}$$

where $j,k=1,\ldots,n$ and the function $F_0[x_1,y_1;...]_{\alpha_1\beta_1...\alpha_n\beta_n}$ is given by

$$F_0[x_1,y_1;...;x_n,y_n]_{\alpha_1\beta_1...\alpha_n\beta_n}=(\gamma_1)_{\alpha_1\beta_1}...(\gamma_n)_{\alpha_n\beta_n}\times\det\frac{1}{(x_i-y_j)} \tag{3.27}$$

which is the wave function of a system of n particles and n antiparticles.

We have to mention here that the wave function (3.25) is the same we obtain working canonically in the Schrodinger representation. In this approach the Dirac fermions $\bar{\psi}$ and ψ are represented by differential operators in the Grassmann variables $\bar{\chi}$ and χ acting over the wave functional (3.25)[11]

$$\bar{\psi}_i=\frac{1}{\sqrt{2}}\left(\bar{\chi}_i+(\gamma_0)_{ji}\frac{\delta}{\delta\chi_j}\right) \tag{3.28}$$

$$\psi_i=\frac{1}{\sqrt{2}}\left(\chi_i+(\gamma_0)_{ij}\frac{\delta}{\delta\bar{\chi}_j}\right). \tag{3.29}$$

They satisfy the canonical commutation relations

$$\{\psi(x)_i,\bar{\psi}(y)_j\}=(\gamma_0)_{ij}\delta(x-y). \tag{3.30}$$

Using the property (3.22) (*i.e.*, $G_F^2\propto I$) we can easily verify that the wave functional (3.25) is annihilated by the second quantized hamiltonian

$$\int d^2x\bar{\psi}(x)\frac{\partial\psi(x)}{\partial x}\ \Psi[\bar{\chi},\chi]=0. \tag{3.31}$$

It might be worthwhile to reanalize the case of free fermions but considering that external sources couple to currents and not directly to fermion fields. What we have in mind is the formulation of two-dimensional fermionic models à la Sugawara, that is, in terms of currents $j_\mu=\bar{\psi}\gamma_\mu\psi$ so that, in our quantum mechanical language wave functionals, after a polarization choice, should depend say on j_0. Then, instead of action (3.14) we start from:

$$S[A_\mu]=\int d^2x(\bar{\psi}i\ \not{\partial}\psi-\bar{\psi}\gamma_\mu\psi A_\mu) \tag{3.32}$$

The corresponding partition function is, after path-integration[14]:

$$Z[A] = \exp(-\frac{1}{2\pi}\int d^2x d^2y A_\mu(\vec{x}) D_{\mu\nu}(\vec{x},\vec{y}) A_\nu(\vec{y})) \tag{3.33}$$

with:

$$D_{\mu\nu}(\vec{x}-\vec{y}) = (\delta_{\mu\nu} - \partial_\mu \bigtriangledown^{-1} \partial_\nu)\delta^2(\vec{x}-\vec{y}) \tag{3.34}$$

and

$$\bigtriangledown^{-1}(\vec{x},\vec{y}) = -\frac{1}{2\pi}\ln|\vec{x}-\vec{y}| \tag{3.35}$$

Following the prescription of section II we can now compute the density-representation ground-state wave-functional. We get:

$$|\Psi_0[\rho]|^2 = \exp\left(-\frac{\pi}{2}\int dx_1 dy_1 \rho(x_1)\hat{D}_{00}^{-1}(x_1,y_1)\rho(y_1)\right). \tag{3.36}$$

The calculation of $\hat{D}_{00}^{-1}$ can be done more easily in momentum space. In terms of the Fourier transform of the vector source $A_\mu(x)$,

$$A_\mu(x) = \int \frac{dp}{2\pi}\tilde{A}_\mu(p)\exp(ipx) \tag{3.37}$$

the partition function (3.33) takes the form

$$Z[A] = \exp(-\frac{1}{2\pi}\int d^2p \tilde{A}_\mu(p)\tilde{D}_{\mu\nu}(\tilde{A}_\nu(-p)) \tag{3.38}$$

where

$$\tilde{D}_{\mu\nu}(p) = \delta_{\mu\nu} - \frac{p_\mu p_\nu}{\vec{p}^2} \tag{3.39}$$

is the Fourier transform of the current-current correlation function (3.34). We are interested in the equal time correlation function of densities which is obtained integrating $\tilde{D}_{00}$ over the frequencies

$$\begin{aligned} < \rho(p)\rho(-p) > |_{\Delta t=0} &= \int dp_0 \frac{1}{\pi}\left(1 - \frac{p_0^2}{p^2+p_0^2}\right). \\ &= |p| \end{aligned} \tag{3.40}$$

Hence the wave function (3.36) can be written as

$$|\Psi_0[\rho]|^2 = \exp(\pi \int dp\rho(p)\frac{1}{|p|}\rho(-p)). \tag{3.41}$$

To return to space-time variables we use the following identity

$$\frac{1}{|p|} = \int \frac{d\omega}{\pi}\frac{1}{\omega^2+p^2}. \tag{3.42}$$

in order to write the Fourier anti-transform of the kernel $\frac{1}{|p|}$ as

$$\mathcal{F}^{-1}\left[\frac{1}{|p|}\right] = 2\int \frac{d\omega dp}{(2\pi)^2}\frac{1}{\omega^2+p^2}e^{i(\omega t+px)}|_{t=0} \tag{3.43}$$

which is nothing but the propagator of a two dimensional massless bosonic theory. Using the well-known result $\frac{1}{\pi}\ln\mu|x|$ for (3.43) we can finally write for the wave function (3.41) in real space

$$|\Psi_0[\rho]|^2 = e^{\int dxdy\rho(x)\ln\mu|x-y|\rho(y)}. \tag{3.44}$$

We can now find the probability of any particular density profile. For instance, we can look for the probability of a configuration in which the N particles are located at coordinates $x_1,\ldots,x_N$ and N antiparticles (holes) are located at $y_1,\ldots,y_N$ (regardless of whether they are left or right movers). For such configurations, the density profile is $\rho(x) = \sum_{j=1}^{n}(\delta(x-x_j)-\delta(x-y_j))$. By substituting this density profile in eq (3.44) we find the result

$$|\Psi_0(\{x_j\};\{y_j\})|^2 = \mathcal{N}\frac{\prod_{i<j}|x_i-x_j|^2|y_i-y_j|^2}{\prod_{i,j}|x_i-y_j|^2} \tag{3.45}$$

