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Fermionic Coset Models as Topological Models

G.L. Rossini* and F.A. Schaposnik†
Departamento de Física, Universidad Nacional de La Plata‡
Argentina

Abstract

By considering the fermionic realization of G/H coset models, we show that the partition function for the $U(1)/U(1)$ model defines a Topological Quantum Field Theory and coincides with that for a 2-dimensional Abelian BF system. In the non-Abelian case, we prove the topological character of G/G coset models by explicit computation, also finding a natural extension of 2-dimensional BF systems with non-Abelian symmetry.

*Partially supported by CONICET, Argentina

†Fellow of the J.S.Guggenheim Memorial Foundation, USA. Investigador CICBA, Argentina

‡Postal address: C.C. 67, (1900) La Plata, Argentina

In the last few years coset models [1]-[2] raised much interest in the study of conformally invariant two-dimensional theories particularly in connection with String theories and with Statistical Mechanics models.[3]

G/H coset models can be realized by gauging a subgroup H of: (i) a Wess-Zumino-Witten (WZW) model with the basic field taking values on a Lie group G [4]-[5] or, alternatively, (ii) a free fermionic model with fermions in the fundamental representation of G [6]-[7].

Recently, Witten [8] analysed the holomorphic factorization of G/H models (in its bosonic realization) showing in particular that the G/G model defines a Topological field theory (i.e. a quantum field theory with metric independent partition function). We discuss this issue in the present note, by considering the *fermionic* realization of coset models.

The coset construction based on fermionic models goes as follows [7]. One starts with two-dimensional free Dirac fermions in the fundamental representation of G, with Lagrangian:

$$L_0 = \bar{\psi} i \partial\!\!\!/ \psi \tag{1}$$

Calling t^a the generators for $H \subset G$ one constructs the associated currents:

$$j_\mu^a = \bar{\psi} t^a \gamma_\mu \psi \tag{2}$$

Then, one imposes the condition that physical states $|phys>$ are singlet under these currents:

$$j_\mu^a |phys >= 0 \tag{3}$$

This is achieved in the path-integral formulation by introducing Lagrange multipliers A_μ^a which play the role of gauge fields in the Lie algebra of H. The partition function for the resulting constrained model reads:

$$Z_{G/H} = \int D\bar{\psi} D\psi \prod_a \delta(J_\mu^a) exp[-\int_M \sqrt{g} d^2x L_0] \tag{4}$$

or

$$Z_{G/H} = \int DA_\mu D\bar{\psi} D\psi exp[-\int_M \sqrt{g} d^2x \bar{\psi}(i\partial\!\!\!/ + A\!\!\!/)\psi] \tag{5}$$

with $\sqrt{g} = (det\, g_{\mu\nu})^{\frac{1}{2}}$ and $g_{\mu\nu}$ a metric on the two-dimensional manifold M. One easily verifies that the constrained model defined by (5) corresponds to a coset model with Virasoro central charge [7]:

$$c_{G/H} = c_G - c_H \tag{6}$$

Of particular interest are G/G models which have been shown to be (in their bosonic formulation) topological field theories [8]. Note that the fermionic version of the G/G model given by eq.(5) is just:

$$Z_{G/G} = \int DA_\mu det(i\partial\!\!\!/ + A\!\!\!/) \tag{7}$$

with A_μ taking values in the Lie algebra of G and hence $Z_{G/G}$ corresponds to the QCD_2 partition function in the infinite coupling constant limit. Of course, an appropriate gauge-fixing is necessary in (7).

As stated above, it is the purpose of this note to study G/G coset models. For the sake of clarity, we shall first consider the $G = U(1)$ case and then consider the non-Abelian extension. In the $U(1)$ case, the fermionic determinant in (7) takes the form [9]:

$$det(i\partial\!\!\!/ + A\!\!\!/) = exp[-\frac{1}{4\pi}\int_M d^2x\epsilon^{\mu\nu}F_{\mu\nu}\varphi] \times det\ i\partial\!\!\!/, \tag{8}$$

where A^μ and φ are related through the decomposition:

$$A^\mu = \frac{1}{\sqrt{g}}\epsilon^{\mu\nu}\partial_\nu\varphi + \partial^\mu\eta \tag{9}$$

