Duality in deformed coset fermionic models

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ABSTRACT

We study the $SU(2)_k/U(1)$-parafermion model perturbed by its first thermal operator. By formulating the theory in terms of a (perturbed) fermionic coset model we show that the model is equivalent to interacting WZW fields modulo free fields. In this scheme, the order and disorder operators of the Z_k parafermion theory are constructed as gauge invariant composites. We find that the theory presents a duality symmetry that interchanges the roles of the spin and dual spin operators. For two particular values of the coupling constant we find that the theory recovers conformal invariance and the gauge symmetry is enlarged. We also find a novel self-dual point.

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i) Introduction

In the present article we reformulate the $SU(2)_k/U(1)$ (Z_k) parafermion model [1, 2] perturbed by its first thermal operator [3, 4], in terms of a constrained fermionic model. This model has recently attracted much attention [5, 6, 7] in connection with the complex sine-Gordon theory, and the study of classical solutions has been extensively developed.

Although this model has been widely studied, almost all the work done about it dealt with its classical aspects. It is the purpose of the present note to give a suitable scenario to study its quantum aspects. We show that the formulation of the model using constrained fermionic models is particularly useful for this purpose. In particular, the connection between the perturbed Z_k parafermion model and a Gross-Neveu-like model is made apparent. We also construct the Z_k primaries in terms of composite fermions, thus giving an explicit realization of the Fateev-Zamolodchikov parafermion algebra [2].

The perturbed system presents a remarkable duality transformation: the theories with coupling constants β and $-\beta/(1 - 2(1 + 2k)\beta)$ are equivalent but the roles of the spin fields σ and the dual spin fields μ are interchanged. This transformation is self-dual at the point $\beta = 1/(2k + 1)$.

We also show the existence of two non-trivial fixed points (mutually related by the duality symmetry), where conformal invariance is explicit and the non-abelian symmetry, originally $SU(k)$, is enlarged to $SU(k) \times SU(k)$.

ii) $SU(2)_k$ Wess-Zumino-Witten (WZW) theory as a fermionic coset

To start with, let us recall the fermionic coset formulation of the $SU(2)_k$ WZW theory [8, 9, 10]. The action is given by

$$S_0 = -\frac{1}{\sqrt{2\pi}} \int d^2x \psi^{\dagger i\alpha} \left((i\not{\partial} + i\not{a})\delta_{ij}\delta_{\alpha\beta} + i\delta_{ij} \not{B}_{\alpha\beta} \right) \psi^{j\beta}, \tag{1}$$

where the fermions $\psi^{i\alpha}$ ($i = 1, 2$, $\alpha = 1, ..., k$) are in the fundamental representation of $U(2k)$ and the $U(1)$ gauge field a_μ and the $SU(k)$ gauge field B_μ act as Lagrange multipliers implementing the constraints

$$j_\mu|phys>= 0, \qquad J_\mu^a|phys>= 0 \tag{2}$$

for j_μ the $U(1)$ and J_μ^a the $SU(k)$ currents respectively. This corresponds to the identification:

$$SU(2)_k \equiv \frac{U(2k)}{SU(k)_2 \times U(1)}, \tag{3}$$

which is understood as an equivalence between the correlation functions of corresponding fields in the two theories.

The fundamental field g and its adjoint $g^\dagger$ of the bosonic $SU(2)_k$ WZW theory are represented in terms of fermions by the bosonization formulae[1] [8]

$$
\begin{aligned}
g^{ij} &= \psi_2^i \psi_2^{j\dagger} \\
g^{ij\dagger} &= \psi_1^i \psi_1^{j\dagger},
\end{aligned}
\tag{4}
$$

where the $SU(k)$ indices are summed out.

Higher spin integrable representations are constructed as symmetrized products of these fundamental fields

$$
\begin{aligned}
g_{j_1,\ldots,j_{2j}}^{(j)\; i_1,\ldots,i_{2j}} &= \mathcal{S}\left(: g_{j_1}^{i_1}\ldots g_{j_{2j}}^{i_{2j}} :\right) \\
&= \mathcal{S}\left(: \psi_2^{i_1}\psi_2^{\dagger j_1}\ldots\psi_2^{i_{2j}}\psi_2^{\dagger j_{2j}} :\right),
\end{aligned}
\tag{5}
$$

where $j = 0, 1/2, 1, \ldots, k/2$. This restriction in the spin of the representation has its origin in the selection rules imposed by the Kac-Moody symmetry [11, 12].

In the fermion representation, the presence of a second index α in the fermion fields $\psi^{i\alpha}$, running from 1 to k, allows for the construction of symmetrized products of at most k bilinears. In this way we obtain only the allowed integrable representations of spin $j = 0, 1/2, 1, \ldots, k/2$ and higher spin representations are forbidden by the Pauli principle.

