Canonical quantization of non-local field equations

D. G. Barci

Instituto de Física, Universidade Federal do Rio de Janeiro
C.P. 68528, Rio de Janeiro, RJ, 21945-970, Brasil.

L. E. Oxman

Departamento de Física, Facultad de Ciencias Exactas y Naturales
Universidad de Buenos Aires, Ciudad Universitaria, 1428,
Buenos Aires, Argentina.

M. Rocca

Departamento de Física, Universidad Nacional de La Plata,
C.C. 67, (1900) La Plata, Argentina.

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Abstract

We consistently quantize a class of relativistic non-local field equations characterized by a non-local kinetic term in the lagrangian. We solve the classical non-local equations of motion for a scalar field and evaluate the on-shell hamiltonian. The quantization is realized by imposing Heisenberg's equation which leads to the commutator algebra obeyed by the Fourier components of the field. We show that the field operator carries, in general, a reducible representation of the Poincare group.

We also consider the Gupta-Bleuler quantization of a non-local gauge field and analyze the propagators and the physical states of the theory.

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1   Introduction

The interest in non-local field theories has always been present in theoretical physics and it has been associated to many different motivations.

In Ref. [1], Wheeler and Feynman considered these theories as a description of the interaction between charged particles where the electromagnetic field does not appear as a dynamical variable. Before the renormalization theory became well established, physicists considered the possibility of formulating a finite theory, in order to describe elementary particle interactions by means of higher order lagrangians or non-local lagrangians. Pais and Uhlembeck [2] were the first in analyzing non-local theories in this context.

More recently, there were efforts to use non-local theories in connection with the understanding of quark confinement and anomalies [3][4], and also in string field theories containing non-local vertices [5][6].

Another aspect of non-local theories, is the possibility of relating them to a regularization scheme. Specifically, the analytic regularization introduced by Bollini and Giambiagi in Ref. [7] can be thought as associated to a non-local kinetic term in the lagrangian. On the other hand, dimensional regularization [8][9] does not admit a lagrangian formulation for non-integer values of the regulating parameter d, and although the Pauli-Villars regularization [10] admits a lagrangian formulation, the corresponding canonical quantization leads to an indefinite-metric Fock-space and the related unitarity problem.

Non-local effective theories, containing non-local kinetic terms, also arise when integrating over some degrees of freedom that belong to an underlying local field theory (see Refs. [11][12]).

Then, among the different non-local theories that can be formulated, it is natural to ask for the possibility of a consistent quantization of theories containing non-local kinetic terms.

At the classical level, Bollini and Giambiagi [13] have studied non-local equations containing arbitrary powers of the d'alambertian operator and in particular they established a non-trivial relation between the space-time dimension and the power of the d'alambertian, in order to satisfy the Huygens principle [14]. In particular, in $(2+1)$ dimensions, the usual wave equation $\Box\phi = j$ leads to a Green function that does not satisfy the Huygens principle, while the non-local equation $\Box^{1/2}\phi = j$ does satisfy this principle.

So, it is not by chance that the pseudo-differential operator $\Box^{1/2}$ also appears in the context of bosonization in $(2+1)$ dimensions. In Ref. [15], Marino has established the following mappings,

$$i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi \quad \leftrightarrow \quad \Phi^{+}\Box^{1/2}\Phi$$

$$i\bar{\psi}\gamma^\mu\partial_\mu\psi \quad \leftrightarrow \quad -\frac{1}{4}F^{\mu\nu}\Box^{-1/2}F_{\mu\nu} + \frac{1}{2}\theta\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho + nqt$$

where ψ is a two-component Dirac spinor, φ is a complex scalar field, A_μ is a $U(1)$ gauge field, and nqt are non-quadratic terms that can be eliminated in a long wavelength approximation. In a very interesting paper by Marino [16] this lagrangian appears when $(3+1)$D QED is projected to a physical plane. The kinetic term of the $(2+1)$ dimensional effective theory is proportional to $F^{\mu\nu}\Box^{-1/2}F_{\mu\nu}$. In the static limit, this term reproduces correctly the $1/r$ Coulomb potential instead of the usual logarithmic behavior of $(2+1)$D QED, this fact was first noted in Ref. [14].

In Refs. [17][15][16], the quantization of higher order and non-local lagrangians has been realized using the functional integral formalism.

This technique is naturally formulated in the euclidean space. In order to go back to Minkowski space it is necessary to perform a Wick rotation. As it was pointed out in many previous works (see Refs. [18][19][20][21][22]), the analytic properties of these theories may be highly non-trivial. So, it would be convenient to quantize these theories following canonical procedures, which are naturally defined in Minkowski space.

In Ref. [23], Amaral and Marino developed the Dirac quantization for theories containing fractional powers of the d'alambertian operator. When using this method some difficulties arise: there is an infinite set of second class constraints and the Poisson brackets between canonical variables are not well defined.

In this work we develop the Schwinger quantization for theories containing a general non-local kinetic term, including the case analyzed in Ref. [23] as a particular case. The method is based on the observation that the hamiltonian must be the generator of time translations, that is, we quantize the theory by imposing Heisenberg's equation. In this way, we do not need to define infinite momenta which lead to an infinite number of second class constraints.