and $\epsilon^{01} = -\epsilon^{10} = 1$. Note that:

$$\frac{1}{2\sqrt{g}}\epsilon^{\mu\nu}F_{\mu\nu} = -\Box\varphi \tag{10}$$

Now, in order to linearize the dependence of the fermionic determinant (8) on A_μ, we introduce a scalar field ϕ through the identity:

$$det(i\partial\!\!\!/ + A\!\!\!/) = \frac{det\ i\partial\!\!\!/}{det^{-\frac{1}{2}}\Box}\int D\phi\exp(-\frac{1}{8\pi}\int_M\phi\Box\phi\sqrt{g}d^2x - \frac{1}{4\pi}\int_M\epsilon^{\mu\nu}F_{\mu\nu}\phi d^2x) \tag{11}$$

integration and Determinants on the r.h.s. of eq.(11) coincide with the partition function for free Dirac fermions and with the one for free bosons, both in the presence of a background metric. Due to the boson-fermion connection in two dimensions, they define equivalent theories. Without lack of generality one can choose $g_{\mu\nu}$ as a conformally flat metric, $g_{\mu\nu} = \exp(\sigma)\delta_{\mu\nu}$. One can then show that [10]:

$$det^{-\frac{1}{2}}\Box = det\ i\partial\!\!\!/ = \exp(S_L[\sigma]) \tag{12}$$

where S_L is the Liouville action for the scalar field σ. (We have disregarded in eq.(12) metric independent constants). Hence, their contribution in eq.(11) cancels out.

With this, the partition function for the $U(1)/U(1)$ coset model can be written as

$$Z_{U(1)/U(1)} = \int DA_\mu D\phi D\bar{c}DcD\pi exp(-S_Q) \tag{13}$$

with

$$S_Q = \frac{1}{8\pi}\int_M \phi\Box\phi\sqrt{g}d^2x + \frac{1}{4\pi}\int_M \epsilon^{\mu\nu}F_{\mu\nu}\phi d^2x + \int_M \{Q, \bar{c}G[A_\mu]\}\sqrt{g}d^2x \tag{14}$$

where the gauge fixing term corresponds to some gauge condition $G[A] = 0$ and BRST transformations $\{Q,\ \}$ are defined as:

$$\begin{array}{ll} \{Q, A_\mu\} = -\partial_\mu c & \{Q, c\} = 0 \\ \{Q, \bar{c}\} = \pi & \{Q, \pi\} = 0 \\ \{Q, \phi\} = 0 & \end{array} \tag{15}$$

Here c and $\bar{c}$ are ghost fields and π is a Lagrange multiplier.

For simplicity of the arguments below, we choose the axial gauge which in light cone coordinates reads:

$$G[A_\mu] \equiv A_- = 0 \tag{16}$$

so that the gauge fixing term becomes

$$\{Q, \bar{c}_+ A_-\} = \pi_+ A_- - \bar{c}_+\partial_- c \tag{17}$$

It is now easy to prove by explicit computation that $Z_{U(1)/U(1)}$ does not depend on the metric. Indeed, the ghost field integration gives as Fadeev-Popov determinant $det\,\partial_-$. As the π integration implements the gauge condition yielding $\delta(A_-)$, the A_- integration is trivial so that one gets

$$Z_{U(1)/U(1)} = det\,\partial_- \int DA_+ D\phi \exp(-\frac{1}{8\pi}\int_M (\phi\Box\phi\sqrt{g} - 4\partial_- A_+\phi)d^2x) \tag{18}$$

Now the A_+ integration imposes the constraint $\partial_-\phi = 0$ and hence the Laplacian in the exponential vanishes. We have finally

$$Z_{U(1)/U(1)} = det\, \partial_- \int D\phi \delta(\partial_-\phi) = 1 \tag{19}$$

We have then proved the metric independence of $Z_{U(1)/U(1)}$, i.e. the topological character of the $U(1)/U(1)$ coset model. Of course, given a theory defined on a manifold M with a *fixed metric*, the corresponding partition function is a number which can be normalized to 1. What eq.(19) means is that this normalization does not change when the metric is varied. (The extension of our proof to an arbitrary gauge condition is trivial.)