Eq (3.45) gives the probability, not the amplitude, for this configuration. We will show below that, up to an overall sign, the full wave function is just the square root of this expression. This can be verified much in the same way as we did for the Grassmann representation: the wave function associated with eq (3.44) is annihilated by the Hamiltonian. In order to prove this statement we must write the Hamiltonian in the density

representation. Of course this is the Sugawara-Sommerfield form of the Hamiltonian which reads

$$\mathcal{H} = 2\pi \int d^2x : \rho(x)^2 + j(x)^2 : \tag{3.46}$$

where the dots denotes normal ordering. In eq (3.46) we have restored a factor of i to the definition of the current $j(x)$, which is now a manifestly hermitian operator. Looking the equal time commutator

$$[\rho(x), j(y)] = -i\frac{1}{\pi}\partial_x \delta(x-y). \tag{3.47}$$

we see that the current $j(x)$ can be represented as a functional differential operator in the variable $\rho(x)$ by

$$j(x) = -i\frac{1}{\pi}\partial_x \frac{\delta}{\delta\rho(x)}. \tag{3.48}$$

Then, the Hamiltonian takes the form

$$\mathcal{H} = 2\pi \int d^2x \ : \rho(x)^2 + \frac{1}{\pi^2}\left(\partial_x \frac{\delta}{\delta\rho(x)}\right)^2 : \tag{3.49}$$

The action of the current over the wave function is given by

$$\begin{aligned} j(x)\Psi_0[\rho] &= -i\frac{1}{\pi}\partial_x \int dy \ln\mu|x-y|\rho(y)\Psi_0[\rho] \\ &= -i\frac{1}{\pi}\int dy \frac{1}{x-y}\rho(y)\Psi_0[\rho] \end{aligned} \tag{3.50}$$

showing that $j(x)$ acts over $\Psi_0[\rho]$ as the Hilbert transform of the density. For the composite operator $: j(x)^2 :$ we obtain

$$: j(x)^2 : \Psi_0[\rho] = -\frac{1}{\pi^2}\int dydz \frac{1}{x-y}\frac{1}{x-z}\rho(y)\rho(z)\Psi_0[\rho] \tag{3.51}$$

and, using the property eq (3.22), we find

$$\int dx : j(x)^2 : \Psi_0[\rho] = -\int dx \rho(x)^2 \Psi_0[\rho] \tag{3.52}$$

which proves our statement

$$\mathcal{H}\Psi_0[\rho] = 0. \tag{3.53}$$

IV. THE THIRRING/LUTTINGER MODEL

In this section we will compute the ground state wave functional for an interacting fermionic model: the Thirring model. As we did with the free fermions we can express the wave functional as a functional either of the fermionic variables $\bar{\chi}, \chi$ or of the charge density $\rho(x)$. We will start with the Grassmannian representation.

The Lagrangian for the Thirring model coupled to external sources $\bar{\eta}, \eta$ is

$$\mathcal{L} = -\bar{\psi} i \not{\partial} \psi - \frac{1}{2} g^2 \left(\bar{\psi} \gamma^\mu \psi\right)^2 - \bar{\eta}\psi - \bar{\psi}\eta. \tag{4.1}$$

We can eliminate the quartic term in (4.1) by making a Hubbard-Stratonovich transformation. The generating functional of the model becomes

$$Z[\bar{\eta}, \eta] = \int D\bar{\psi} D\psi D A_\mu e^{\int \bar{\psi}(i\not{\partial} + g\not{A})\psi - \frac{1}{2}A^2 - \bar{\eta}\psi - \bar{\psi}\eta \; d^2x}. \tag{4.2}$$

The integral over the fermionic variables is straightforward

$$Z[\bar{\eta}, \eta] = \int DA_\mu \;\; \det \not{D}[A] e^{\int \bar{\eta}(\not{D}[A])^{-1}\eta - \frac{1}{2}\int A^2} \tag{4.3}$$

where $(\not{D}[A])^{-1}$ is the inverse of the Dirac operator. In order to perform the integration in the auxiliary field A_μ we use the following decomposition

$$A_\mu = \frac{1}{g}\left(\partial_\mu \omega - \epsilon_{\mu\nu}\partial_\nu \phi\right) \tag{4.4}$$

In terms of the fields ϕ and ω the Green function $(\not{D}[A])^{-1}$ takes the simple form

$$(\not{D}[A])^{-1} = e^{\gamma_5\phi + i\omega}|_{\vec{x}} \frac{i}{2\pi} \frac{\gamma_\mu (x_\mu - y_\mu)}{(\vec{x} - \vec{y})^2} e^{\gamma_5\phi - i\omega}|_{\vec{y}}. \tag{4.5}$$

For the determinant of the Dirac operator we use the result (3.33)-(3.34) in terms of the fields ϕ and ω. The partition function (4.3) becomes

$$Z[\bar{\eta}, \eta] = \int D\phi D\omega e^{\int \bar{\eta}(\not{D}[A])^{-1}\eta} e^{-\frac{1}{2\pi}S[\phi,\omega]} \tag{4.6}$$

where

$$S[\phi,\omega]=\int\ \{(1+\frac{\pi}{g^2})\partial_\mu\phi\partial_\mu\phi+\frac{\pi}{g^2}\partial_\mu\omega\partial_\mu\omega\}\ d^2x. \tag{4.7}$$

(The value of the fermionic determinant is given by equations (3.33)-(3.34) up to a gauge-breaking term

$$\alpha\int A^2d^2x \tag{4.8}$$

depending on the arbitrary regularization parameter α. This means that the quantization of the Thirring Lagrangian (4.1) leads to a one parameter family of quantum theories. Our election $\alpha=0$, corresponding to a gauge invariant definition of the fermionic determinant, is justified by the fact that this value reproduces the standard solutions of the Thirring model).

Finally if we expand the exponential term containing the Grassmann variables, we can perform the bosonic integrals term by term. Instead of do this is more convenient to use first the formula (3.3) for the square of the wave functional and leave the bosonic integral to the end. Hence the square of the wave functional can be written as

$$|\Psi[\bar\chi,\chi]|^2=\int D\phi D\omega D\bar\eta D\eta e^{\int\bar\eta(\not{D}[A])^{-1}|_{\Delta t=0})\eta}\ e^{-\frac{1}{2\pi}S[\phi,\omega]}\ e^{-i\int(\bar\chi\eta+\bar\eta\chi)} \tag{4.9}$$

where

$$(\not{D}[A])^{-1}|_{\Delta t=0}=e^{\gamma_5\phi(x,t)+i\omega(x,t)}\frac{i}{2\pi}\frac{\gamma_1}{(x-y)}e^{\gamma_5\phi(y,t)-i\omega(y,t)} \tag{4.10}$$

is the equal time propagator.