Here we will be interested in Lorentz-invariant actions of the form

$$S = \int d^\nu x d^\nu y \ \varphi(x)V(x-y)\varphi(y) \tag{1}$$

Note that the Klein-Gordon equation $\Box\phi + m^2\phi = 0$ projects onto an irreducible representation of the Poincare group labeled by the mass m. However, for non-local fields, the action (1) do not lead to a Klein-Gordon type equation, and therefore the on-shell field will carry, in general, a reducible representation of the Poincare group. This fact will be reflected in the mass spectrum of the theory.

Changing variables in (1) we can write this action in a different way

$$
\begin{aligned}
S &= \int\int d^\nu x d^\nu z \; \varphi(x) V(z) \varphi(x+z) \\
&= \int d^\nu x \; \varphi(x) \left\{ \int d^\nu z V(z) e^{z.\partial} \right\} \varphi(x) \\
&= \int d^\nu x \; \varphi(x) f(\Box) \varphi(x)
\end{aligned}
\tag{2}
$$

where we used that $V(z)$ is a function of z^2 (due to Lorentz covariance) and therefore its Fourier transform is a function of k^2 only.

If $f(\Box)$ is a polynomial in $\Box$, we are in the case of a higher order theory. The Shwinger quantization of different models belonging to this case has been analyzed in Refs. [18][19][20][24]. But, if $f(\Box)$ cannot be expanded in a finite series, we are in the case of the non-local theories that we will consider in this paper.

In the higher derivative case, the model is completely defined by the roots of the polynomial, that is, by giving the set of "masses" that participate in the model. Similarly, we will see that, in non-local theories, the physical content depends on the zeros and cuts of the function f. That is, the choice of the cuts that leads to a given analytic determination of f is a physical data that must be fixed "a priori"; indeed, the mass spectrum will be given by the singularities of f^{-1}.

In section §2 we develope the mode expansion for a field obeying a general non-local equation. In section §3 we evaluate the canonical hamiltonian, we define the vacuum state and evaluate the propagators.

In section §4 we quantize the non-local gauge field and we define the physical states using the Gupta-Bleuler method to deal with the gauge invariance of the theory. Then we evaluate the field propagator. Finally, we interpret the mass spectrum and the form of the propagators in the case of the $(2+1)$D non-local gauge theory given by $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}\Box^{-\frac{1}{2}}F^{\mu\nu}$, introduced in Ref. [16].

2   Mode expansion for a field obeying a non-local equation

In this section we will obtain the mode expansion for the on-shell field that corresponds to the quadratic lagrangian

$$
\mathcal{L} = \frac{1}{2}\phi f(\Box)\phi
\tag{3}
$$

where $f(z)$ is a general analytic function having a cut contained in the negative real axis. For instance, when $f(z) = (z + m_0^2)^\alpha$ and α is non integer, the cut is taken along the real axis,

running from $-m_0^2$ to $-\infty$. Note that in the Fourier-transformed space the operation $\square$ amounts to a multiplication by $-k^2$ ($k^2 = k_0^2 - \mathbf{k}^2$), implying that the functions $f(-k^2)$ we are considering have a cut contained in the space $k^2 \geq 0$ of time-like or light-like vectors.

As it occurs with the poles of f^{-1} (isolated singularities) that they are associated with a possible isolated massive mode of the field, we will see that the cuts of f^{-1} (a continuum of singularities) are associated with a continuum of massive modes that the field can support; then, the determination we are using for $f(z)$ simply corresponds to a determination of our physical system. That is, the cut contained in $k^2 \geq 0$, fixes the possible modes in the continuum to be normal massive modes ($k^2 > 0$) or light-like modes ($k^2 = 0$). Note that a light-like mode will be present in the continuum if $z = 0$ is a brunch point of $f(z)$; this is the case, for $f(z) = (z + m_0^2)^\alpha$, when $m_0 = 0$.

In order to obtain the Euler-Lagrange equation for the system (3) we first expand f in a power series: $f(\square) = \sum_n a_n \square^n$, and then apply higher order lagrangian procedures (see Ref. [25]):

$$\sum_l \square^l \frac{\partial \mathcal{L}}{\partial \square^l \phi} =$$
$$= \sum_l a_l \square^l \phi = 0 \tag{4}$$

By summing up the series we obtain the non-local equation of motion:

$$f(\square)\phi = 0 \quad . \tag{5}$$

Now, by Fourier transforming the spatial part of ϕ,

$$\phi(x) = \int d\mathbf{k} \phi_{\mathbf{k}}(t) e^{-i\mathbf{k}.\mathbf{x}} \tag{6}$$

we obtain:

$$f\left(\partial_t^2 + \mathbf{k}^2\right)\phi_{\mathbf{k}}(t) = 0 \quad . \tag{7}$$

In the simple case of a Klein-Gordon equation, where $f = \square + m_0^2$, a general solution to (7) can be written as

$$\phi_{\mathbf{k}}(t) = \frac{1}{2\pi i} \int_{L_+ + L_-} dk_0 \, e^{ik_0 t} \times \left[\frac{1}{k_0^2 - \omega_0^2}\right] a(k_0, \mathbf{k}) \tag{8}$$

where $\omega_0 = \sqrt{\mathbf{k}^2 + m_0^2}$, and L_+ (resp. L_-) is a loop sorrounding the pole at ω_0 (resp. $-\omega_0$) in the positive (resp. negative) sense. The function $a(k_0, \mathbf{k})$ is supposed to be analytic in k_0 and after the loop integration we are left with to different functions of $\mathbf{k}$ ($a(\omega_0, \mathbf{k}), a(-\omega_0, \mathbf{k})$) representing the arbitrarieness in the initial conditions.