Before going to the perturbed case, we will first construct the partition function of the Z_k parafermion theory as the fermionic coset $SU(2)_k/U(1)$ and construct the primary fields in terms of the constrained fermions. The construction of these primary fields is one of our new results. As far as we know, an explicit representation of the primary fields in Z_k parafermion models starting from first principles has not been presented before, although some work related to classical aspects has been done [13].

iii) Z_k formulated as the coset $SU(2)_k/U(1)$

To carry out this program we have to further mod out a $U(1)$ subgroup of the unconstrained $SU(2)$ group in the $SU(2)_k$ theory (1). We will do this by freezing the component of the $SU(2)$ current in the direction of t^3,

$$
j_\mu^3 =: \psi^{i\alpha\dagger}\gamma_\mu t_{ij}^3 \psi^{j\alpha} :.
\tag{6}
$$

To implement this constraint in the path-integral formulation we have to add to the action a term of the form

$$
-\frac{i}{\sqrt{2\pi}}\int d^2 x\, \psi^{i\alpha\dagger}\, b^3 t_{ij}^3 \psi^{j\alpha},
\tag{7}
$$

---

[1] Our conventions are $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ and γ_i are the Pauli matrices.

and integrate over the gauge field b_μ^3, thus obtaining

$$\mathcal{Z}_{SU(2)_k/U(1)} = \int Da_\mu DB_\mu Db_\mu^3 D\psi^\dagger D\psi e^{-\mathcal{S}}, \tag{8}$$

where

$$\mathcal{S} = \frac{1}{\sqrt{2}\pi} \int d^2x \psi^{\dagger i\alpha} \left((i\slashed{\partial} + i\slashed{a})\delta_{ij}\delta_{\alpha\beta} + i\delta_{ij}\slashed{B}_{\alpha\beta} + i\slashed{b}^3 t_{ij}^3 \delta_{\alpha\beta} \right) \psi^{j\beta}. \tag{9}$$

The Virasoro central charge of this model can be evaluated using standard methods (see the appendix) and is given by

$$c = \frac{2(k-1)}{k+2}, \tag{10}$$

which corresponds to the central charge of the Z_k-parafermion theory.

The parafermion models and their integrable perturbations have been widely studied [2, 3, 4] using Conformal Field Theory techniques. It was shown in particular that the primary fields of the Z_k-parafermion theory, $\phi_{m,\bar{m}}^{(j)}$, are closely related to the primaries of the $SU(2)_k$ WZW theory, $\Phi_{m,\bar{m}}^{(j)}$ [2]. Indeed, they are connected by the relation

$$\Phi_{m,\bar{m}}^{(j)}(z,\bar{z}) = \phi_{2m,2\bar{m}}^{(2j)}(z,\bar{z}) : \exp\left(i(\frac{m}{\sqrt{k}}\varphi(z) + \frac{\bar{m}}{\sqrt{k}}\bar{\varphi}(\bar{z})) \right) :, \tag{11}$$

where $\varphi(z)$ and $\bar{\varphi}(\bar{z})$ are the holomorphic and antiholomorphic components of an auxiliary massless free boson field.

However, although this relation has proven to be useful, it should be desirable to have an explicit representation of the primaries in the Z_k-parafermion models.

Let us first recall which are the more relevant primaries in Z_k parafermion models.

Among the primaries $\phi_{m,\bar{m}}^{(j)}$, there are the "spin" fields $\sigma_p = \phi_{p,\bar{p}}^{(p)}$ and the "dual spin" fields $\mu_p = \phi_{p,-\bar{p}}^{(p)}$, with conformal dimensions

$$h_p = \bar{h}_p = \frac{p(k-p)}{2k(k+2)} \tag{12}$$

where the duality (Krammers-Wannier) symmetry corresponds to the interchange

$$\sigma \leftrightarrow \mu. \tag{13}$$

Other relevant primaries are the so-called parafermion currents, $\Psi_p^{PF} = \phi_{2p,0}^{(0)}$ and $\bar{\Psi}_p^{PF} = \phi_{0,2p}^{(0)}$, $p = 1, 2..., k-1$, with conformal dimensions $(h, \bar{h})$ given by

$$(\frac{p(k-p)}{k}, 0) \quad \text{and} \quad (0, \frac{p(k-p)}{k}) \tag{14}$$

respectively and the "thermal" operators, $\epsilon_j = \phi_{0,0}^{(2j)}$, $j = 1, 2, ... \leq k/2$, with dimensions $h_j = \bar{h}_j = j(j+1)/(k+2)$. These thermal operators are the only ones explicitly represented by eq.(11) in terms of the WZW $SU(2)_k$ primaries.

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We will now show how these primary fields can be constructed as gauge invariant fermion composites in the theory described by the partition function (8).

In order to identify the physical fields in this theory, we have to seek for gauge invariant operators, which will be constructed using the gauge invariant fermions [14].

