La Plata-Th 98/22

Bosonic description of a Tomonaga-Luttinger model with impurities

Victoria Fernández[a], Kang Li[a,b] and Carlos Naón[a]

December 1998

Abstract

We extend a recently proposed non-local version of Coleman's equivalence between the Thirring and sine-Gordon models to the case in which the original fermion fields interact with fixed impurities. We explain how our results can be used in the context of one-dimensional strongly correlated systems (the so called Tomonaga-Luttinger model) to study the dependence of the charge-density oscillations on the range of the fermionic interactions.

Pacs: 05.30.Fk
11.10.Lm
71.10.Pm

---

[a] Depto. de Física. Universidad Nacional de La Plata. CC 67, 1900 La Plata, Argentina.
E-Mail: naon@venus.fisica.unlp.edu.ar

[b] Department of Physics, Zhejiang University, Hangzhou, 310028, P. R. China.

In a recent paper [1] a *non-local* generalization of Coleman's equivalence between the massive Thirring and sine-Gordon models [2] was established. Specifically it was proved the identity of the vacuum to vacuum functionals corresponding to the following (Euclidean) Lagrangian densities:

$$\mathcal{L}_T = i\bar{\Psi}\partial\!\!\!/\Psi + \frac{1}{2}g^2 \int d^2y J_\mu(x) V_{(\mu)}(x,y) J_\mu(y) - m\bar{\Psi}\Psi \tag{1}$$

and

$$\mathcal{L}_{SG} = \frac{1}{2}(\partial_\mu\phi(x))^2 + \frac{1}{2}\int d^2y \partial_\mu\phi(x) d_{(\mu)}(x,y)\partial_\mu\phi(y) - \frac{\alpha_0}{\beta^2}\cos\beta\phi, \tag{2}$$

where $J_\mu(x)$ is the usual fermion current, $V_{(\mu)}(x,y)$ and $d_{(\mu)}(x,y)$ $(\mu = 0,1)$ are arbitrary functions of $|x-y|$ whose Fourier transforms ($\hat{V}_{(\mu)}(p)$ and $\hat{d}_{(\mu)}(p)$) must be related by

$$\frac{1}{\frac{g^2}{\pi}(p_0^2\hat{V}_{(1)} + p_1^2\hat{V}_{(0)}) + p^2} = \frac{\beta^2}{4\pi(p^2 + \hat{d}_{(0)}p_0^2 + \hat{d}_{(1)}p_1^2)} \tag{3}$$

For the sake of clarity, we have omitted the p-dependence of the potentials. When $V_{(\mu)}(x,y) = \delta^2(x-y)$ and $d_{(\mu)}(x,y) = 0$ one reobtains the usual local equivalence (let us also mention that with the present convention attractive potentials correspond to positive values of $V_{(\mu)}$). Of course, the usual identity

$$m = \frac{\alpha_0}{\beta^2} \tag{4}$$

also holds in this *non-local* context.

The purpose of this letter is to extend the above result in two directions. First of all we shall consider a coordinate-dependent mass in our non-local version of the Thirring model. At the same time we will also add a local interaction between the fermion current and a classical background field C_μ. In other words, we shall modify equation (1) above by setting $m = 0$ and adding

$$\mathcal{L}_{imp} = \bar{\Psi}C\!\!\!/\Psi - m(x)\bar{\Psi}\Psi \tag{5}$$

where the name $\mathcal{L}_{imp}$ refers to the fact that this density can be used to study the interaction between electrons and impurities [3]. Indeed, if one uses (1) (with $m = 0$) in order to describe the forward scattering of one-dimensional spinless electrons [4] [5], then the first (second) term in the

rhs of (5) models the forward (backward) scattering between electrons and impurities. In particular, if we choose

$$C_0(x) = V\delta(x_0)\delta(x_1 - d) = m(x) \tag{6}$$

where V is a constant, together with

$$C_1(x) = 0, \tag{7}$$

our model coincides with the one recently considered in ref.[6] to study Friedel charge-density oscillations in a 1d Tomonaga-Luttinger liquid [7] with an impurity located at $x_1 = d$. Moreover, the bosonic vacuum to vacuum functional we shall derive here can also be used to compute the 2-point density function and the electron propagator as functionals of the electron-electron potential. The former was computed in ref.[8] (for a Coulomb potential and without impurities) where a Wigner crystal structure was revealed. Apart from academic interest, these connections are the main practical motivation for the present computation. Before making contact with this solid-state application, we now start by considering the general case ($C_\mu(x)$ and $m(x)$ arbitrary). Using a convenient representation of the functional delta and introducing the vector field A_μ (Please see ref. [1] for details), the partition function of the model under consideration can be written as