Now, we will construct a general solution to the equation (5). In order to do so, we proceed by analogy to the Klein-Gordon case. In eq. (8), the presence of the pole (isolated singularity) prevents the paths of integration to be deformed to a point. On the other hand, when we apply the operator $\partial_t^2 - \omega_0^2$ in (8), an additional factor $-k_0^2 + \omega_0^2 = -(k_0 - \omega_0)(k_0 + \omega_0)$ is produced in the integrand, the poles are canceled, and the loops can be shrinked to a point, showing that (8) is a solution to the equation $(\partial_t^2 - \omega_0^2)\, \phi_{\mathbf{k}}(t) = 0$.

In the general case, according to the physical determination of f, the singularities are contained in $k^2 \geq 0$. For a fixed value of $\mathbf{k}$, the cuts in the k_0 variable are contained in the intervals $(-\infty, -\omega), (+\omega, +\infty)$, where $\omega = \sqrt{\mathbf{k}^2}$. For instance, in the case $f(z) = (z + m_0^2)^\alpha$, the cuts are given by the points in $(-\infty, -\omega_0), (+\omega_0, +\infty)$, $\omega_0 = \sqrt{\mathbf{k}^2 + m_0^2}$. Then, confronting (8), we propose the general solution to eq. (7):

$$\phi_{\mathbf{k}}(t) = i \int_{\Gamma_+ + \Gamma_-} dk_0 e^{ik_0 t} \left[\frac{1}{f(-k^2)} \right] a(k_0, \mathbf{k}) \tag{9}$$

where Γ_+ (resp. Γ_-) is a path surrounding in the positive (resp. negative) sense, all the singularities of $1/f(-k^2)$, which are present in the positive (resp. negative) k_0-axis. Actually, when we apply the operator $f(\partial_t^2 + \mathbf{k}^2)$ on (9), we obtain

$$f\left(\partial_t^2 + \mathbf{k}^2\right) i \int_{\Gamma_+ + \Gamma_-} dk_0 e^{ik_0 t} \frac{1}{f(-k^2)} a(k_0, \mathbf{k}) =$$
$$i \int_{\Gamma_+ + \Gamma_-} dk_0 e^{ik_0 t} \frac{f(-k^2)}{f(-k^2)} a(k_0, \mathbf{k}) = 0 \tag{10}$$

then, while the integration paths in (9) are not homotopic to zero due to the presence of both types of singularities, poles and cuts in f^{-1}, the application of f turns the integrand an analytic function of k_0.

From (6) and (9) we obtain the expression for the on-shell field associated with equation (5):

$$\phi(x) = i \int_{\Gamma_+ + \Gamma_-} dk e^{ikx} \frac{1}{f(-k^2)} a(k) \tag{11}$$

where dk integrates over the whole of the space-time, with k_0 moving along $\Gamma_+ + \Gamma_-$ ($a(k) = a(k_0, \mathbf{k})$).

Now, we can introduce the distributions $\delta^+ G(k)$ and $\delta^- G(k)$ ($k_0 \in \Re$):

$$\delta^+ G(k) \equiv -i\, \delta\, [1/f]\, \theta(k_0) \quad , \quad \delta^- G(k) \equiv -i\, \delta\, [1/f]\, \theta(-k_0) \tag{12}$$

where

$$\delta\, [1/f] = \left[\frac{1}{f(-(k_0 + i\epsilon)^2 + \mathbf{k}^2)} - \frac{1}{f(-(k_0 - i\epsilon)^2 + \mathbf{k}^2)} \right] \tag{13}$$

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so as to obtain

$$\phi(x) = \int dk e^{ikx} [\delta^+ G(k) + \delta^- G(k)] a(k) \tag{14}$$

and using $\delta^- G(-k) = \delta^+ G(k)$, results:

$$\phi(x) = \int dk [e^{ikx} a(k) + e^{-ikx} \bar{a}(k)] \delta^+ G(k) \tag{15}$$

where we have also taken into account that $\delta^+ G(k)$ is a real distribution, and used $a(-k) = \bar{a}(k)$ (when k_0 is real) so as to work with a real $\phi(x)$.

The functional $\delta[1/f]$ is the "discontinuity functional" at the cut of $1/f$, and is analogue to the functional $2\pi i \delta(k^2 - m_0^2)$, when $f(-k^2) = -k^2 + m_0^2$ This can be seen by using $(k^2 - m_0^2 - i\epsilon)^{-1} = P((k^2 - m_0^2)^{-1}) + i\pi\delta(k^2 - m_0^2)$ in eq. (12).

3   The canonical hamitonian and the field propagator

If we take the lagrangian

$$\mathcal{L} = \phi(\sum_{k=0}^{n} a_k \Box^k)\phi \tag{16}$$

we can compute the corresponding hamiltonian as the magnitud which is conserved due to the time-translation symmetry of the system. By using lagrangian procedures for higher order field theories, the hamiltonian is obtained from the density (see Ref. [25]):

$$T^{00} = \sum_{s,t=0}^{n} \{\Box^s \frac{\partial \mathcal{L}}{\partial \phi^{(s+t+1)}} \ddot{\phi}^{(t)} - \partial^0 \Box^s \frac{\partial \mathcal{L}}{\partial \phi^{(s+t+1)}} \dot{\phi}^{(t)}\} - \mathcal{L} \tag{17}$$

where $\phi^{(t)} = \Box^t \phi$. From $\frac{\partial \mathcal{L}}{\partial \phi^{(s+t+1)}} = a_{s+t+1}\phi$, it results

$$T^{00} = \sum_{s,t=0}^{n} a_{s+t+1}(\Box^s \phi \Box^t \ddot{\phi} - \Box^s \dot{\phi} \Box^t \dot{\phi}) - \mathcal{L} \tag{18}$$