The integration in the sources $\bar\eta,\eta$ is Gaussian and gives the result

$$|\Psi[\bar\chi,\chi]|^2=\int D\phi D\omega e^{-\int\bar\chi((\not{D}[A])^{-1}|_{\Delta t=0})^{-1}\chi}\ e^{-\frac{1}{2\pi}S[\phi,\omega]} \tag{4.11}$$

(we have omitted in (4.11) a factor $\det[(\not{D}[A])^{-1}|_{\Delta t=0}]$. It can be proved, using for example a coherent-state definition of the functional integral, that this factor is constant and does not play any role in our problem).

The kernel $((\not{D}[A])^{-1}|_{\Delta t=0})^{-1}$ is computed identically to the one of the free fermion case giving the result

$$((\not{D}[A])^{-1}|_{\Delta t=0})^{-1} = e^{-i(\gamma_5\phi(x,t)-\omega(x,t))}\frac{2i}{\pi}\frac{\gamma_1}{(x-y)}e^{-i(\gamma_5\phi(y,t)+\omega(y,t))}. \tag{4.12}$$

Finally the integration over the fields ϕ and ω can be done term by term in the expansion of the exponential of the Grassmann variables and we have for the wave functional

$$|\Psi[\bar{\chi},\chi]|^2 = \sum_{n=0}^{\infty}\left(\frac{2i}{\pi}\right)^n\frac{1}{n!^2}\int\left(\prod_{i=1}^{n}dx_i dy_i\right)\prod_{j=1}^{n}\left(\bar{\chi}\alpha_j(x_j)\chi_{\beta_j}(y_j)\right)F[\{x_j,y_j\}]_{\{\alpha_k\beta_k\}} \tag{4.13}$$

where the indices $\alpha_i,\ \beta_i$ label the Dirac components and $j,k=1,\ldots,n$. The function $F[x_1,y_1;...]_{\alpha_1\beta_1...\alpha_n\beta_n}$ has two contributions: a factor corresponding to the free fermions contributions (the function F_0 of equations (3.26) and (3.27)) and an equal-time correlation function of vertex operators with the bosonic action (4.7), coming from the interaction.

$$\begin{aligned}F[\{x_j,y_j\}]_{\{\alpha_k\beta_k\}} &= F_0[\{x_j,y_j\}]_{\{\alpha_k\beta_k\}}\times \\ &\langle\exp(\sum_{j=1}^{n}s_j\phi(x_j,t))\ \exp(\sum_{j=1}^{n}t_j\phi(y_j,t))\rangle\ \langle\exp(-i\sum_{j=1}^{n}\omega(x_j,t))\exp(i\sum_{j=1}^{n}\omega(y_j,t))\rangle\end{aligned} \tag{4.14}$$

where $j,k=1,\ldots,n$. The variables s_i (t_i) are equal to 1 if α_i $(\beta_i)=1$ and equal to -1 if α_i $(\beta_i)=2$. The correlation function (4.14) can be evaluated using the Wick theorem with the bosonic propagators

$$\langle 0|\phi(\vec{x})\phi(\vec{y})|0\rangle = -\frac{1}{2(1+\frac{\pi}{g^2})}\ln|\vec{x}-\vec{y}| \tag{4.15}$$

and

$$\langle 0|\omega(\vec{x})\omega(\vec{y})|0\rangle = -\frac{g^2}{2\pi}\ln|\vec{x}-\vec{y}|. \tag{4.16}$$

For example, the term $F[x_1,y_1;...;x_n,y_n]_{12...12}$ is given by

$$F[x_1,y_1;...;x_n,y_n]_{12...12} = [(\gamma_1)_{12}]^n\det\frac{1}{(x_i-y_j)}\ \frac{\prod_{i<j}|x_i-x_j|^\mu\prod_{i<j}|y_i-y_j|^\mu}{\prod_{i,j}|x_i-y_j|^\mu} \tag{4.17}$$

where the exponent μ is

$$\mu = -\frac{1}{2}\frac{(g^2/\pi)^2}{1+g^2/\pi}. \tag{4.18}$$

As we mentioned before we can also compute the wave functional of the Thirring model in the density representation. The first step is to obtain the generating functional of current correlation functions of this model. Making a Hubbard-Stratonovich transformation as we did in equation (4.2), the required generating functional becomes

$$\begin{aligned} Z[B_\mu] &= \int D\bar{\psi}D\psi DA_\mu e^{\int \bar{\psi}(i\not{\partial}+g\not{A}-\not{B})\psi - \frac{1}{2}A^2 \ d^2x} \\ &= \int DA_\mu \det\left(i\not{\partial}+g\not{A}-\not{B}\right) \ e^{-\frac{1}{2}\int A^2 \ d^2x} \end{aligned} \tag{4.19}$$

where B_μ is the external source. After a shift in the integration variables $A_\mu \to A_\mu - \frac{1}{g}B_\mu$ and using the result (3.33)-(3.34) for the fermionic determinant we get

$$Z[B_\mu] = \int DA_\mu e^{-\frac{g^2}{2\pi}\int d^2xd^2y \ A_\mu(\vec{x})D_{\mu\nu}(\vec{x},\vec{y})A_\nu(\vec{y})} \ e^{-\frac{1}{2}\int d^2x(A_\mu+\frac{1}{g}B_\mu)^2} \tag{4.20}$$

with $D_{\mu\nu}(\vec{x},\vec{y})$ defined in equation (3.34). We can perform the Gaussian integration in A_μ and the partition function takes the form

$$Z[B_\mu] = e^{\int d^2xd^2y \ B_\mu(\vec{x})\{\frac{1}{2g^2}\tilde{D}^{-1}_{\mu\nu}(\vec{x},\vec{y}) - \frac{1}{2g^2}\delta_{\mu\nu}\delta^2(\vec{x}-\vec{y})\}B_\nu(\vec{y})} \tag{4.21}$$

where $\tilde{D}^{-1}_{\mu\nu}(\vec{x},\vec{y})$ is the Green function of the operator

$$\begin{aligned} \tilde{D}_{\mu\nu}(\vec{x},\vec{y}) &= \frac{g^2}{\pi}D_{\mu\nu}(\vec{x},\vec{y}) + \delta_{\mu\nu}\delta^2(\vec{x}-\vec{y}) \\ &= ((1+\frac{g^2}{\pi})\delta_{\mu\nu} - \frac{g^2}{\pi}\partial_\mu \bigtriangledown^{-1} \partial_\nu)\delta^2(\vec{x}-\vec{y}). \end{aligned} \tag{4.22}$$