The fields in (4) and (5) are already gauge invariant under gauge transformations associated with the gauge fields a_μ and B_μ, but they vary under gauge transformations associated with the new gauge field, b^3_μ, introduced in eq.(7). In order to ensure invariance also under these transformations, we will define the gauge invariant fermions as

$$\hat{\psi} = e^{-i \int_x^\infty dz^\mu b^3_\mu} \psi. \tag{15}$$

Using these fields we can construct the gauge invariant version of the g-field and its adjoint in eq.(4) [15]

$$\begin{aligned}
\hat{g}_{ij} &= \hat{\psi}_2^i \hat{\psi}_2^{j\dagger}, \\
\hat{g}_{ij}^\dagger &= \hat{\psi}_1^i \hat{\psi}_1^{j\dagger}
\end{aligned} \tag{16}$$

and using as a guideline the identification made in ref.[2], i.e. eq.(11), we will study the fields

$$\begin{aligned}
\sigma_1 &\equiv \hat{g}_{1,1} + h.c. \ , \\
\mu_1 &\equiv \hat{g}_{2,1}^\dagger + h.c.
\end{aligned} \tag{17}$$

These fields have dimensions (see the appendix)

$$h = \bar{h} = \frac{k-1}{2k(k+2)} \tag{18}$$

and satisfy the dual algebra [14]

$$\sigma_1(x_1)\mu_1(x_2) = e^{\frac{i2\pi}{k}\Theta(x_1-x_2)}\mu_1(x_2)\sigma_1(x_1). \tag{19}$$

Eqs. (18) and (19) lead one to identify the fields σ and μ defined in (17) with the order and disorder operators in the Z_k parafermion theory.

The other spin fields, $(p > 1)$, are obtained by using eq.(5) but with the gauge invariant fermions, and following the same lines as above, it can be shown that the fields

$$\begin{aligned}
\sigma_p &\equiv \ : \underbrace{\hat{g}_{1,1}\hat{g}_{1,1}\ldots\hat{g}_{1,1}}_{p-times} : + \ h.c. \ , \\
\mu_p &\equiv \ : \underbrace{\hat{g}_{2,1}^\dagger \hat{g}_{2,1}^\dagger \ldots \hat{g}_{2,1}^\dagger}_{p-times} : + \ h.c. \ ,
\end{aligned} \tag{20}$$

have dimensions given by eq.(12) and satisfy the algebra

$$\sigma_p(x_1)\mu_{p'}(x_2) = e^{\frac{i2pp'\pi}{k}\Theta(x_1-x_2)}\mu_{p'}(x_2)\sigma_p(x_1), \tag{21}$$

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as required.

The Krammers-Wannier symmetry corresponds in this context to the interchange of the fermion components:

$$\psi_2^{1\alpha} \leftrightarrow \psi_1^{1\alpha\dagger} , \qquad \psi_2^{1\alpha\dagger} \leftrightarrow \psi_1^{2\alpha}. \tag{22}$$

Using eq.(16) we can write the duality transformation as

$$\hat{g}_{ij} \to (\hat{g}\gamma_1)_{ij}^\dagger , \tag{23}$$

that resembles the one proposed in ref.[7] but involves the gauge invariant version of the $SU(2)$ field and is valid for any k.

Besides the bilinears defined in eq.(16), we can construct another class of bilinears in terms of the gauge invariant fermions, which have no local counterpart in the bosonic formulation. Let us consider the fields

$$\hat{\psi}_1^2 \hat{\psi}_2^{1\dagger} , \qquad \hat{\psi}_2^2 \hat{\psi}_1^{1\dagger}. \tag{24}$$

These fields have conformal dimensions $(h, \bar{h})$ (see the appendix)

$$(\frac{k-1}{k}, 0) \quad \text{and} \quad (0, \frac{k-1}{k}), \tag{25}$$

respectively and then can be identified with the basic parafermion currents,

$$\Psi_1^{PF} = \hat{\psi}_1^2 \hat{\psi}_2^{1\dagger} + h.c. \quad \text{and} \quad \bar{\Psi}_1^{PF} = \hat{\psi}_2^2 \hat{\psi}_1^{1\dagger} + h.c. \tag{26}$$

The other parafermion currents are constructed as

$$\Psi_p^{PF} \equiv : \underbrace{\hat{\psi}_1^2 \hat{\psi}_2^{1\dagger} \hat{\psi}_1^2 \hat{\psi}_2^{1\dagger}...\hat{\psi}_1^2 \hat{\psi}_2^{1\dagger}}_{p-times} : + h.c. ,$$

$$\bar{\Psi}_p^{PF} \equiv : \underbrace{\hat{\psi}_2^2 \hat{\psi}_1^{1\dagger} \hat{\psi}_2^2 \hat{\psi}_1^{1\dagger}...\hat{\psi}_2^2 \hat{\psi}_1^{1\dagger}}_{p-times} : + h.c. , \tag{27}$$

with dimensions given respectively by

$$(\frac{p(k-p)}{k}, 0) \quad \text{and} \quad (0, \frac{p(k-p)}{k}). \tag{28}$$

Finally, let us stress that the order and disorder operators and the parafermionic currents, as defined by eqs.(20) and (27) respectively, satisfy the conformal operator product expansions:

$$\sigma_p(z, \bar{z})\mu_p(0,0) \propto z^{\frac{p(k-p)(k+1)}{k(k+2)}} \bar{z}^{-\frac{p(k-p)}{k(k+2)}} \left(\Psi_p^{PF}(0) + O(z) \right),$$

$$\Psi_p^{PF}(z)\Psi_q^{PF}(0) \propto z^{-\frac{2pq}{k}} \left(\Psi_{p+q}^{PF}(0) + O(z) \right), \tag{29}$$

as expected [2]. These operator expansions are easily calculated in the present formulation by using the decoupled approach (see the appendix).