Now, if we take $\mathcal{L}_n = \phi \Box^n \phi$ we have

$$T_n^{00} = \sum_{s+t=n-1} (\Box^s \phi \Box^t \ddot{\phi} - \Box^s \dot{\phi} \Box^t \dot{\phi}) - \mathcal{L}_n \tag{19}$$

where s and t run from 0 to $n-1$. If we Fourier transform the field ϕ (here we are not using the on-shell field yet), we obtain:

$$H_n = \int d\mathbf{x} \int dk dk' e^{ikx} e^{ik'x} \phi(k)\phi(k')(k_0'^2 - k_0 k_0') \sum_{s+t=n-1} (i^2)^{s+t+1} (k^2)^s (k'^2)^t - \int d\mathbf{x}\mathcal{L}_n \tag{20}$$

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Also we have

$$\sum_{s+t=n-1} (i^2)^{s+t+1}(k^2)^s(k'^2)^t = (k^2)^{n-1} \sum_{s+t=n-1} (-1)^n (k^2)^{s-n+1}(k'^2)^t = \frac{(-k^2)^n - (-k'^2)^n}{k^2 - k'^2} \qquad (21)$$

and therefore

$$H_n = \int d\mathbf{x} \int dk dk' e^{ikx} e^{ik'x} \phi(k)\phi(k')(k_0'^2 - k_0 k_0') \frac{(-k^2)^n - (-k'^2)^n}{k^2 - k'^2} - \int d\mathbf{x} \mathcal{L}_n \qquad (22)$$

We will now evaluate the hamiltonian for (3). By taking the development

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad (23)$$

we have

$$\mathcal{L} = \phi f(\Box)\phi = \sum_{n=0}^{\infty} a_n \phi \Box^n \phi = \sum_{n=0}^{\infty} a_n \mathcal{L}_n \qquad (24)$$

In this way we get the expression

$$H = \sum_{n=0}^{\infty} a_n H_n =$$

$$= \int d\mathbf{x} \int dk dk' e^{ikx} e^{ik'x} \phi(k)\phi(k')(k_0'^2 - k_0 k_0') \frac{f(-k^2) - f(-k'^2)}{k^2 - k'^2} - \int d\mathbf{x} \mathcal{L}$$

$$(25)$$

We can see that in spite of the local validity of the development we are using for f, the final result can be expressed in terms of f, irrespective of the particular coefficients of the series.

Now, let us consider the field on-shell, that is, we use in (25) the expression (11):

$$H = -\int d\mathbf{x} \int_{\Gamma_+ + \Gamma_-} dk \int_{\Gamma_+' + \Gamma_-'} dk' e^{ikx} e^{ik'x} \frac{a(k)a(k')}{f(-k^2)f(-k'^2)} (k_0'^2 - k_0 k_0') \frac{f(-k^2) - f(-k'^2)}{k^2 - k'^2} \qquad (26)$$

Integrating in $\mathbf{x}$ and then in $\mathbf{k}'$ we get

$$H = -\int d\mathbf{k} \int_{\Gamma_+ + \Gamma_-} dk_0 \int_{\Gamma_+' + \Gamma_-'} dk_0' \frac{e^{i(k_0 + k_0')t}}{k_0 + k_0'} a(k_0, \mathbf{k})a(k_0', -\mathbf{k}) \frac{k_0'}{f(-k_0'^2 + \mathbf{k}^2)}$$

$$+ \int d\mathbf{k} \int_{\Gamma_+ + \Gamma_-} dk_0 \int_{\Gamma_+' + \Gamma_-'} dk_0' \frac{e^{i(k_0 + k_0')t}}{k_0 + k_0'} a(k_0, \mathbf{k})a(k_0', -\mathbf{k}) \frac{k_0'}{f(-k_0^2 + \mathbf{k}^2)}$$

$$(27)$$

Now, for a given $\mathbf{k}$, the paths Γ_+ and Γ_+' (resp. Γ_- and Γ_-') surround the points that belong to the cut contained in the interval $(+\omega, +\infty)$ (resp. $(-\infty, -\omega)$). If in addition we choose the path Γ_+' (Γ_-') surrounding all the points of the paths $-\Gamma_-$ and Γ_+ (resp. $-\Gamma_+$ and Γ_-), we have:

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In the first term of (27), the integrand has no cuts in the variable k_0 and only presents a pole at $-k_0'$ ($a(k_0, \mathbf{k})$ is analytic in k_0). But for the paths we are considering, we can see that when k_0' belongs to $\Gamma_+' + \Gamma_-'$, the point $-k_0'$ is not encircled neither by Γ_+ nor by Γ_-. Therefore, making the k_0-integral, we obtain that the first term is equal to zero.