We can compute formally

$$\begin{aligned} \tilde{D}^{-1}_{\mu\nu} &= \left(1+\frac{g^2}{\pi}\right)^{-1} \ \frac{1}{\delta_{\mu\nu} - \left(1+\frac{g^2}{\pi}\right)^{-1}\partial_\mu \bigtriangledown^{-1} \partial_\nu} \\ &= \left(1+\frac{g^2}{\pi}\right)^{-1} \ [\delta_{\mu\nu} + \sum_{n=1}^{\infty}\left(1+\frac{g^2}{\pi}\right)^{-n}\left(\partial_\mu \bigtriangledown^{-1} \partial_\nu\right)^n]. \end{aligned} \tag{4.23}$$

and using the fact that that the operator $\partial_\mu \bigtriangledown^{-1} \partial_\nu$ is idempotent, we get for $\tilde{D}^{-1}_{\mu\nu}$

$$\tilde{D}^{-1}_{\mu\nu} = \frac{1}{1+\frac{g^2}{\pi}} \left(\delta_{\mu\nu} + \frac{g^2}{\pi} \partial_\mu \bigtriangledown^{-1} \partial_\nu \right) \delta^2(\vec{x}-\vec{y}). \tag{4.24}$$

Finally adding the second term in the exponential of equation (4.21) we obtain

$$Z[B_\mu] = \exp\left\{ -\frac{1}{2\pi} \frac{1}{1+\frac{g^2}{\pi}} \int d^2x d^2y B_\mu(\vec{x}) D_{\mu\nu}(\vec{x},\vec{y}) B_\nu(\vec{y}) \right\} \tag{4.25}$$

recovering the usual result for the Thirring model: the current correlation functions are only modified (respect to the free theory) by the presence of a constant factor. At this point is straightforward to write the ground state wave functional following the same steps we did in the free theory. The result is

$$|\Psi_0[\rho]|^2 = \exp\left\{ (1+\frac{g^2}{\pi}) \int dx dy \rho(x) \ln \mu |x-y| \rho(y) \right\} \tag{4.26}$$

We can check this result by looking if the wave functional is annihilated by the Hamiltonian. For the Thirring model the Hamiltonian in the Sugawara-Sommerfield form reads

$$\mathcal{H} = 2\pi(1+\frac{g^2}{\pi}) \int d^2x : \rho(x)^2 + j(x)^2 : \tag{4.27}$$

where we have restored a factor of i to the current $j(x)$. The equal time commutator between the density and the current is

$$[\rho(x), j(y)] = -i \frac{1}{\pi} \frac{1}{1+\frac{g^2}{\pi}} \partial_x \delta(x-y). \tag{4.28}$$

and the current operator can be represented as

$$j(x) = -i \frac{1}{\pi} \frac{1}{1+\frac{g^2}{\pi}} \partial_x \frac{\delta}{\delta \rho(x)} \tag{4.29}$$

It is easy to verify that, also in this case, the wave functional of eq (4.26) is a zero mode of the Hamiltonian

$$\mathcal{H}\Psi_0[\rho] = 0. \tag{4.30}$$

As a final test of our wave functional, we can verify the vacuum expectation current algebra in the Schrodinger representation. For example for the two point correlation function of currents we have

$$\begin{aligned} <0|j(x)j(y)|0> &= -\int D\rho\Psi_0[\rho]\frac{1}{(\pi+g^2)^2}\partial_{\rho(x)}\partial_y\frac{\delta}{\delta\rho(y)}\Psi_0[\rho] \\ &= -\frac{1}{\pi^2}\partial_x\partial_y\int dudv\ln|u-x|\ln|v_y|\int D\rho\rho(u)\rho(v)|\Psi_0[\rho]|^2 \\ &= \frac{1}{2\pi^2}\frac{1}{1+\frac{g^2}{\pi}}\frac{1}{(x-y)^2} \end{aligned} \tag{4.31}$$

which is the correct result.

A final comment about the wave functionals is in order. When we tested our wave functionals looking if they were annihilated by the hamiltonian we *assumed* that they were real (no phases appeared when we took the square root). Our assumptions were confirmed when we obtained the right results, equations (3.31), (3.52) and (4.30). However , in the same spirit of the method developed in section I, it is possible to obtain, for a wide class of models, the *whole* ground state wave functional, *i.e.,* with the phase factors. Let us explain how does this idea work for the case of the Thirring model.

We write generically the generating functional (4.19) as

$$Z[B_\mu] = \int D\bar{\psi}D\psi\exp\left(-S[\bar{\psi},\psi] - \int d^2x j_\mu B_\mu\right) \tag{4.32}$$

where $S[\bar{\psi},\psi]$ is the Thirring action. We can write this equation as a vacuum expectation value

$$Z[B_\mu] = \langle 0|\exp\{-\int d^2x j.B_1\}\exp\{-\int d^2x\rho.B_0\}|0\rangle \tag{4.33}$$

and choose the fields B_0 and B_1 in the form

$$B_0(x,t) = B_0(x)\delta(t-\tau)\ \text{ and }\ B_1(x,t) = B_1(x)\delta(t-\tau). \tag{4.34}$$

Inserting in equation (4.34) the following resolution of the identity

$$I = \int D\hat{\rho}\ |\hat{\rho}\rangle\langle\hat{\rho}| \tag{4.35}$$

we arrive to the result

$$Z[B_\mu] = \int D\hat{\rho}\langle 0|\exp\{-\int d^2x j.B_1\}|\hat{\rho}\rangle \ \exp\{-\int d^2x \hat{\rho}.B_0\}\Psi_0[\hat{\rho}]. \tag{4.36}$$

Now we can use the representation (4.29) for the current in the density representation and we have

$$\exp\{-\int d^2x j(x)B_1(x)\}|\hat{\rho}\rangle = |\hat{\rho} + (\pi + g^2)^{-1}\partial_x B_1\rangle. \tag{4.37}$$

Hence the partition function (4.33) takes the form

$$Z[B_\mu] = \int D\hat{\rho}\ \Psi_0^*[\hat{\rho} + (\pi + g^2)^{-1}\partial_x B_1]\Psi_0[\hat{\rho}]\exp\{-\int d^2x \hat{\rho}.B_0\} \tag{4.38}$$

and leads to the quantity

$$\Psi_0^*[\rho + (\pi + g^2)^{-1}\partial_x B_1]\Psi_0[\rho] = \int \mathcal{D}B_0 Z[B_\mu]\exp\{\int d^2x \hat{\rho}.B_0\}. \tag{4.39}$$

This last equation allows us to read-off the phase factor of the wave functional Ψ_0. Using equations (4.21)-(4.24) we can verify that for the Thirring model the wave functional (4.26) is real.

V. COSET MODELS

As a last example of ground state wave functional of a two dimensional system, we will consider a Conformal Coset Model. The fermionic version of the Coset Model is obtained by projecting out a suitable subgroup H of the original fermionic group G. This projection is carried out, in the path-integral approach, introducing Lagrange multipliers which forces the H-currents to vanish. The Lagrange multiplier fields can be interpreted as gauge fields. The total effect of the gauge fields is to constrain the fermion currents in H to zero when acting on the physical states[15,16,17]. The Coset Model $U(2)/SU(2)$ can be mapped onto

the Heisenberg Model. We will exploit this connection in Section VI. The generalization to other coset of the form $U(N)/SU(n)$ is straightforward.