The same result can be more elegantly obtained by connecting the partition function in eqs.(13-14) with that of an Abelian BF system[1] (see [11] and references therein). To see this, let us perform in eq.(14) the following change of variables

$$A_\mu \to A_\mu + \frac{\sqrt{g}}{4}\epsilon_{\mu\nu}\partial^\nu\phi \tag{20}$$

which leaves invariant the path-integral measure in (13). After this change $Z_{U(1)/U(1)}$ takes the form:

$$S_Q = \frac{1}{4\pi}\int_M \epsilon^{\mu\nu} F_{\mu\nu}\phi d^2x + \int_M \{Q, \bar{c}\tilde{G}[A_\mu]\}\sqrt{g}d^2x \tag{21}$$

Eq.(21) is the quantum action corresponding to a BF system for a scalar field ϕ and a gauge field A_μ, with classical action

$$S_{BF} = \frac{1}{4\pi}\int_M \epsilon^{\mu\nu} F_{\mu\nu}\phi d^2x \tag{22}$$

and the appropriate gauge fixing (note that in the case we were working in the Landau gauge, the gauge fixing functional $G[A_\mu] = \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}A^\mu)$ would remain unchanged after performing the change of variables). Then, taking into account the invariance of the path-integral measure DA_μ, eq.(13) becomes the partition function for the BF system:

$$Z_{U(1)/U(1)} = Z_{BF} \tag{23}$$

[1]We thank M. Henneaux for suggesting this connection.

It is well known [11] that the partition function of an Abelian BF system defined on an n-dimensional manifold M_n is a topological invariant; moreover, it gives some power of the Ray-Singer torsion of M_n, which is one in even-dimensional manifolds. In fact, one can prove this last result just by following analogous steps of those leading from eq.(13) to eq.(19). Thus, the relation (23) is consistent with our result in eq.(19).

Let us now extend our derivation to the case of a non-Abelian group G (which will be taken as a compact Lie group). The fermion determinant appearing in the partition function $Z_{G/G}$ (eq.(7)) can be written, in the $A_- = 0$ gauge, in the form [12]:

$$det(i\partial\!\!\!/ + A\!\!\!/) = exp(I[h]) \times det\; i\partial\!\!\!/ \tag{24}$$

with h a G-valued field related to A_+ through:

$$A_+ = h^{-1}\partial_+ h \tag{25}$$

and $I[h]$ the WZW action:

$$I[h] = \frac{1}{8\pi} tr \int_M d^2x \sqrt{g}\partial^\mu h^{-1}\partial_\mu h + \frac{1}{12\pi} tr \int_B d^3y \epsilon^{ijk} h^{-1}\partial_i h h^{-1}\partial_j h h^{-1}\partial_k h \tag{26}$$

Here B is a three dimensional manifold such that $\partial B = M$. The third coordinate in B, which we call t, will be taken as usual as $t \in [0,1]$. Then $h(x,t)$ is an extension of $h(x)$ over B such that $h(x,1) = h(x)$ and $h(x,0) = 1$, the unit element of G.

The dependence of the fermionic determinant (24) on A_μ can be linearized, as in the Abelian case, by introducing a G-valued scalar field g. Indeed, using the Polyakov-Wiegmann identity [13]

$$I[hg] = I[g] + I[h] - \frac{1}{4\pi} tr \int_M h^{-1}\partial_+ h \partial_- g g^{-1} d^2x \tag{27}$$

one can easily see that

$$det(i\partial\!\!\!/ + A\!\!\!/) = \frac{det\; i\partial\!\!\!/}{Z_{WZW}} \int Dg\; \exp(-I[g] + \frac{1}{4\pi} tr \int_M (A_+ \partial_- g g^{-1}) d^2x) \tag{28}$$

where

$$Z_{WZW} = \int Du \, \exp(-I[u]) \tag{29}$$

is the WZW partition function and the argument in the exponential in eq.(28) is minus the gauged WZW action in the $A_- = 0$ gauge.