In the second term, the integrand has no cuts in the variable k_0' and only presents a pole at $-k_0$. Again, taking into account the paths we choosed, we see that for k_0 belonging to Γ_+, $-k_0$ is encircled by Γ_-', while for k_0 belonging to Γ_-, $-k_0$ is encircled by Γ_+'. Then, making the k_0'-integral first and using Cauchy's theorem, results:

$$
\begin{aligned}
H &= 2\pi i \int d\mathbf{k} \int_{\Gamma_+} dk_0 \, a(k_0, \mathbf{k}) a(-k_0, -\mathbf{k}) \frac{k_0}{f(-k_0{}^2 + \mathbf{k}^2)} \\
&\quad - 2\pi i \int d\mathbf{k} \int_{\Gamma_-} dk_0 \, a(k_0, \mathbf{k}) a(-k_0, -\mathbf{k}) \frac{k_0}{f(-k_0{}^2 + \mathbf{k}^2)}
\end{aligned}
\tag{28}
$$

which is time independent as it was expected.

Changing variables $k_0 \to -k_0$ in the second integral and recalling that Γ_+ (Γ_-) goes in the positive (negative) sense, we get:

$$
H = 2\pi i \int d\mathbf{k} \int_{\Gamma_+} dk_0 [a(k_0, \mathbf{k}) a(-k_0, -\mathbf{k}) + a(-k_0, -\mathbf{k}) a(k_0, \mathbf{k})] \frac{k_0}{f(-k_0{}^2 + \mathbf{k}^2)}
$$

or in terms of the distribution $\delta^+ G$:

$$
H = 2\pi \int dk \, k_0 \, \delta^+ G(k) [a(k_0, \mathbf{k}) \bar{a}(k_0, \mathbf{k}) + \bar{a}(k_0, \mathbf{k}) a(k_0, \mathbf{k})]
\tag{29}
$$

We can see that the hamiltonian H is real ($\delta^+ G$ is real); we note also, from (12) and (13) that the weight $\delta^+ G$ is essentially a function of k^2:

$$
\delta^+ G(k) \equiv i \left[\frac{1}{f(-k^2 + i\epsilon)} - \frac{1}{f(-k^2 - i\epsilon)} \right] \theta(k_0)
\tag{30}
$$

Then, from (15), (29) and (30), the modes which are present in the spectrum of the theory are those having k^2 at the cut, where the function f^{-1} has a jump. These modes participate with a weight $\delta^+ G$. We will denote the mass spectrum of the model, or equivalently the support of $\delta^+ G$, by $\mathcal{S}$.

Now we will obtain the quantum version for the system (3). Recalling that H is conserved due to the time-translation symmetry of the system, in order to quantize the theory we require, after the replacement $a \to \hat{a}$, $\bar{a} \to \hat{a}^\dagger$, that the hamiltonian be the generator of time-translations, that is, we impose Heisenberg's equation:

$$
[\phi, H] = i \dot{\phi}
\tag{31}
$$

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Using (15) it results as usual:

$$[a, H] = k_0 a \quad , \quad [a^\dagger, H] = -k_0 a^\dagger \tag{32}$$

then, for $k_0 > 0$, $a(k)$ (resp. $a^\dagger(k)$) is a lowering operator (resp. raising operator). The vacuum is the Poincare-invariant state and is given by:

$$a(k)|0\rangle = 0 \quad , \quad k_0 > 0 \tag{33}$$

where k^2 belongs to $\mathcal{S}$, the support of $\delta^+ G$. Then, by taking normal ordering, we set the vacuum energy to zero, and the hamiltonian results (up to a factor)

$$H = \int dk \, k_0 \, \delta^+ G(k) a^\dagger(k_0, \mathbf{k}) a(k_0, \mathbf{k}) \tag{34}$$

Using now (15), (34) and (32) we obtain the algebra ($k_0 > 0$, $k_0' > 0$):

$$\delta^+ G(k)[a(k), a^\dagger(k')] = \delta(k - k') \quad , \quad [a(k), a(k')] = 0 \tag{35}$$

which is valid for $k^2, k'^2 \in \mathcal{S}$.

The two-point correlation function is:

$$\begin{aligned}
\langle 0|\phi(x)\phi(y)|0\rangle &= \int dk dk' \delta^+ G(k) \delta^+ G(k') \times \\
&\quad \langle 0|(e^{ikx} a(k) + e^{-ikx} a^\dagger(k))(e^{ik'y} a(k') + e^{-ik'y} a^\dagger(k'))|0\rangle \\
&= \int dk dk' e^{ikx - ik'y} \delta^+ G(k) \delta^+ G(k') \langle 0|a(k) a^\dagger(k')|0\rangle \\
&= \int dk e^{ik(x-y)} \delta^+ G(k)
\end{aligned} \tag{36}$$

which can also be written as:

$$\langle 0|\phi(x)\phi(y)|0\rangle = \Delta_+(x - y) \quad , \quad \Delta_+(x) = i \int_{\Gamma_+} dk \frac{e^{ikx}}{f(-k^2)} \tag{37}$$

We clearly see from (37) that $\Delta_+(x)$ is a solution to the homogeneus equation:

$$f(\Box)\Delta_+(x) = i \int_{\Gamma_+} dk e^{ikx} = 0 \tag{38}$$

On the other hand, the propagator is:

$$\begin{aligned}
\langle 0|T\{\phi(x)\phi(y)\}|0\rangle &= \theta(x_0 - y_0)\Delta_+(x - y) + \theta(y_0 - x_0)\Delta_+(y - x) \\
&= \theta(x_0 - y_0) \int dk e^{ik(x-y)} \delta^+ G(k) + \theta(y_0 - x_0) \int dk e^{ik(x-y)} \delta^- G(k) \\
&= i \int_{\Gamma_F} dk \frac{e^{ik(x-y)}}{f(-k^2)}
\end{aligned} \tag{39}$$