By following the methods of the previous sections we construct the wave functions by first choosing a reasonable set of labels. We will choose to work in the density representation. Since the group $U(2)$ has several generators, we must choose one (or several of them). The constraints impose the condition that all the $SU(2)$ generators annihilate the physical states. For the coset $U(2)/SU(2)$, this leaves only one $U(2)$ available generator, the "charge" $\bar{\psi}\gamma_0\psi$. In section VI we will show that this operator is essentially the same as the z-component of the spin in the Heisenberg model. Thus, we are led to consider, once again, the generating functionals of the density correlation functions. The Lagrangian of the coset fermion model is given by

$$\mathcal{L} = \sum_{j=1}^{2} \bar{\psi}^j i\partial\!\!\!/ \psi^j + \sum_{a=1}^{3} \lambda_\mu^a j_\mu^a \tag{5.1}$$

where ψ^j , $(j = 1, 2)$ are two Dirac fermions, λ_μ^a are Lagrange multipliers and

$$j_\mu^a = \sum_{i,j=1}^{2} \bar{\psi}^i \gamma_\mu \sigma_{ij}^a \psi^j \tag{5.2}$$

are the $SU(2)$ Noether currents (the σ^a are the Pauli matrices). These currents annihilate the physical states.

The generating functional of the correlation functions of the two diagonal densities in $U(2)$ is given by

$$\begin{aligned} Z[a^1, a^2] &= \int D\bar{\psi} D\psi D\lambda \, \exp\left\{ - \int d^2x \mathcal{L} - \int d^2x (\rho_1 a^1 + \rho_2 a^2) \right\} \\ &= \int D\bar{\psi} D\psi D\lambda \, \exp\left\{ - \int d^2x \bar{\psi}\gamma \cdot (i\partial + \lambda + a^1 P_1 + a^2 P_2)\psi \right\} \end{aligned} \tag{5.3}$$

where $P_1 = \frac{I+\sigma^3}{2}$, $P_2 = \frac{I-\sigma^3}{2}$ and $\lambda_\mu = \sum_{a=1}^{3} \lambda_\mu^a \sigma^a$. We can eliminate in (5.3) the σ^3 part of the coupling with the sources by a shift in the variable λ_μ^3. Then the partition

function (5.3) can be written as

$$
\begin{aligned}
Z[a^1,a^2] &= \int D\lambda \det\left(\gamma\cdot(i\partial+\lambda+\frac{a^1+a^2}{2})\right) \\
&= \exp\left\{-\frac{1}{\pi}\int d^2x d^2y \frac{a^1(x)+a^2(x)}{2} D_{00}(x,y) \frac{a^1(y)+a^2(y)}{2}\right\} \int D\lambda\ \exp\{-W[\lambda]\}
\end{aligned}
\tag{5.4}
$$

where $D_{00}(x,y)$ is given in equation (3.34) and $W[\lambda]$ is the Wess-Zumino-Witten action[18]. Eq (5.4) shows that the non-abelian structure factorizes and decouples. This is a consequence of the simple structure of this coset. In more complicated coset, such as those in the sequence of minimal models, the non-abelian structure does not decouple.

The factorized structure of eq (5.4) has important consequences for the wave functions. Indeed, once the generating function is known, the methods of the past sections tell us that the wave functional of the $U(2)/SU(2)$ Coset Model is

$$
\begin{aligned}
|\Psi[\rho_1,\rho_2]|^2 &= \int Da^1 Da^2\ Z[a^1,a^2]\ \exp\left\{\int dx\ (\rho_1 a^1+\rho_2 a^2)\right\} \\
&= \int Da^1 Da^2\ \exp\left\{-\frac{1}{\pi}\int dxdy\ \frac{a^1+a^2}{2} D_{00} \frac{a^1+a^2}{2}\right\} \\
&\qquad \exp\left\{\int dx\ \left[(\rho_1+\rho_2)\frac{a^1+a^2}{2}+(\rho_1-\rho_2)\frac{a^1-a^2}{2}\right]\right\} \\
&= \delta[\rho_1-\rho_2]\ \exp\left\{2\int dxdy\ \rho_1(x)\ln|x-y|\rho_1(y)\right\}.
\end{aligned}
\tag{5.5}
$$

The factor $\delta[\rho_1-\rho_2]$ is a result of the constraint and it simply means that the two densities have to be equal to each other. The second factor implies that the wave function of the coset, as a functional of one of the densities (say ρ_1) is equal to the *square* of the wave function for free fermions (see eq (3.44)). Thus, the Jastrow exponent ν for the $U(2)/SU(2)$ Coset Model is $\nu=2$. In section VI we will make use of this result in the context of the spin chains.

It is straightforward to generalize the result of eq (5.5) for the slightly more general case of the coset $U(N)/SU(n)$. In this case, the constraint affects only $n-1$ diagonal generators of $U(N)$ (*i.e.,* those which span the Cartan subalgebra of $SU(n)$). If we define

an array of densities $\rho_1, \ldots, \rho_N$, the constraint forces the first n densities to be equal to each other and, say equal to ρ_1, while the other $N-n$ densities remain unconstrained. The wave function $\Psi_0[\rho_1, \ldots, \rho_N]$ becomes

$$\begin{aligned} |\Psi_0[\rho_1, \ldots, \rho_N]|^2 &\equiv |\Psi_0[\rho_1, \ldots, \rho_1, \rho_{n+1}, \ldots \rho_N]|^2 \\ &= \exp\left\{ n \int dxdy\ \rho_1(x) \ln|x-y| \rho_1(y) \right\} \\ &\quad \prod_{i=n+1}^{N} \exp\left\{ \int dxdy\ \rho_i(x) \ln|x-y| \rho_i(y) \right\} \end{aligned} \tag{5.6}$$

From this results we conclude that for all the simple coset the Jastrow exponents are always integers. Non-integer exponents can only arise as the consequence of the non--abelian structure.

VI. APPLICATION TO ONE-DIMENSIONAL STRONGLY CORRELATED FERMI SYSTEMS AND ANTIFERROMAGNETS

We will now discuss the applicability of the results of previous sections to a number of models of one-dimensional many-body physics. A number of these models are exactly solvable through the Bethe Ansatz. These systems are said to be integrable and for this to happen a number of conditions have to be satisfied. While these conditions do not affect the low energy behavior, they do change the microscopic properties of these systems. The Bethe Ansatz wave functions thus appear to depend on a number of microscopic properties which are physically irrelevant but are essential for integrability to hold. We will show in this section that the wave functions of these systems also have universal features. These universal properties, independent as they are of the short distance physics, hold separately from the integrability conditions. Most of the identifications of models listed below have been known for some time. We present them here with the sole purpose of making the connections clear.