The determinant appearing in the r.h.s. of eq.(28) corresponds to the partition function for free Dirac fermions in the fundamental representation of G while Z_{WZW} is the partition function for the equivalent bosonic theory. One can again show [14], exploiting the (non-Abelian) boson-fermion equivalence in two dimensions, that these partition functions are identical:

$$det \, i\partial\!\!\!/ = Z_{WZW} \tag{30}$$

Putting all this together we have for the G/G coset model partition function:

$$Z_{G/G} = \int DA_+ DA_- Dg D\bar{c}_+ Dc D\pi_+ \, \exp(-S_Q) \tag{31}$$

with

$$S_Q = I[g] - \frac{1}{4\pi} tr \int_M (A_+ \partial_- g g^{-1}) d^2x + tr \int_M \{Q, \bar{c}_+ A_-\} \sqrt{g} d^2x \tag{32}$$

Here the gauge fixing corresponds to the gauge condition $A_- = 0$ and BRST transformations are defined as

$$\begin{array}{ll} \{Q, A_\mu\} = -D_\mu c & \{Q, c\} = \frac{1}{2}[c, c] \\ \{Q, \bar{c}_+\} = \pi_+ & \{Q, \pi_+\} = 0 \\ \{Q, g\} = -[g, c] & \end{array} \tag{33}$$

with $\bar{c}_+$ and c ghost fields and π_+ a Lagrange multiplier, all of them taking values in the Lie algebra of G. The explicit form of the gauge fixing term in eq.(32) is

$$\int_M \{Q, \bar{c}_+ A_-\} \sqrt{g} d^2x = \int_M (\pi_+ A_- - \bar{c}_+ D_- c) \sqrt{g} d^2x \tag{34}$$

As in the Abelian case, we can now perform the explicit computation of eq.(31). The ghost field integration yields the Fadeev-Popov determinant $det \, D_-$, while the π_+ integration implements the gauge condition $\delta(A_-)$. The A_- integration then sets $det \, D_- = det \, \partial_-$ and one then ends with

$$Z_{G/G} = \int DA_+ Dg \, det \, \partial_- \, \exp(-I[g] + \frac{1}{4\pi} tr \int_M A_+ \partial_- g g^{-1} d^2x) \tag{35}$$

We see that the A_+ integration in eq.(35) imposes the constraint $\partial_- g g^{-1} = 0$ for each point on the manifold M. Moreover, one can find an appropriate extension of $g(x)$ over B such that $\partial_- g g^{-1} = 0$ for every point in B. With this eq.(35) becomes

$$Z_{G/G} = det\, \partial_- \int Dg\, \delta(\partial_- g g^{-1}) \tag{36}$$

The integration over g is most easily performed by writing $g = \exp(\alpha)$ (α in the Lie algebra of G) and integrating over α. Using

$$\frac{\delta(\partial_- g g^{-1})}{\delta \alpha}|_{\partial_- g g^{-1}=0} = \partial_- \tag{37}$$

we then finally get

$$Z_{G/G} = 1 \tag{38}$$

Hence, as in the Abelian case, we have proved that $Z_{G/G}$ is metric independent thus defining a topological quantum field theory (This proof should be extended to an arbitrary gauge without difficulty).

It is important to stress at this point that for G/H coset models with $H \neq G$, an identity analogous to (30) is not valid. Indeed, for $H \neq G$, g should belong to subgroup H and A_μ to its Lie algebra, while fermions should still be in the fundamental representation of G. Then, following the steps described above, one should arrive to a relation of the form (30) with $det\, i\partial\!\!\!/$ still being the partition function for free Dirac fermions in the fundamental representation of G while Z_{WZW} would correspond to a partition function of H-valued WZW fields. Hence, these two partition functions would not cancel each other as they do for $H = G$ and $Z_{G/H}$ would be *metric-dependent*.

Let us now discuss the non-Abelian analogue of the steps leading to the equivalence between the Abelian coset model and a BF system. After some algebra, S_Q in eq.(32) can be written as

$$\begin{aligned} S_Q = \frac{1}{4\pi} \int_M d^2x \int_0^1 dt\, tr[g^{-1}(x,t)\partial_t g(x,t) \\ \partial_-(g^{-1}(x,t)A_+(x)g(x,t) + g^{-1}(x,t)\partial_+ g(x,t))] + \\ +tr \int_M \{Q, \bar{c}_+ A_-\} \sqrt{g} d^2x \end{aligned} \tag{39}$$