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where the path

$$\Gamma_F = \theta(x_0 - y_0)\Gamma_+ + \theta(y_0 - x_0)\Gamma_- \tag{40}$$

can be seen to be equivalent to a path that runs above the cut which is contained in the interval of negative frequencies $(-\infty, -\omega)$, and runs bellow the cut contained in the interval of positive frequencies $(+\omega, +\infty)$. It is also clear that $F(x) = \langle 0|T\{\phi(x)\phi(0)\}|0\rangle$ is i times a Green function:

$$f(\Box) \int_{\Gamma_F} dk \frac{e^{ikx}}{f(-k^2)} = \int_{\Gamma_F} dk e^{ikx} \tag{41}$$

now, the path Γ_F in (41) is equivalent to R (the real axis) and the integral gives $(2\pi)^4\delta(x)$. Then, from (39), we see that the propagator is the inverse of the kinetic operator $f(\Box)$ with Feynman's prescription to avoid the singularities.

As an example we can take, $f(z) = z^{1-\alpha}$, $(0 < \alpha < 1)$ determined by the cut $\mathcal{R}e\, z < 0$ so that $f(-k^2)$ has the cut in the positive k^2-axis:

$$\mathcal{L} = \phi\Box^{1-\alpha}\phi \tag{42}$$

In this case, we have a weight function δ^+G given by $2(k^2)_+^{\alpha-1}\sin\pi\alpha\,\theta(k_0)$, where the distribution x_+^λ is defined by

$$x_+^\lambda = \begin{cases} x^\lambda & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Note that in this case δ^+G is positive definite $(0 < \alpha < 1)$; then the contribution to the energy of every mode is positive definite and the resulting quantization does not present indefinite metric problems (cf. (35)). In the singular case where $\alpha = 0$, the distribution $(k^2)_+^{\alpha-1}$ has a pole (see Ref. [26]), the residue is proportional to $\delta(k^2)$ and the weight function results $\delta(k^2)\theta(k_0) = \frac{1}{2\omega}\delta(k_0-\omega)$, obtaining the usual zero-mass dispersion relation. Note also that the commutators (35) which are valid for $k^2, k'^2 \in \mathcal{S}$, can be written as $\delta^+G(k')\delta^+G(k)[a(k), a^\dagger(k')] = \delta^+G(k')\delta(k-k')$ (this is what one really gets from Heisenberg's equation). Then, in the usual Klein-Gordon case, where $\delta^+G(k) = 1/(2\omega)\delta(k_0 - \omega)$, we can integrate in k_0, k'_0 to obtain the usual commutation relation $[a(\omega, \mathbf{k}), a^\dagger(\omega', \mathbf{k}')] = 2\omega\delta(\mathbf{k} - \mathbf{k}')$. When we are in a region where k^2 belongs to a cut, then δ^+G is a well defined non-zero function of k^2 and we can write the commutators according to (35).

In the general case there is not a unique dispersion relation for the modes of the field but a continuum of massive modes (those that belong to the cut), each of which with the corresponding weight.

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4 The non-local gauge field

Here we will consider the quantization of a non-local abelian gauge theory defined by the lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}g(\Box)F^{\mu\nu} + \mathcal{L}_{GF} \tag{43}$$

$\mathcal{L}_{GF}$ is the gauge fixing term:

$$\mathcal{L}_{GF} = \frac{\xi}{2}A_\mu g(\Box)\partial^\mu\partial^\nu A_\nu \tag{44}$$

Using the gauge $\xi = 1$, we get the Euler-Lagrange equation for (43):

$$\Box g(\Box)A^\mu = 0 \tag{45}$$

Taking $f(\Box) = \Box g(\Box)$ we obtain the on-shell field (cf. (15) and (12))

$$A_\mu(x) = -i\int dk \sum_{\lambda=0}^{3}\epsilon_\mu^{(\lambda)}(k)[e^{ikx}a^{(\lambda)}(k) + e^{-ikx}\bar{a}^{(\lambda)}(k)]\,\delta\,[1/f]\,\theta(k_0) \tag{46}$$

where $\delta[1/f]$ is given by

$$\delta[1/f] = \delta\left[\frac{1}{(-k^2)g(-k^2)}\right] \tag{47}$$

The polarization vectors $\epsilon^{(\lambda)}(k)$ are defined, in a $d+1$-dimensional space-time, according to:

$$\epsilon^{(0)} = n \quad , \quad (n.n = 1 \ , \ n_0 > 0) \tag{48}$$

$$\epsilon^{(i)}.k = \epsilon^{(i)}.n = 0 \ , \ \epsilon^{(i)}.\epsilon^{(j)} = -\delta_{ij} \quad , \quad (i = 1,...,d-1) \tag{49}$$

$$\epsilon^{(d)} = [(k.n)^2 - k^2]^{-\frac{1}{2}}(k - (k.n)n) \tag{50}$$

where n and k are independent vectors. The equation in (45) leads at the classical level, taking appropriate boundary conditions, to the requirement $\partial.A = 0$.