There is a considerable number of systems in one-dimensional many-body physics whose long distance behavior is governed by the Thirring/Luttinger model, *i.e.,* they are in the same universality class. The reason behind this universality has been known for a long time [19,20,21]: if spin exchange is ignored, the low energy physics of all these systems reduces to the dynamics of fermion excitations with momentum near the two Fermi points (right and left movers) interacting *via* backscattering processes (*i.e.,* processes which exchange left and right movers). The basic assumption here is that, if the interactions are not too strong, the only states which participate in the dynamics are those which are sufficiently close to the Fermi "surface". With this assumptions, it is natural to separate the electron field into its fast and slow components. At weak coupling, the fast components can be neglected (their only effect is to renormalize the coupling constant and the energy scale for the slow components). If we denote by $\psi(x)$ the "electron" field operator, the left $\psi_L(x)$ and right $\psi_R(x)$ moving components of the Fermi field are defined through the decomposition

$$\psi(x) = e^{ik_F x}\,\psi_R(x) + e^{-ik_F x}\,\psi_L(x) \tag{6.1}$$

and can be arranged in the form of a two-component spinor $\psi_\alpha(x)$ ($\alpha = 1, 2$), with $\psi_1 = \psi_R$ and $\psi_2 = \psi_L$. At low energies the non-interacting dispersion curve can be linearized in the vicinity of the Fermi points. The result is a spinor Fermi field with an effective low energy Hamiltonian which is identical to the Hamiltonian of the relativistically invariant Thirring/Luttinger model. This identification holds provided that (a) all energies are rescaled so that the Fermi velocity is set to be one , $v_F = 1$, and (b) all irrelevant operators are neglected (*i.e.,* operators with scaling dimensions larger than 2).

a) The Anisotropic Antiferromagnetic (XXZ) Heisenberg spin chain:

The best known example is the anisotropic Heisenberg antiferromagnet (the XXZ model) as a function of anisotropy. The physics of the Thirring/Luttinger model coincides with the long distance physics of the XXZ model[19] if umklapp processes can be neglected

[21,22]. In this model spin-$\frac{1}{2}$ degrees of freedom are arranged on a linear chain with N sites. The Hamiltonian is

$$H = J \sum_{j=1}^{N} \left(\sigma_1(j)\, \sigma_1(j+1) + \sigma_2(j)\, \sigma_2(j+1) + \lambda\, \sigma_3(j)\, \sigma_3(j+1) \right) \tag{6.2}$$

where $\{\sigma_k\}$ $(k = 1, 2, 3)$ are the three Pauli matrices, J is the exchange constant and λ measures the anisotropy ($\lambda = 1$ is the Heisenberg antiferromagnet). This model is integrable through the Bethe Ansatz [23,24]. The identification with the Thirring/Luttinger model was done by Luther and Peschel[19] (a detailed argument is reviewed in references[20,25]). The mapping is done in two steps. First, a Jordan-Wigner transformation is used to map the spin model to a model of spinless fermions $c(j)$ defined on the same one-dimensional lattice. The Hamiltonian for the equivalent fermion model turns out to be

$$H = \frac{J}{2} \sum_{j=1}^{N} \left(c^\dagger(j)\, c(j+1) + \text{h.c.} + 2\lambda (n(j) - \frac{1}{2})(n(j+1) - \frac{1}{2}) \right) \tag{6.3}$$

where $n(j) = c^\dagger(j)c(j)$ is the fermion occupation number, and it is related with the z-component of the spin through $\sigma_3(j) = 2n(j) - 1$. Hence, the state with zero (one) fermion at j is the same as the state with one spin down (up) at j. The sector with $S_z = 0$ is identified with the half-filled sector of the fermion model. The continuum limit of this fermion model in this sector is easily seen to be equivalent to the Thirring/Luttinger model once the Fermi field is split into its left and right components. For small λ (*i.e.,* close to the XY-model limit), the coupling constant for the g Thirring model is seen to be given by $g^2 = \lambda$ (in units such that $v_F = 1$). The relation between the two coupling constants is not universal and, in general, it is non-linear[19]. The umkalpp operators are irrelevant close to the XY limit (small λ) but become marginal at $\lambda = 1$, the isotropic point. In the Thirring/Luttinger representation these operators become marginal at $g^2 = \pi$. For $g^2 > \pi$ these operators become relevant and cause the system to become massive.

Since the charge density of the Thirring model is proportional to the average of the fermion occupation numbers on two consecutive sites (*i.e.,* the average of the z-components

of the spins), the wave function of the Thirring/Luttinger model in the density representation has to coincide with the wave function of the Heisenberg chain in the σ_3 representation. This is the conventional representation of the Bethe Ansatz wave functions in which the coordinates represent spin flips. On the other hand, the Grassmann representation of the Thirring wave functions have coordinates which represent *fermions*, not spin flips. The elementary excitations of the Heisenberg chain are fermions which, in the spin picture, are solitons. In section IV we found that the wave functions obtained through these two pictures yield different exponents. With this identifications, we can now make use of our results from section IV and conclude that, for large separation of its coordinates, the wave function of the XXZ model also has a Jastrow form with the exponent that we found for the Thirring model. This exponent is accurate only close to the XY limit. The behavior of the wave function *at* the isotropic Heisenberg limit can be determined as a limit $g^2 \to \pi$. The exponent of the Jastrow factor of the wave function, $\nu = (1 + g^2/\pi)$, has the limiting value $\nu = 2$. Interestingly enough this is precisely the same exponent found by Haldane and Shastri in the $1/|x|^2$ spin chain.

b) Sutherland's model:

Sutherland's model is a system of non-relativistic spinless fermions, at fixed density, interacting *via* a pair interaction potential $V(|x-x'|) = A/|x-x'|^2$, where A is a coupling constant. In second quantized form the Hamiltonian for the Fermi field $\psi(x)$ is

$$H = \int dx \, \frac{1}{2M} (\partial_x \psi^\dagger(x)) \, (\partial_x \psi(x)) + \frac{1}{2} \int dx \int dx' \rho(x) V(|x-x'|) \rho(x') \qquad (6.4)$$

where $\rho(x) = \psi^\dagger(x)\psi(x)$ is the density operator and we have set $\hbar = 1$ and M is the mass.