iv) Perturbed Z_k parafermion model

We are now going to study the model discussed above perturbed by the primary operator corresponding to the first thermal operator, $\epsilon_1 = \phi^{(2)}_{00}$ [2]. In terms of WZW fundamental fields ϵ_1 can be written as (see eq.(11)):

$$\epsilon_1 = Tr\left(: gt^3 g^\dagger \bar{t}^3 :\right) , \tag{30}$$

and has conformal dimensions $h = \bar{h} = 2/(2+k)$.

Using the bosonization dictionary (4), the interaction term can be written as:

$$S_{int} = -\frac{\beta}{4\pi} \int d^2x Tr\left(K_z^3 K_{\bar{z}}^3\right), \tag{31}$$

where $K_\mu^{3\alpha\beta} = \psi^{i\alpha\dagger}\gamma_\mu t^3 \psi^{i\beta}$. Note that in the present case the gauge invariant fields introduced in eq.(16) are not needed since the expression for ϵ_1 is already gauge invariant in terms of the $SU(2)_k$ fields.

At this stage we want to point out the connection between the Z_k parafermion model and a Gross-Neveu-like model which is apparent in the fermionic coset formulation. Indeed, the whole action $S_0 + S_{int}$, (eqs.(9) and (31)), which represents the perturbed parafermion model, corresponds to gauged fermions with an $SU(2)$ Gross-Neveu-like interaction term. The connection between the perturbed parafermion model and a Gross-Neveu-like model has been suggested [5].

We now introduce an auxiliary field C_μ^3 through the identity

$$\exp\left(\frac{\beta}{4\pi}\int d^2x Tr(K_\mu^3 K_\mu^3)\right) = \int DC_\mu^3 \exp\left(-\frac{1}{\pi\beta}\int d^2x Tr(C_\mu^3)^2 + \frac{1}{\pi}\int d^2x Tr K_\mu^3 C_\mu^3\right) \tag{32}$$

so the action can be written as:

$$
\begin{aligned}
S &= -\frac{1}{\sqrt{2}\pi} \int d^2x \psi^{\dagger i\alpha}\left((i\slashed{\partial} + i\slashed{a})\delta_{ij}\delta_{\alpha\beta} + i\delta_{ij}\,\slashed{B}_{\alpha\beta} + i\,\slashed{b}^3 t_{ij}^3\delta_{\alpha\beta} - (\slashed{C}^3)^{\beta\alpha}t_{ij}^3\right)\psi^{j\beta} \\
&+ \frac{1}{\pi\beta}\int d^2x Tr(C_\mu^3)^2.
\end{aligned}\tag{33}
$$

Though the auxiliary field C_μ belongs to the Lie algebra of $U(k)$, its $U(1)$ component decouple from the rest. In the Dirac operator it can be absorbed by the field b_μ^3 and in the interaction it just decouples from the fermion fields, leading to a Gaussian integral which indeed does not modify the conformal algebra. Therefore we will simply discard the $U(1)$ piece of C_μ^3.

The action (33) is invariant under independent $U(1)$ gauge transformations for the fields a_μ and b_μ and also has the local $SU(k)$ symmetry

$$
\begin{aligned}
B_\mu &\to g^{-1}B_\mu g - g^{-1}\partial_\mu g , & C_\mu &\to g^{-1}C_\mu g, \\
\psi^i &\to g\psi^i, & \psi^{\dagger i} &\to \psi^{\dagger i}g^{-1}.
\end{aligned}\tag{34}
$$

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So the combination $B_\mu \pm C_\mu$ transforms as a gauge field. The chiral version of these symmetries are anomalous. We will see later that for special values of the coupling constant this symmetry is actually enlarged and the theory becomes invariant under a local $SU(k) \times SU(k)$ symmetry.