At the quantum level we have the algebra ($k_0 > 0$):

$$-i\delta[1/f][a^{(\lambda)}(k), a^{(\lambda')\dagger}(k')] = -\eta^{\lambda\lambda'}\delta(k - k') \quad , \quad [a^{(\lambda)}(k), a^{(\lambda')}(k')] = 0 \tag{51}$$

Now, according to the Gupta-Bleuler quantization we impose the gauge condition by defining the space of physical states that satisfy the Lorentz condition $\partial.A^-|phys\rangle = 0$, where A^- is the annhilation part of A; this is equivalent to:

$$L(k)|phys\rangle = 0$$

$$L(k) = (a^{(0)}(k) - \alpha a^{(d)}(k)) \quad , \quad \alpha = \frac{\sqrt{(k.n)^2 - k^2}}{(k.n)} \tag{52}$$

A basis is obtained from the vacuum $|0\rangle$ by repeated application of $a^{(i)\dagger}(k)$, $i = 1, ..., d - 1$ and

$$M^\dagger(k) = a^{(0)\dagger}(k) - \alpha^{-1}a^{(d)\dagger}(k) \tag{53}$$

(these operators commute with $L(k)$).

For $k^2 = 0$, we have $\alpha = 1$. In this case, $M^\dagger = L^\dagger$ and then, $L^\dagger$ and L commute. A state containg at least an $M^\dagger$-type particle ($k^2 = 0$) has zero-norm. If we add such a state to any physical state, the norm of the latter is not modified.

For $k^2 > 0$, the operator L annhilates particles having polarization $k/\sqrt{k^2}$, which is proportional to $\epsilon^{(0)}(k) + \alpha\epsilon^{(d)}(k)$. The physical massive states have particles with polarizations $\epsilon^{(i)}$ ($i = 1, ..., d - 1$) and the polarization $\epsilon'^{(d)}$, which is proportional to $\epsilon^{(0)}(k) + \alpha^{-1}\epsilon^{(d)}(k)$, carried by the particles created by $M^\dagger$. Here, $\alpha \neq 1$ and $\epsilon'^{(d)}$, k are independent vectors satisfying $\epsilon'^{(d)}.k = 0$. The physical states containing $M^\dagger$-type particles (none of them having zero mass) have non-zero norm.

If we take a matrix element of the vector field between two physical states and we add to one of them a state containing a zero mass $M^\dagger$-type particle, then, this matrix element changes by a gauge transformation; on the other hand, when we add a massive $M^\dagger$-type particle the change is non-trivial. In this regard, note that in a matrix element of the form

$$g_\mu = \langle phys|A_\mu M^\dagger(k)|\psi\rangle \tag{54}$$

only when $k^2 = 0$ we have $\langle phys|M^\dagger(k) = \langle phys|L^\dagger(k) = 0$ and we can write

$$g_\mu = \langle phys|[A_\mu, M^\dagger(k)]\||\psi\rangle \tag{55}$$

to obtain from (46), (53) and (51):

$$g_\mu = \partial_\mu\alpha(x) \quad , \quad \alpha(x) = -(k.n)^{-1}e^{ik.x}\langle phys|\psi\rangle \tag{56}$$

Then, the states containing at least a zero-mass $M^\dagger$-type particle are associated to gauge transformations. The coexistence of massive states and gauge symmetry is possible because of the presence of the zero mass modes. For instance, if $k^2 = 0$ does not belong to the cut of g, then $1/f$ has a pole at $k^2 = 0$; for $g(\Box) = \Box^{-\alpha}$, $(0 < \alpha < 1)$ we have $f = \Box^{1-\alpha}$ and $k^2 = 0$ is a brunch point of $1/f$. In both cases the on-shell field is a superposition of fields with a given mass spectrum that contains the zero-mass. Then, the gauge transformation $A_\mu \to A_\mu + \partial_\mu\alpha$ can be thought as operating on the zero-mass sector of the vector field.

We end this section discussing the interpretation of the mass spectrum and the form of the propagators when we take the $(2+1)$ dimensional gauge theory corresponding to $g(\Box) = \Box^{-\frac{1}{2}}$:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}\Box^{-\frac{1}{2}}F^{\mu\nu} + \mathcal{L}_{GF} + j^{\mu}A_{\mu} \tag{57}$$

In Ref. [16], Marino has shown that this lagrangian describes, at the tree-level, the projection of QED from $(3+1)$ to $(2+1)$ dimensions. That is, the $(2+1)$D effective action (obtained by path-integrating over the gauge field)

$$\int dt\,dx\,dy\,j^{\mu}(x)G^{(2+1)}_{\mu\nu}(x-y)j^{\nu}(y) \quad , \quad \Box^{\frac{1}{2}}G^{(2+1)}_{\mu\nu}(x) = \eta_{\mu\nu}(2\pi)^3\delta(x) \tag{58}$$

corresponds to the $(3+1)$D effective action

$$\int d^4x\,J^{\mu}(x)G^{(3+1)}_{\mu\nu}(x-y)J^{\nu}(y) \quad , \quad \Box G^{(3+1)}_{\mu\nu}(x) = \eta_{\mu\nu}(2\pi)^4\delta(x) \tag{59}$$

when the currents are constrained to live on the plane $z=0$:

$$J^{\mu} = \delta(z)j^{\mu} \quad (\mu = 0,1,2) \quad , \quad J^3 = 0 \tag{60}$$

In other words, the projected non-local $(2+1)$D theory given by (57) displays a static coulombic interaction ($\sim 1/R$) between charges, instead of the logarithmic behavior present in $(2+1)$ dimensions, when $\mathcal{L} = -(1/4)F_{\mu\nu}F^{\mu\nu}$.