By means of a suitable generalization of Bethe's Ansatz, Sutherland[4] was able to find the complete spectrum of this model. The wave functions turned out to have a very simple form. For a system with periodic boundary conditions it is natural to think of the particles as being on a ring of circumference L. For the pair potential $V(|x|) = \frac{g\pi^2}{L^2} \left[\sin(\frac{\pi|x|}{L})\right]^{-2}$,

which has the required periodicity properties, he found that the wave function for the ground state in the N-particle sector has the Jastrow form

$$\Psi = \prod_{i>j} \left[\psi(x_i - x_j)\right]^{\lambda} \tag{6.5}$$

with

$$\lambda = \frac{1}{2}\left[1 + (1+2g)^{\frac{1}{2}}\right] \tag{6.6}$$

We will use the methods of the previous sections to calculate the asymptotic long distance behavior of the wave function. Up to a redefinition of the coupling constant, we will be able to reproduce Sutherland's result, eq (6.6).

For a general pair potential $V(|x-x'|)$, Sutherland's model is equivalent (at low energies) to a Thirring-like model with Hamiltonian

$$\begin{aligned} H = &\int dx \ \left(\bar{\psi}(x) i\gamma_1 \partial_x \psi(x) - \frac{g}{2}(\bar{\psi}(x)\gamma_\mu \psi(x))^2\right) + \\ &\frac{1}{2}\int dx \int dx' \ (\psi^\dagger(x)\psi(x)) u(|x-x'|) \ (\psi^\dagger(x')\psi(x')) \end{aligned} \tag{6.7}$$

where have separated in the Hamiltonian the interactions into forward scattering backward scattering processes. The forward scattering processes are weighed by the low momentum components of the pair potential (low compared with $2k_F$) $\frac{1}{v_F}\ V(|x|) = u(|x|)$. The backward scattering processes, which involve momentum transfers near $2k_F$, are represented by the Thirring term with the coupling constant $g^2 \approx -\frac{1}{2v_F}\tilde{V}(2k_F)$, where v_F is the Fermi velocity and $\tilde{V}(2k_F)$ is the Fourier transform of the pair potential at $2k_F$. This separation is accurate for processes in which the single particle energies differences are much smaller than an energy cutoff D in which range the dispersion curve is accurately linear. This restricts the validity of this identification to the weak coupling regime. For larger values of the coupling constant, it is necessary to renormalize the parameters of the Thirring model.

With these caveats we can, in principle, use the wave functions of section IV, and use these results in the long distance limit of the Sutherland model. But, before we do that,

we must find out if the density-density coupling could possible change the results. There are several ways to see that the pair potential cannot affect the long distance behavior. Firstly, by direct bosonization, it is easy to show that, for potentials $u(|x|)$ which whose Fourier transform $\tilde{u}(q)$ satisfies $q\ \tilde{u}(q) \to 0$ (as $q \to 0$), the effective bosonized action coincides with the bosonized action of the Thirring model, up to a finite renormalization of the Thirring coupling constant (*i.e.,* the backscattering amplitude). Instead of following that approach, we can compute the wave function for the generalized theory of eq (6.7) using the same methods as in the previous sections. An explicit calculation in the density representation yields the wave function for the ground state to be

$$|\Psi_0[\rho]|^2 = \mathcal{N} \exp\left\{ (1+\frac{g^2}{\pi}) \int \frac{dp}{2\pi}\, \rho(p) \frac{1}{|p|} \left(1 - \frac{\tilde{u}(p)}{\pi(1+\frac{g^2}{\pi})} \right)^{\frac{1}{2}} \rho(-p) \right\} \tag{6.8}$$

where $\rho(p)$ is the Fourier transform of the density. This is the exact wave function for this model (within the approximation of a linearized dispersion curve). By direct inspection of this result it is apparent that the Jastrow exponent λ is determined by the low momentum behavior of the potential $\tilde{u}(q)$. Any delta function contribution to $u(|x|)$ is a constant term in the Fourier transform at all momenta. Thus, delta function pieces shift (*i.e.,* renormalize) *both* the Thirring coupling or backscattering amplitude ($\propto \tilde{V}(2k_F)$) *and* the $q \to 0$ behavior of $\tilde{u}(q)$, the forward scattering amplitude. Thus, such effects are essentially unobservable since these effects cancel each other out. If the behavior $\tilde{u}(q) \to$ const. is excluded, we have to consider a polynomial (*i.e.,* analytic) behavior, such as $\tilde{u}(q) \approx A\ |q|$. This is in fact the behavior for Sutherland's model where the constant is $A = -\pi g$ (for $V(|x|) = ga/|x|^2$ and a a short distance cutoff).

Obviously, this analysis fails if the Fourier transform of the pair potential is singular in the $q \to 0$ limit in the form $\tilde{u}(q) \approx A\ /q^{\sigma}$ (with $A < 0$). For this type of behavior, the kernel in the integral of eq (6.8) behaves, at large separations, like $-$const.$|x-x'|^{\sigma/2}$. Thus, while the wave function retains its Jastrow form, the physics is completely different: for

all potentials with $\sigma > 0$ (*i.e.,* they decay more *slowly* than $1/|x|^2$ at long distances), the wave function vanishes like $\exp(-\text{const.}|x|^{\sigma/2})$ at long distances. This indicates that, for these potentials, the spectrum of these systems does not contain particle-like excitations. Indeed, the case $\sigma = 2$ (*i.e.,* $\tilde{u} = -e^2/q^2$ corresponds to the one-dimensional Coulomb interaction This case is exactly equivalent to electrodynamics in one space dimension,the Schwinger model. The Schwinger model is known to be a *confining* theory and it does not contain any states in its spectrum with fermionic quantum numbers.

Thus, we conclude that, for a large class of pair potentials, the wave function at long distances not only has the Jastrow form, but the Jastrow exponent can be calculated directly from the Thirring/Luttinger model. In section IV we got a wave function with the same Jastrow form but with the exponent $\lambda_{Th} = 1 + g^2/\pi$ (here g is the coupling constant of the Thirring model, *i.e.,* $g \approx -\frac{1}{2v_F}\tilde{V}(2k_F)$). In the Sutherland model, the Fourier transform of the pair potential $V(|x|) = g/|x|^2$ is $\tilde{V}(|p|) = -(g/v_F)|p|$. Hence, we must identify the Thirring model g^2/π with Sutherland's coupling constant g (for $v_F = 1$). This identification is valid for g small and, in this limit, the exponent of the wave function calculated in the Thirring model agrees with Sutherland's exponent eq (6.6). However, what matters here is that the *form* of the wave function is the same. The actual value of the exponent depends on the identification of the coupling constants, which is not universal. This problem is well known in the context of the Heisenberg spin chain.

c) The Haldane-Shastri model:

The Haldane-Shastri (HS) model is a non-local modification of the Heisenberg spin chain with Hamiltonian

$$H_{HS} = \sum_{j,k=1}^{N} J(|j-k|)\,(\sigma_1(j)\,\sigma_1(k) + \sigma_2(j)\sigma_2(k) + \lambda\,\sigma_3(j)\,\sigma_3(k)) \tag{6.9}$$

with $J(|j-k|) = A/|j-k|^2$ (for $j \neq k$). It has the same physics as the Heisenberg spin chain except that the Bethe Ansatz has a much simpler structure: the wave functions have

a Jastrow form. But, since the interactions are non-local, the mapping of the HS model to the Thirring/Luttinger model cannot be done directly, contrary to what we discussed for the Heisenberg spin chain.