The key observation now is that the fields in eq.(33) can be almost completely decoupled through transformations:

$$
\begin{aligned}
a_\mu &= \partial_\mu \eta_a - \epsilon_{\mu\nu}\partial_\nu \phi_a , & b_\mu &= \partial_\mu \eta_b - \epsilon_{\mu\nu}\partial_\nu \phi_b \\
\delta_{ij}B_z + t_{ij}^3 C_z &= \begin{pmatrix} U_+^{-1}\partial_z U_+ & 0 \\ 0 & U_-^{-1}\partial_z U_- \end{pmatrix}_{ij}, & \delta_{ij}B_{\bar z} + t_{ij}^3 C_{\bar z} &= \begin{pmatrix} V_+^{-1}\partial_{\bar z} V_+ & 0 \\ 0 & V_-^{-1}\partial_{\bar z} V_- \end{pmatrix}_{ij} \\
\psi_1 &= \begin{pmatrix} e^{-(\phi_a+\phi_b)-i(\eta_a+\eta_b)}U_+^{-1} & 0 \\ 0 & e^{-(\phi_a-\phi_b)-i(\eta_a-\eta_b)}U_-^{-1} \end{pmatrix} \chi_1, \\
\psi_2 &= \begin{pmatrix} e^{(\phi_a+\phi_b)-i(\eta_a+\eta_b)}V_+^{-1} & 0 \\ 0 & e^{(\phi_a-\phi_b)-i(\eta_a-\eta_b)}V_-^{-1} \end{pmatrix} \chi_2
\end{aligned}
$$

$$(35)$$

where $\phi_{a,b}$, $\eta_{a,b}$ are scalar fields and $U_\pm$ and $V_\pm$ are $SU(k)$ matrix valued fields.

Taking into account the jacobians of the above transformations and those arising from the gauge-fixing procedure [16], we can rewrite the whole partition function as a product of decoupled sectors (see the appendix)

$$
\mathcal{Z} = \mathcal{Z}_{ff}\mathcal{Z}_{bos}\mathcal{Z}_{gh}\mathcal{Z}_{int}, \tag{36}
$$

where $\mathcal{Z}_{ff}$ is the partition function for free fermions, $\mathcal{Z}_{bos}$ the partition function for free scalar bosons, $\mathcal{Z}_{gh}$ the ghost partition function arising in the gauge-fixing procedure and

$$
\begin{aligned}
\mathcal{Z}_{int} = & \int DU_\pm DV_\pm \exp\left\{ (1+2k)\left(\Gamma[U_+ V_+^{-1}] + \Gamma[U_- V_-^{-1}]\right) \right. \\
& \left. - \frac{1}{2\beta\pi}\int d^2x Tr\left[\left(U_+^{-1}\partial_z U_+ - U_-^{-1}\partial_z U_-\right)\left(V_+^{-1}\partial_{\bar z}V_+ - V_-^{-1}\partial_{\bar z}V_-\right)\right] \right\}, \tag{37}
\end{aligned}
$$

where $\Gamma[u]$ is the WZW action [17]

$$
\Gamma[u] = \frac{1}{16\pi}\int d^2x Tr\left(\partial_\mu u \partial^\mu u^{-1}\right) + \frac{1}{24\pi}\int d^3y \epsilon_{ijk} Tr\left(u^{-1}\partial_i u u^{-1}\partial_j u u^{-1}\partial_k u\right). \tag{38}
$$

Hence, modulo decoupled conformally invariant sectors, we have an effective theory of interacting WZW fields.

Using the Polyakov-Wiegmann identity:

$$
\Gamma[uv^{-1}] = \Gamma[u] + \Gamma[v^{-1}] - \frac{1}{2\pi}\int d^2x Tr\left(u^{-1}\partial_z u v^{-1}\partial_{\bar z}v\right) \tag{39}
$$

we can rewrite the interaction action in a more symmetric form:

$$
S_{eff} = -(1 - \frac{1}{2\tilde\beta})(2k+1)\left(\Gamma[U_+ V_+^{-1}] + \Gamma[U_- V_-^{-1}]\right) - \frac{1}{2\tilde\beta}(2k+1)\left(\Gamma[U_+ V_-^{-1}] + \Gamma[U_- V_+^{-1}]\right), \tag{40}
$$

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where $\tilde{\beta} = (2k+1)\beta$. In this form it is obvious that S_{eff} is invariant under the following local $SU(k)$ transformation

$$U_+ \to U_+ g, \qquad U_- \to U_- g, \tag{41}$$
$$V_+ \to V_+ g, \qquad V_- \to V_- g, \tag{42}$$

which is the representation of the gauge symmetry, eq.(34), on the $SU(k)$ valued fields.

This last form of the action, consisting in four WZW actions can give an illusory appearance of manifest conformal invariance. However it is not the case since the arguments of the WZW actions are not all independent fields but constrained through the relation

$$(U_+ V_+^{-1})^{-1} U_+ V_-^{-1} (U_- V_-^{-1})^{-1} U_- V_+^{-1} = 1. \tag{43}$$

The interaction action eq.(40) has an invariance under the duality transformation:

$$\begin{aligned} U_+ &\to U_- \\ U_- &\to U_+ \\ \tilde{\beta} &\to -\frac{\tilde{\beta}}{1 - 2\tilde{\beta}}. \end{aligned} \tag{44}$$

This transformation, modulo hermitian conjugation, interchanges the $SU(2)$ components of the fermions in consistency with eq.(22). Thus it corresponds to the Krammers-Wannier duality $\sigma \leftrightarrow \mu$. This symmetry has no effects over the parafermionic currents. Using a bosonic realization of the deformed parafermionic theory the authors of refs.[5, 7] found a *classical* invariance under the transformation $\beta \to -\beta$. However we have to stress that our result is valid at the quantum level: is an invariance of the partition function of the theory. The duality transformation of ref.[5, 7] agrees with the leading order term of our eq.(44) for $\tilde{\beta}$ small.