$$Z = \int DA_\mu det(i\not{\partial} + g\not{A} + \not{C} - m(x))e^{-S[A]}, \tag{8}$$

with

$$S(A) = \frac{1}{2}\int d^2x d^2y V^{-1}_{(\mu)}(x,y)A_\mu(x)A_\mu(y), \tag{9}$$

where $V^{-1}_{(\mu)}$ is such that

$$\int d^2z\ V^{-1}_{(\mu)}(z,x)V_{(\mu)}(y,z) = \delta^2(x-y). \tag{10}$$

As it is known, the massive-like determinant in equation (8) cannot be exactly solved, even in the local case. However, we were able to write the vacuum to vacuum functional in such a way that non-local terms are not present in the determinant. Then, following [1], we can decouple A_μ and C_μ from fermions by performing chiral and gauge transformations in the fermionic path-integral measure, with parameters $\Phi(x)$ and $\eta(x)$ respectively:

$$\Psi(x) = exp[-g(\gamma_5\Phi(x) + i\eta(x))]\chi(x), \tag{11}$$

(and a similar expression for $\bar{\Psi}(x)$) and writing

$$A_\mu(x) = \epsilon_{\mu\nu}\partial_\nu\Phi(x) - \partial_\mu\eta(x) - \frac{1}{g}C_\mu(x). \tag{12}$$

Taking into account the non-trivial Fujikawa Jacobian [9] associated to the fermionic transformation, which gives a local kinetic term for $\Phi(x)$, we obtain

$$Z = \int D\bar{\chi}D\chi D\Phi D\eta e^{-S_{eff}} \tag{13}$$

where

$$S_{eff} = S_{0B} + \int d^2x(\bar{\chi}i\partial\!\!\!/\chi - m(x)\bar{\chi}e^{-2g\gamma_5\Phi}\chi), \tag{14}$$

$$\begin{aligned} S_{0B} = \; & \tfrac{g^2}{2\pi}\int d^2x(\partial_\mu\Phi)^2 \\ & +\tfrac{1}{2}\int d^2xd^2y\epsilon_{\mu\lambda}\epsilon_{\mu\sigma}V^{-1}_{(\mu)}(y,x)\partial_\lambda\Phi(x)\partial_\sigma\Phi(y) \\ & +\tfrac{1}{2}\int d^2xd^2yV^{-1}_{(\mu)}(y,x)\partial_\mu\eta(x)\partial_\mu\eta(y) \\ & -\int d^2xd^2y[V^{-1}_{(0)}(y,x)\partial_0\eta(x)\partial_1\Phi(y) \\ & -V^{-1}_{(1)}(y,x)\partial_1\eta(x)\partial_0\Phi(y)] \\ & -\tfrac{1}{g}\int d^2xd^2yV^{-1}_{(\mu)}(x,y)C_\mu(y)(\epsilon_{\mu\sigma}\partial_\sigma\Phi(x) - \partial_\mu\eta(x)) \\ & +\tfrac{1}{2g^2}\int d^2xd^2yV^{-1}_{(\mu)}(x,y)C_\mu(x)C_\mu(y). \end{aligned} \tag{15}$$

Note that the last term of S_{OB} is field independent (remember that $C_\mu(x)$ is a classical function), thus its contribution can be absorbed in a path-integral normalization constant $N[C_\mu]$ which is relevant if one is interested in functional derivatives of Z with respect to $C_\mu(x)$. Since here we are mainly concerned with vacuum to vacuum functionals we shall disregard this factor. Its contribution will be recovered at the end of this work, where the evaluation of the fermion current v.e.v. will be sketched. Let us also mention that one should be more careful with this normalization when considering the finite temperature version of the present calculation, which could be easily done by following the lines of ref. [10]. On the other hand, the first term in (15) comes from the contribution of the previously mentioned fermionic Jacobian. Exactly as it was done in [1], the partition function for the model can be formally expanded in powers of $m(x)$. In fact, the x dependence of this perturbative parameter, together with the appearance of $C_\mu(x)$ in the bosonic action are the new features of the present computation. As far as these functions are well-behaved one can assume the existence of every term in the following series:

$$Z = \sum_{n=0}^{\infty} \frac{1}{n!} < \prod_{j=1}^{n} \int d^2x_j m(x_j) \bar{\chi}(x_j) e^{-2g\gamma_5 \Phi(x_j)} \chi(x_j) >_0 \tag{16}$$