Now, defining a field $\tilde{A}_+(x,t)$ over B as

$$\tilde{A}_+(x,t) = g^{-1}(x,t)A_+(x)g(x,t) + g^{-1}(x,t)\partial_+ g(x,t) \tag{40}$$

(compare with the transformation in eq.(20) for the Abelian case, setting $g(x,t) = \exp(it\phi(x))$ and noting that $\partial_+ = \frac{1}{\sqrt{g}}\partial^-$ in a conformally flat metric) we can write

$$S_Q = \frac{1}{4\pi}\int_M d^2x \int_0^1 dt\, tr[\partial_- \tilde{A}_+(x,t) g^{-1}(x,t)\partial_t g(x,t)] + \\ + tr\int_M \{Q, \bar{c}_+ A_-\}\sqrt{g} d^2x \qquad (41)$$

Note that, as in the WZW model, though $\tilde{A}_+(x,t)$ appears in the first integral in eq.(41), S_Q is a functional of $\tilde{A}_+$ on M, i.e. a functional of $\tilde{A}_+(x,1)$. So we can change variables from $A_+(x)$ to $\tilde{A}_+(x,1)$ in the path-integral (31). From eq.(40) we see that the Jacobian associated with this change is trivial, and hence we get

$$Z_{G/G} = \int D\tilde{A}_+ DA_- Dg D\bar{c}_+ Dc D\pi_+ \exp(-S_Q) \qquad (42)$$

with S_Q given by eq.(41). Note that in terms of the integration variable $\tilde{A}_+(x,1)$, we can write

$$\tilde{A}_+(x,t) = u^{-1}(x,t)\tilde{A}_+(x,1)u(x,t) + u^{-1}(x,t)\partial_+ u(x,t) \qquad (43)$$

with $u(x,t) = g^{-1}(x,1)g(x,t)$. Comparing expression (41) with the one obtained for the Abelian case (eq.(21)), we see that it is sensible to write

$$S_Q = \tilde{S}_{BF} + tr\int_M \{Q, \bar{c}_+ A_-\}\sqrt{g} d^2x \qquad (44)$$

with

$$\tilde{S}_{BF} = \frac{1}{4\pi}\int_M d^2x \int_0^1 dt\, tr[\partial_- \tilde{A}_+(x,t) g^{-1}(x,t)\partial_t g(x,t)] \qquad (45)$$

representing the natural extension of the 2-dimensional Abelian BF system defined by action (22) to the non-Abelian case. With this interpretation not only we have again

$$Z_{G/G} = \tilde{Z}_{BF} = 1 \qquad (46)$$

but also parallel the route followed when one extends the bosonization recipe from the Abelian to the non-Abelian case. Both in the bosonization procedure and in our proof above, the basic objects in the non-Abelian case are constructed from group elements g and one needs an extension of the original

2-dimensional manifold M to the ball B in order to have a closed expression for the Lagrangians (eqs. (26) and (45)). One can then conclude that the non-Abelian version of BF systems discussed in the literature, consisting in writing an action like in eq.(22) but with ϕ and $F_{\mu\nu}$ in the Lie algebra of G, is just the counterpart, when studying BF systems, of taking $\int \partial_\mu \phi^a \partial^\mu \phi^a d^2x$ as the bosonized form of a 2-dimensional fermionic theory with symmetry group G. As it is well-known, a major limitation of this bosonization procedure is, however, that the non-Abelian symmetry is not preserved by the bosonization. In view of the connection of fermionic and bosonic versions of coset models, we then prefer to consider $\tilde{S}_{BF}$ as defined in eq.(45) as the natural non-Abelian extension of BF systems.