It is now straightforward to see the existence of two critical points: At the value of the coupling constant $1/\tilde{\beta}_1 = 0$, the effective partition function can be written as:

$$\mathcal{Z}_{int}|_{\beta_1} = \int DU_\pm DV_\pm \exp\{-(2k+1)\left(\Gamma[U_+ V_+^{-1}] + \Gamma[U_- V_-^{-1}]\right)\}, \tag{45}$$

which corresponds to a Conformal Field Theory whose Virasoro central charge is given by $c = 2(1 + 2k)(k^2 - 1)/(k + 1)$

At the value $\tilde{\beta}_2 = \frac{1}{2}$, we have its dual counterpart:

$$\mathcal{Z}_{int}|_{\beta_2} = \int DU_\pm DV_\pm \exp\{-(2k+1)\left(\Gamma[U_+ V_-^{-1}] + \Gamma[U_- V_+^{-1}]\right)\}, \tag{46}$$

with the same value for the Virasoro charge.

For these two critical values of the coupling constant the system acquires an extra symmetry. In fact consider for example the first critical theory, characterized by the coupling β_1. Its effective action is invariant under the transformations

$$U_+ \to U_+ g, \qquad V_+ \to V_+ g, \tag{47}$$
$$U_- \to U_- h, \qquad V_- \to V_- h. \tag{48}$$

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where g and h are independent $SU(k)$ elements.

This doubling of the symmetry has an important consequence on the parafermionic primary fields. While the order operator σ is invariant under this symmetry, the disorder operator μ, being proportional to $U_- V_+^{-1}$ is not. Thus the symmetry prevent us for the inclusion of disorder operators in the vacuum expectation values. The order operator σ is still a primary field in the new critical point but with a vanishing conformal dimension. The properties of the second fixed point are similar but with the roles of σ and μ interchanged.

Taking into account the remaining degrees of freedom (fermions, scalars and ghosts, (see the appendix)), we can construct the total conformal algebra of the critical theories. In both cases the Virasoro central charge vanishes.

A remarkable curiosity is the fact that the system (40) has another self-dual point, besides the one corresponding to the original parafermionic theory, (i.e. $\beta = 0$), which is reached when the coupling constant takes the value $\beta = \frac{1}{(1+2k)}$. At this novel self-dual point the $SU(k)$ valued effective action reads

$$S_{eff}|_{SD} = -(k + \frac{1}{2}) \left(\Gamma[U_+ V_+^{-1}] + \Gamma[U_- V_-^{-1}] + \Gamma[U_+ V_-^{-1}] + \Gamma[U_- V_+^{-1}] \right). \tag{49}$$

At first sight this action does not seem to present conformal invariance. However, this statement needs a deep analysis as interacting WZW action can present very non-trivial fixed points [18].

v) Conclusions and discussion

We have studied in this paper the $SU(2)_k/U(1)$ parafermionic theory in terms of a fermionic coset model. This formulation reveals to be particularly fruitful for the analysis of the conformal properties of the model. In particular the order/disorder operators and the parafermionic currents can be explicitly constructed and they take a very simple form in terms of gauge invariant fermionic fields. The computation of the conformal OPE's is then straightforward in this scheme.

The inclusion of a deformation in the direction of the first thermal operator can be implemented by perturbing the coset model with a Gross-Neveu type interaction

$$S_{int} = -\frac{\beta}{4\pi} \int d^2x \ \psi^{\alpha\dagger} \gamma_\mu t^3 \psi^\beta \ \psi^{\beta\dagger} \gamma_\mu t^3 \psi^\alpha. \tag{50}$$

After some manipulations within the path integral we have factored out from the whole partition function a fermionic and two scalars free partition functions. The remaining degrees of freedom contain all the information about the interaction and correspond to a system of coupled WZW actions. This last system presents a remarkable duality symmetry: the theories with couplings constants β and $-\frac{\beta}{1-2\beta(1+2k)}$ are identical. Under this transformation the roles of the order an disorder operators are reversed, so it corresponds to a generalization

of the usual Krammers-Wannier duality. Unlike the duality symmetry of refs. [5, 7], our result is valid at the quantum level. The result of refs. [5, 7] coincides with the tree level approximation of eq. (44).

We also found two critical values of the coupling constant for which the theory recovers conformal invariance,

$$\beta_1 = \frac{1}{2(1+2k)} \qquad \text{and} \qquad \beta_2 = \infty \quad \text{(its dual counterpart).} \tag{51}$$

For both critical actions the original symmetry is enlarged and the system is invariant under an extra local $SU(k)$ symmetry. At both fixed points the theory represents a CFT with total central charge equal to 0.