It is not difficult to convince oneself that the same manipulations used in [1], in order to compute fermionic and bosonic v.e.v's at every order of perturbation theory, also work in the present case. This allows to write

$$\begin{aligned} Z \quad &= \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \int \prod_{i=1}^{k} d^2x_i d^2y_i m(x_i) m(y_i) exp\{ - \int \frac{d^2p}{(2\pi)^2} [\frac{2\pi}{p^2} \\ &- \frac{2\pi g^2 (\hat{V}_{(0)}^{-1} p_0^2 + \hat{V}_{(1)}^{-1} p_1^2)}{g^2 (\hat{V}_{(0)}^{-1} p_0^2 + \hat{V}_{(1)}^{-1} p_1^2) p^2 + \pi \hat{V}_{(0)}^{-1} \hat{V}_{(1)}^{-1} p^4}] D(p, x_i, y_i) D(-p, x_i, y_i) \} \\ &exp\{ - \int \frac{d^2p}{(2\pi)^2} \frac{2i \hat{V_{(\mu)}}^{-1}(-p) \hat{C}_\mu(-p)}{\Delta(p)} (C(p) p_\mu - 2\epsilon_{\mu\nu} p_\nu B(p)) D(-p, x_i, y_i) \} \end{aligned} \tag{17}$$

where, for simplicity, we have gone to momentum space and defined

$$A(p) = \frac{g^2}{2\pi}\, p^2 + \frac{1}{2} [\hat{V}_{(0)}^{-1}(p) p_1^2 + \hat{V}_{(1)}^{-1}(p) p_0^2], \tag{18}$$

$$B(p) = \frac{1}{2} [\hat{V}_{(0)}^{-1}(p) p_0^2 + \hat{V}_{(1)}^{-1}(p) p_1^2], \tag{19}$$

$$C(p) = [\hat{V}_{(0)}^{-1}(p) - \hat{V}_{(1)}^{-1}(p)] p_0 p_1, \tag{20}$$

$$\Delta = C^2(p) - 4A(p)B(p), \tag{21}$$

and

$$D(p, x_i, y_i) = \sum_i (e^{ipx_i} - e^{ipy_i}). \tag{22}$$

In these expressions $\hat{\Phi}, \hat{\eta}$ and $\hat{V}_{(\mu)}$ are the Fourier transforms of Φ, η and $V_{(\mu)}$ respectively. Equation (17) is our first non-trivial result. It is the extension of the result presented in [1] to the case $m \to m(x)$ and $C_\mu(x) \neq 0$.

Let us now briefly make our second relevant observation which concerns the way of constructing a purely bosonic model that gives the same expansion as in (17). It is straightforward to show that such model can be obtained from equation (2) by setting $\alpha_0 = 0$ and adding

$$\mathcal{L}'_{imp} = F_\mu(x) \partial_\mu \phi(x) - \frac{\alpha_0(x)}{\beta^2} \cos \beta\phi \tag{23}$$

where $F_\mu(x)$ represents a couple of classical functions to be related to the C_μ's and $\alpha_0(x)$ is the x-dependent version of α_0, to be related, of course, to $m(x)$. Again, going to momentum space and employing standard procedures to evaluate each v.e.v., the partition function Z' corresponding to this generalized sine-Gordon model can be expressed as

$$Z' = \sum_{k=0}^{\infty} [\frac{1}{k!}]^2 \int \prod_{i=1}^{k} d^2x_i d^2y_i \frac{\alpha(x_i)}{\beta^2} \frac{\alpha(y_i)}{\beta^2}$$
$$exp\{-\frac{\beta^2}{4} \int \frac{d^2p}{(2\pi)^2} \frac{D(p,x_i,y_i)D(-p,x_i,y_i)}{\frac{1}{2}p^2 + \frac{1}{2}(\hat{d}_{(0)}(p)p_0^2 + \hat{d}_{(1)}(p)p_1^2)}\}$$
$$exp\{\beta \int \frac{d^2p}{(2\pi)^2} \frac{\hat{F}_\mu(-p)p_\mu}{p^2 + p_0^2\hat{d}_0(p) + p_1^2\hat{d}_1(p)} D(-p,x_i,y_i)\} \tag{24}$$