Let us end this work by discussing a second supersymmetry (appart from BRST symmetry), which can be implemented in the $U(1)/U(1)$ model (and presumably extended to the non-Abelian case). For that purpose, we first choose the Landau gauge $G[A_\mu] = \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}A^\mu)$ to write S_Q in eq.(14) as:

$$S_Q = \frac{1}{8\pi}\int_M \phi\Box\phi\sqrt{g}d^2x + \frac{1}{4\pi}\int_M \epsilon^{\mu\nu}F_{\mu\nu}\phi d^2x + \\ + \int_M (\pi\frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}A^\mu) - \bar{c}\Box c)\sqrt{g}d^2x \tag{47}$$

Calling Q^* the generator associated to this second supersymmetry, and following Soda [16] in his analysis of two-dimensional Maxwell theory, we define transformation laws in the form:

$$\begin{array}{ll} \{Q^*, A_\mu\} = -\sqrt{g}\epsilon_{\mu\nu}\partial^\nu c & \{Q^*, c\} = 0 \\ \{Q^*, \bar{c}\} = -\frac{1}{2\pi}\phi & \{Q^*, b\} = 0 \\ \{Q^*, \phi\} = 0 & \end{array} \tag{48}$$

Now, we note that S_Q, defined in eq.(47), satisfies not only $\{Q, S_Q\} = 0$ but also:

$$\{Q^*, S_Q\} = 0 \tag{49}$$

Indeed:

$$\{Q^*, \frac{1}{4\pi}\int_M d^2x\epsilon^{\mu\nu}F_{\mu\nu}\phi\} = -\frac{1}{2\pi}\int_M d^2x\phi\epsilon^{\mu\nu}\partial_\mu(\sqrt{g}\epsilon_{\nu\alpha}\partial^\alpha c) \tag{50}$$

or

$$\{Q^*, \frac{1}{4\pi}\int_M d^2x \epsilon^{\mu\nu} F_{\mu\nu}\phi\} = \frac{1}{2\pi}\int_M \sqrt{g} d^2x \phi \Box c \tag{51}$$

while

$$\{Q^*, \int_M \sqrt{g} d^2x \bar{c} \Box c\} = -\frac{1}{2\pi}\int_M \sqrt{g} d^2x \phi \Box c \tag{52}$$

and the other terms in S_Q are Q^*-invariant separatedly.

Let us note that S_Q can be written in the form:

$$S_Q = \frac{1}{4\pi}\int_M d^2x \epsilon^{\mu\nu} F_{\mu\nu}\phi + \{Q, V\} + \{Q^*, W\} \tag{53}$$

with

$$V = \int_M d^2x \bar{c} \partial_\mu (\sqrt{g} A^\mu) \quad W = -\tfrac{1}{4}\int_M \sqrt{g} d^2x \bar{c} \Box \phi \tag{54}$$

and that all metric dependence in S_Q is in the two last terms in the r.h.s. of eq.(53). One then has:

$$\frac{1}{Z_{U(1)/U(1)}} \frac{\delta Z_{U(1)/U(1)}}{\delta g_{\mu\nu}} =< \{Q, \frac{\delta V}{\delta g_{\mu\nu}}\} > + < \{Q^*, \frac{\delta W}{\delta g_{\mu\nu}}\} > \tag{55}$$

The l.h.s. in eq.(55) is zero due to the topological character of the model. This, together with the condition:

$$Q|phys >= 0 \tag{56}$$

on physical states $|phys>$ means that:

$$< \{Q^*, \frac{\delta W}{\delta g_{\mu\nu}}\} >= 0 \tag{57}$$

It is interesting to note that in his analysis of the two-dimensional Maxwell model, which is not in principle a topological one by itself [17, 18, 19], Soda had to impose $Q^*|phys >= 0$ in order to define a topological theory. In contrast, in the $U(1)/U(1)$ case, we have shown that (57) holds due to the topological character of the coset model, without imposing Soda's condition. Let us finally mention that the steps leading to (55) can be repeated using an explict invariant measure as for example Fujikawa's measure (see [20]).

In summary, we have been able to show that G/G models are topological by starting from their fermionic realization. That is, $Z_{G/G}$ is independent of the metric of the 2-dimensional manifold M on which the model is defined. We have also established a connection with BF systems provided in the non-Abelian case one considers a new class of such models.

Since Topological Quantum Field Theories are characterized by observables which depend only on the global features of M, it should be of interest to study in detail correlation functions for the fermionic realization of G/G model as a way of obtaining novel representations of global invariants. From the point of view of Quantum Field Theory, the connection of G/G models with QCD_2 at strong coupling opens a new route to the analysis of 2-dimensional Yang-Mills theory with matter, using Topological Quantum Field Theory tools. We hope to report on these issues elsewhere.

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