A question that remains open is the evolution of the deformed parafermionic theory under the renormalization group. The perturbation from the parafermionic theory is in a relevant direction (the operator responsible of the deformation has dimension $\frac{4}{k+2} < 2$). Thus the perturbation drives away the theory from the critical point along an infrared renormalization group trajectory. As the central charge of the parafermionic theory is greater than zero we can expect, invoking Zamolodchikov's c-Theorem, that the model evolves under the renormalization group to the conformal points parametrized by β_1 and β_2, depending on the sign of the original perturbation. The proof of this conjecture is a very interesting problem though highly non trivial due to its non perturbative nature. A perturbative expansion for sufficiently large k could bring some light to the answer, as one of the fixed points is of order $1/k$ (but note however that the central charges are finitely separated for any k). Work in this direction is in progress.

Appendix

In this appendix we will evaluate the central charge and construct the primary fields in the Z_k-parafermion theory formulated as a fermionic coset.

The partition function is given by eqs.(8) and (9) and it can be decoupled through transformations

$$
\begin{aligned}
a_\mu &= \partial_\mu \eta_a - \epsilon_{\mu\nu}\partial_\nu\phi_a & b_\mu &= \partial_\mu \eta_b - \epsilon_{\mu\nu}\partial_\nu\phi_b \\
B_z &= 2i(\partial_z U)U^{-1} & B_{\bar{z}} &= 2i(\partial_{\bar{z}}V)V^{-1} \\
\psi_1 &= e^{-\phi_a - i\eta_a}e^{(-\phi_b - i\eta_b)t_3}U\chi_1 & \psi_2^\dagger &= \chi_2^\dagger\, U^{-1}e^{\phi_a + i\eta_a}e^{(\phi_b + i\eta_b)t_3} \\
\psi_2 &= e^{\phi_a - i\eta_a}e^{(\phi_b - i\eta_b)t_3}V\chi_1 & \psi_1^\dagger &= \chi_1^\dagger\, V^{-1}e^{-\phi_a + i\eta_a}e^{(-\phi_b + i\eta_b)t_3},
\end{aligned}
\tag{A.1}
$$

where $B_z = \frac{1}{2}(B_0 - iB_1)$, $B_{\bar{z}} = \frac{1}{2}(B_0 + iB_1)$.

Taking into account the gauge fixing procedure and the Jacobians associated to (A.1) [16] one arrives at the desired decoupled form for the partition function:

$$\mathcal{Z} = \mathcal{Z}_{ff}\mathcal{Z}_{fb}\mathcal{Z}_{WZW}\mathcal{Z}_{gh}, \tag{A.2}$$

where

$$
\begin{aligned}
\mathcal{Z}_{ff} &= \int \mathcal{D}\chi^\dagger \mathcal{D}\chi \exp(-\frac{1}{\pi}\int(\chi_2^\dagger\bar{\partial}\chi_1 + \chi_1^\dagger\partial\chi_2)d^2x), \\
\mathcal{Z}_{fb} &= \int \mathcal{D}\phi_a\mathcal{D}\phi_b \exp(\frac{k}{\pi}\int\phi_a\Delta\phi_a d^2x + \frac{k}{\pi}\int\phi_b\Delta\phi_b d^2x), \\
\mathcal{Z}_{WZW} &= \int \mathcal{D}\tilde{U} \exp\left((2k+2)\Gamma[\tilde{U}]\right).
\end{aligned}
\tag{A.3}
$$

$\Gamma[\tilde{U}]$ is the WZW action [17] (38) for the gauge invariant combination $\tilde{U} = U^{-1}V$, and Z_{gh} corresponds to the Fadeev-Popov ghosts partition function, whose explicit form will not be needed.

The central charge is now easily evaluated as the sum of four independent contributions coming from the different sectors, $c_{ff} = 2k$, $c_{fb} = 2$, $c_{WZW} = 2(k+1)(k^2-1)/(k+2)$ and $c_{gh} = -2(k^2+1)$, thus giving

$$c = \frac{2(k-1)}{k+2}, \tag{A.4}$$

which coincides with the central charge of the Z_k parafermion model.

After decoupling the gauge field b_μ^3 the components of the fermions defined in (15) can be written as

$$
\begin{aligned}
\hat{\psi}_1^{\;i}(z) &= e^{\varphi_b(z)}\tilde{\psi}_1^i(z) & \hat{\psi}_1^{\;i\dagger}(\bar{z}) &= \tilde{\psi}_1^{i\dagger}(\bar{z})e^{\bar{\varphi}_b(\bar{z})} \\
\hat{\psi}_2^i(\bar{z}) &= e^{-\bar{\varphi}_b(\bar{z})}\tilde{\psi}_2^i(\bar{z}) & \hat{\psi}_2^{\;i\dagger}(z) &= \tilde{\psi}_2^{i\dagger}(z)e^{\varphi_b(z)},
\end{aligned}
\tag{A.5}
$$

where $\tilde{\psi}$ stands for the fermions decoupled from the gauge field b_μ, and

$$
\begin{aligned}
\varphi_b(z) &= \phi_b + i\int_x^\infty dz_\mu\epsilon_{\mu\nu}\partial_\nu\phi_b \\
\bar{\varphi}_b(\bar{z}) &= \phi_b - i\int_x^\infty dz_\mu\epsilon_{\mu\nu}\partial_\nu\phi_b
\end{aligned}
\tag{A.6}
$$

are the chiral (holomorphic and anti-holomorphic) components of the free boson ϕ_b.