This can be seen by noting that, for $z=0$, the Maxwell propagator (in the Feynman gauge) can be written as:

$$iG^{(3+1)}_{\mu\nu}(x)|_{z=0} = i\eta_{\mu\nu}\int_0^{+\infty}dk_z^2\,(k_z^2)^{-\frac{1}{2}}\int d^3k\frac{e^{ik_{(3)}x_{(3)}}}{(-k_{(3)}^2 + k_z^2 + i\epsilon)} \tag{61}$$

$$= i\pi\eta_{\mu\nu}\int d^3k\frac{e^{ik_{(3)}x_{(3)}}}{(-k_{(3)}^2 + i\epsilon)^{\frac{1}{2}}} \tag{62}$$

where $k_{(3)}$ and $x_{(3)}$ are the $(2+1)$D part of the four-vectors k and x, respectively; and we used $\int_0^{\infty}ds\,s^{-1/2}(1+\beta s)^{-1} = \beta^{-1/2}B(\frac{1}{2},\frac{1}{2})$. On the other hand, from (51), we obtain the propagator corresponding to (57), computed as the vacuum expectation value of the T-product for two fields (cf. (39)):

$$i\eta_{\mu\nu}\int_{\Gamma_F} d^3k\frac{e^{ik_{(3)}x_{(3)}}}{(-k_{(3)}^2)^{\frac{1}{2}}} \tag{63}$$

which is proportional to the projected propagator given in (62), as the $i\epsilon$ prescription is equivalent to integrating over the path Γ_F that runs above the cut $(-\infty, -\omega)$, and runs bellow the cut $(+\omega, +\infty)$.

Then, in the case of the $(2+1)$ dimensional non-local gauge theory defined by (57), we see that the form of the propagator coincides with that obtained by projecting QED from $(3+1)$ to $(2+1)$ dimensions. Also, we can see that the existence of a continuous mass spectrum in the non-local theory can be traced back from the dispersion relation $k^2 = 0$, present in $(3+1)$D QED, which in the projected theory reads as $k_{(3)}^2 = k_z^2$, giving a mass spectrum that goes from zero to infinity (cf. eq. (61)).

5   Conclusions

In this work, we have solved a general class of non-local field equations characterized by a non-local kinetic term. The obtained mode expansion for the on-shell field coincides with that proposed in Ref. [23], where fractional powers of the d'alambertian operator are considered.

We have seen that a non-local equation projects the field onto a reducible representation of the Poincare group. That is, the on-shell field carries a representation which is the direct sum of irreducible representations labeled by the mass. The possible values of k^2 that appear in the representation are the singularities of the function $f^{-1}(-k^2)$ that characterizes the kinetic term. Each mode is associated with a corresponding weight $\delta^+ G = \sigma(k^2)\theta(k_0)$, where $\sigma(k^2) = -i\,\delta\left[1/f\right]$ (cf. eq. (12)). Then, we have seen that the analytic determination of $f(-k^2)$ fixes our physical system, that is, the mass spectrum of the model. These facts can be displayed more clearly by using

$$\delta^+ G = \int ds\, \sigma(s)\delta(k^2 - s)\theta(k_0)$$

in eq. (15), to write the field as

$$\phi(x) = \int ds\, \sigma(s)\phi_s(x) \quad , \quad \phi_s(x) = \int dk [e^{ikx} a(k) + e^{-ikx}\bar{a}(k)]\delta(k^2 - s)\theta(k_0)$$

ϕ_s is the expression for the field that corresponds to particles having mass squared s. Similar expressions can be obtained for the hamiltonian and propagators, as both quantities are linear in $\delta^+ G$ (cf. (34) and (36)).

In order to quantize the theory we have first computed the hamiltonian and verified that it is conserved when the field is on-shell. Then, we have imposed Heisenberg's equation and have obtained the commutation rules obeyed by the Fourier components of the field.

If the model is characterized by a kinetic term that leads to a mass spectrum that contains normal modes only ($k^2 \geq 0$) and $\delta^+ G(k)$ (see eq. (12)) has a positive definite sign, then the energy is positive definite, we can define the vacuum state in the usual way, and we can construct

a Fock-space with a positive definite metric (cf. eq. (35)). In this case the obtained propagators are of the Feynman type and it is a simple matter to make a Wick rotation in order to make contact with the path integral formulation. This is the case for the theory defined by (42) where the lagrangian $\mathcal{L} = \phi \square^{1-\alpha} \phi$ leads to the mass weight function $2(k^2)_+^{\alpha-1} \sin \pi\alpha \, \theta(k_0)$.

Applying the canonical formalism to a non-local gauge theory we obtained a continuum mass spectrum that contains the zero mass modes. That is, the vector field is a continuum superposition of modes that has a zero mass component. The gauge invariance is preserved due to the presence of these component. When a gauge transformation is performed, we can consider that the zero mass component of the field changes, while the rest of the modes remain unchanged. Finally we considered a particular case, by specializing to $(2+1)$ dimensions and considering a kinetic term $F_{\mu\nu} \square^{-1/2} F^{\mu\nu}$. In this case, we have interpreted the mass spectrum and have shown that the form of the propagator coincides with that obtained in the context of the projected effective action of Ref. [16]. Essentialy the massive modes of the electromagnetic field take into account that we are making a model where the matter is confined to live on a physical plane while the electromagnetic field is not confined.

Acknowledgments

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