The HS model can be viewed as a strong coupling limit of a generalization of a Hubbard model. Consider a system of spin-$\frac{1}{2}$ fermions on a one-dimensional lattice of length N with creation and annihilation operators $c^\dagger_\sigma(j)$ and $c_\sigma(j)$ defined at each site j $(= 1, \ldots, N)$ and for each spin orientation $\sigma = \uparrow, \downarrow$. The Hamiltonian

$$H = \sum_{\sigma=\uparrow,\downarrow} \sum_{j,k=1}^{N} \Big[t(|j-k|) c^\dagger_\sigma(j) c_\sigma(k) + \text{h.c.} \Big] + \sum_{j=1}^{N} U n_\uparrow(j)\, n_\downarrow(j) \tag{6.10}$$

is a Hubbard model with a non-local hopping term with amplitude $t(|j-k|)$. Standard arguments[20] show that, for a half-filled system and in the strong coupling limit, ($U \gg \max_{j,k}(t(|j-k|))$, this Hubbard model is also equivalent to a Heisenberg antiferromagnet. The exchange interactions are now also non-local and have an exchange constant $J(|j-k|) = [t(|j-k|)]^2/U$. Hence, the $J/|x|^2$ interaction of the HS model requires that the equivalent Hubbard model should have a hopping term $t(|x|) \approx \sqrt{JU}/|x|$. Hence, our results on wave functions also apply to the long distance behavior of the spin component of the wave functions of the Hubbard models.

However, all quantum antiferromagnets in all dimensions and for all lattices, can also be regarded as the strong coupling limit of a gauge theory[26]. For a one-dimensional chain with exchange constant $J(|x_j - x_k|)$ the correspondence goes as follows: a set of fermion degrees of freedom are defined on each site of the chain with the corresponding fermion creation and annihilation operators, $\psi^\dagger_a(j)$ and $\psi_a(j)$, $a = 1, 2$. The spin operators are defined to be $S^+(j) = \frac{1}{2}\epsilon_{ab}\psi_a(j)\psi_b(j)$, $S^-(j) = S^+(j)^\dagger$ and $S_3(j) = \psi_a(j)^\dagger\psi_a(j) - 1$. These operators are invariant under the *local* $SU(2)$ gauge transformation $\psi_a(j) \to \psi'_a(j) = U_{ab}(j)\psi_b(j)$. In terms of these fermions, the spin operators have a local symmetry, the quantum antiferromagnets must be equivalent to an $SU(2)$ gauge theory. Indeed, it is

straightforward to show that they are the strong coupling limit of an $SU(2)$ lattice gauge theory[27,26]. The gauge degrees of freedom are 2×2 $SU(2)$ matrices. In the absence of gauge fields, the free fermions have a global $U(2)$ invariance. The effect of gauge fields is to set all $SU(2)$ charges and currents to zero on the physical state[25]. Hence, in the (naïve) continuum limit, this is a theory of constrained (two-component) Dirac fermions in $1+1$ dimensions with the effective Lagrangian $\mathcal{L} = \bar{\psi} i \not{D} \psi$, where D_μ is the covariant derivative in the fundamental (spinor) representation of local $SU(2)$. This result can be derived by a straightforward application of the methods used above. The constraint implies that the physical degrees of freedom live in the coset $G = U(2)/SU(2)$. (We have assumed that any non-locality is sufficiently weak so that the mapping holds; it certainly works for the usual nearest neighbor Heisenberg model and it applies to the HS model as well). More complex coset models are found for higher spin representations as well as for groups other than $SU(2)$ (for a review on quantum chains see reference [28]).

Since all the local $SU(2)$ currents and densities annihilate the physical states, it is obvious that they cannot be used to label the states. Thus, the labels of the wave functions have to be the eigenvalues of the unconstrained currents. The coset $G = U(2)/SU(2)$ contains a global $U(1)$ subgroup whose generator is, precisely, $\psi^\dagger \psi$ which, as we discussed above, is essentially the same as the spin operator S_3 in the Heisenberg model. Hence, the wave function of the constrained fermion, on the diagonal generators of the coset $G = U(2)/SU(2)$, has precisely the same information as the Bethe wave functions for the Heisenberg model. The results of section V show that these wave functions also have the Jastrow-like factorized form and that the exponent is also equal to 2 (see eq (5.5)), in agreement with the results of Haldane and Shastri.

VII. CONCLUSIONS

In this paper we have used path-integral methods to derive the ground state wave functions of a number of two-dimensional fermion field theories and related systems in one--dimensional many body physics. We derived the exact wave function for the Thirring/Luttinger and Coset fermion models and used these results to derive the universal behavior of the wave functions of the Heisenbergmodels and of the Sutherland model. We found explicit forms for the wave functions in the density and in the Grassmann representations. For the field-theoretic models, the wave functions can be calculated exactly. We showed that these wave functions always have the Jastrow factorized form and calculated the exponent. The exponents obtained with our methods agree with the exponents derived from the Bethe Ansatz for the Sutherland model and the Haldane-Shastri spin chain. We also showed that they also apply to all the systems which are in the same universality class. Thus, the long distance form of the ground state wave function for the nearest neighbor Heisenberg antiferromagnet has the same exponent $\nu = 2$ as the HS model. We also verified that this result also applies to the fully $SU(2)$ invariant $U(2)/SU(2)$ constrained (Coset) fermion field theory. Also, up to a finite renormalization of the coupling constant, the Thirring/Luttinger form of the wave function agrees with the wave functions of both the Sutherland model and of the anisotropic (XXZ) antiferromagnet.

Our results confirm that the wave functions of these theories also have a universal content. In previous studies, the universality was always investigated in the context of the Green functions. Naturally, the universality of the Green function requires that the wave function should also have universal component. However, due to the complexity of the wave functions, their universality is not usually investigated. Our methods can be extended with only minor effort to the computation of the wave functions of the excited states. More interesting, however, is the study of the properties of the wave functions to

more complex theories with Kac-Moody level $k > 1$. This work is in progress.

ACKNOWLEDGMENTS

We would like to thank Michael Stone and Ana Lopez for useful discussions. This work was supported in part by the National Science Foundation through grants No.DMR91-22385 at the University of Illinois, by the NSF-CONICET International Cooperation Program through the grant No.INT-8902032 University of Illinois, by CONICET (Argentina) and by CIC (Argentina).

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