We are now in position to evaluate the conformal dimensions and the algebra obeyed by the fields defined in (17) and (20).

In fact, in the completely decoupled picture the fields defined in (17) can be written as

$$
\begin{aligned}
\sigma_1 &\equiv \hat{g}_{1,1} =: \chi_2^{1\alpha}\tilde{U}^{-1\;\alpha\beta}\chi_2^{\dagger 1\beta} :: e^{2\phi_a} :: e^{2\phi_b} : + \; h.c. \;, \\
\mu_1 &\equiv \hat{g}_{2,1}^\dagger =: \chi_1^{2\alpha}\tilde{U}^{\alpha\beta}\chi_1^{\dagger 1\beta} :: e^{-2\phi_a} :: e^{\varphi_b-\bar{\varphi}_b} : + \; h.c. \;,
\end{aligned}
\tag{A.7}
$$

and their conformal dimensions are then given by the sum of the contributions $h = 1/2$, coming from free fermions, $h = -(k^2-1)/(2k(k+2))$, coming from the WZW field $\tilde{g}$, and twice $h = -1/4k$, coming from the vertex operators of the free bosons ϕ_a and ϕ_b. This sum leads to the result quoted in (18).

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For general values of p, eq.(20), the decoupled expression is

$$
\begin{aligned}
\sigma_p &\equiv: e^{2p\phi_a} :: e^{2p\phi_b} :: \chi_2^{1\alpha_1}...\chi_2^{1\alpha_p} :: \chi_2^{\dagger 1\gamma_1}...\chi_2^{\dagger 1\gamma_p} :: \tilde{U}_{\mathcal{A}}^{-1\ \alpha_1\gamma_1...\alpha_p\gamma_p} :+ h.c.\ , \\
\mu_p &\equiv: e^{-2p\phi_a} :: e^{p(\varphi_b-\bar{\varphi}_b)} :: \chi_1^{2\alpha_1}...\chi_1^{2\alpha_p} :: \chi_1^{\dagger 1\gamma_1}...\chi_1^{\dagger 1\gamma_p} :: \tilde{U}_{\mathcal{A}}^{\alpha_1\gamma_1...\alpha_p\gamma_p} :+ h.c.\ ,
\end{aligned}
\tag{A.8}
$$

where

$$
U_{\mathcal{A}}^{\alpha_1\gamma_1...\alpha_r\gamma_r} \equiv [: U^{\alpha_1\gamma_1} \ldots U^{\alpha_r\gamma_r} :]_{\mathcal{A}}
\tag{A.9}
$$

and the subscript $\mathcal{A}$ means antisymmetrization of the left and right indices.

The conformal dimensions of σ_p and μ_p are now given by the sum of the contributions $h = p/2$, coming from free fermions, $h = -p(k-p)(k+1)/(2k(k+2))$, coming from the WZW field $\tilde{U}_{\mathcal{A}}$ and twice $h = -p/4k$, coming from the vertex operators. The result coincides with eq.(12).

The order/disorder algebras (19) and (21) are easily calculated using the decoupled expressions, (A.7) and (A.8), and making use of the canonical commutation relations of the decoupled fields [14].

In fact, this algebra has its origin in the odd combinations of the holomorphic and anti-holomorphic components of the free boson ϕ_b in eqs.(A.7) and (A.8). Indeed, one can easily check that

$$
: e^{2p\phi_b(x_1)} :: e^{p'(-\varphi_b+\bar{\varphi}_b)(x_2)} := e^{\frac{i2pp'\pi}{N}\Theta(x_1-x_2)} : e^{p'(-\varphi_b+\bar{\varphi}_b)(x_2)} :: e^{2p\phi_b(x_1)} :,
\tag{A.10}
$$

being the other factors commuting.

Finally, the fields defined in (26) are given in terms of the decoupled fields by the expressions

$$
\begin{aligned}
\psi_1^{PF} &\equiv \hat{\psi}_1^2\hat{\psi}_2^{1\dagger} =: e^{-2\varphi_b} : \chi_1^2\chi_2^{1\dagger} + h.c.\ , \\
\bar{\psi}_1^{PF} &\equiv \hat{\psi}_2^2\hat{\psi}_1^{1\dagger} =: e^{-2\bar{\varphi}_b} : \chi_2^2\chi_1^{1\dagger} + h.c.
\end{aligned}
\tag{A.11}
$$

Their conformal dimensions are easily evaluated and give the result quoted in eq.(14). For the more general parafermion currents, eq.(27), the evaluation of the conformal dimensions proceeds similarly.

It is straightforward, from the decoupled expressions to derive the whole set of conformal operator product expansions quoted in eq.(29).

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