By comparing equation (24) with equation (17), we find that both series are identical if not only equation (3) holds, but one also has:

$$m(x) = \frac{\alpha_0(x)}{\beta^2} \tag{25}$$

and

$$2i\hat{V}_{(\mu)}^{-1}(-p)\hat{C}_\mu(-p)\frac{p_\mu C(p) - 2\epsilon_{\mu\nu}p_\nu B(p)}{\Delta(p)} = \beta \frac{\hat{F}_\mu(-p)p_\mu}{p^2 + \hat{d}_{(0)}p_0^2 + \hat{d}_{(1)}p_1^2} \tag{26}$$

Therefore, we have obtained an equivalence between the partition functions Z and Z' corresponding to the non-local Thirring and sine-Gordon models with extra interactions defined above, in equations (5) and (23) respectively. Note that, apart from eq.(3) which was already obtained in [1] and eq.(25) which is a trivial generalization of eq.(4), eq.(26) should be considered the specific original contribution of the present work. As stated in the introductory paragraph, this path-integral identification allows to make contact with recent descriptions of 1d electronic systems (the so called Tomonaga-Luttinger model) in the presence of fixed (not randomly distributed) impurities [6]. We shall devote the remainder of this letter to briefly discuss this possibility. To be specific we shall consider the mean value of the fermionic current defined by:

$$\langle j_\mu \rangle = -\frac{1}{Z}\frac{\delta Z}{\delta C_\mu} \tag{27}$$

This is an interesting object because j_0 is the charge density whereas j_1 is the electric current. Please recall that the usual version of the Tomonaga-Luttinger model, where only density-density fluctuations are taken into account, corresponds to the choice [1]

$$\hat{V}_{(1)} = 0 \tag{28}$$

Concerning the impurity, it is evident that after performing the functional derivative in eq.(27) one has to use equations (6) and (7) for C_0 and C_1. At this point the way of exploiting the above depicted procedure, i.e. the equivalence between Z and Z', as a bosonizing scheme, becomes apparent. Indeed, one has just to enforce condition (28) in (3) and (26) and then use eq.(27) with Z' instead of Z. For the most relevant case of j_0 one has to keep in mind that there will be an extra term coming from the disregarded C_0-dependent factor in Z (Please remember the comments on $N[C_\mu]$ that followed equation (15)). However this is not a big trouble, since the corresponding contribution can be obtained by inspection yielding

$$\langle j_0(x) \rangle = -\frac{1}{Z'} \frac{\delta Z'}{\delta C_0(x)} + \frac{1}{g^2} V V_{(0)}^{-1}(x_1 - d) \tag{29}$$

We want to stress that this expression gives the charge density of a Tomonaga-Luttinger electronic liquid, with a non-magnetic impurity located at $x_1 = d$, as functional of the electron-electron potential. Thus our present proposal can be considered as an alternative path-integral approach to explore the joint effect of impurities and potentials on the charge density behavior. The explicit calculation of the first term would allow to check the result recently reported in ref.[6], where a strong dependence of the density oscillations on the range of the interaction was obtained . We hope to consider this issue in a forthcoming article [11].

Acknowledgments

K.L. thanks Prof. R.Gamboa Saraví for hospitality in Universidad Nacional de La Plata (UNLP), Argentina.
V.F. and C.N. are partially supported by Universidad Nacional de La Plata (UNLP) and Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET), Argentina. The authors also recognize the support of the Third World Academy of Sciences (TWAS).

References

[1] Kang Li and Carlos Naón, J. Phys. A **31** 7929(1998).

[2] S.Coleman, Phys. Rev. D **11**, 2088 (1975).

[3] C.L.Kane and M.P.A.Fisher,Phys.Rev.B**46** 15, 233, (1992).

[4] J.Voit, Rep. on Progr. in Phys. **58**, 977 (1995).

[5] C.Naón, M.C.von.Reichenbach and M.L. Trobo, Nucl. Phys. **B 435** [fs] 567 (1995).

[6] Q. Yuan, H.Chen, Y.Zhang and Y.Chen, Phys.Rev.B**57** 24, 1, (1998).

[7] S.Tomonaga, Prog. Theor. Phys. **5**, 544 (1950).
J.Luttinger, J. Math. Phys. **4**, 1154 (1963).
E.Lieb and D.Mattis, J. Math. Phys. **6**, 304 (1965).

[8] H.J.Schulz, Phys.Rev.Lett.**71**, 1864, (1993).

[9] K.Fujikawa, Phys. Rev. D **21** 2848 (1980).

[10] M.Manías, C.Naón y M.L.Trobo, Physics Letters B **416**, 157 (1998).

[11] Victoria Fernández, Kang Li and Carlos Naón, in preparation.