A Family of unitary higher order equations *

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Abstract

A scalar field obeying a Lorentz invariant higher order wave equation, is minimally coupled to the electromagnetic field. The propagator and vertex factors for the Feynman diagrams, are determined. As an example we write down the matrix element for the Compton effect. This matrix element is algebraically reduced to the usual one for a charged Klein-Gordon particle. It is proved that the n^{th} order theory is equivalent to n independent second order theories. It is also shown that the higher order theory is both renormalizable and unitary for arbitrary n.

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1 Introduction

In a previous work (ref.[1]), we considered the interaction of tachyons with the electromagnetic field. As the former can not exist in free particle states (ref.[2],[3]), we took a fourth order wave equation implying two modes of propagation for a scalar field φ . One of the two modes corresponds to a normal Klein-Gordon particle. The other is a tachyon mode (ref. [4]). When φ is coupled to electromagnetism by using the gauge covariant derivative:

$$\partial_\mu \to \mathcal{D}_\mu = \partial_\mu - ieA_\mu \tag{1}$$

we found that all matrix elements of the fourth order theory:

$$\left(\Box'^2 - m^4\right)\varphi = 0 \tag{2}$$

$$\Box'^2 = \mathcal{D}^\mu \mathcal{D}_\mu \tag{3}$$

are equivalent to a second order theory in which the bradyon (a bradyon is a particle whose momentum P_μ satisfies $P_\mu P^\mu = -m^2$) and the tachyon are independent fields obeying:

$$\left(\Box' - m^2\right)\varphi_1 = 0 \tag{4}$$

$$\left(\Box' + m^2\right)\varphi_2 = 0 \tag{5}$$

The bradyon mode is equivalent to a normal charged Klein-Gordon particle. The tachyon mode can only be found in closed loops connected to photon lines.

The fourth order equation belongs to a family found in reference [5], when studing supersymmetry in spaces of arbitrary dimensions. In this work the fields obey higher order equations of motion. The order increasing with de dimensionality of the space-time. Those equations have the form:

$$\left(\Box^n - m^{2n}\right)\varphi = 0 \tag{6}$$

The usual Klein-Gordon equation is a member of the family ($n = 1$). For $n = 2$ we have the equation examined in reference [4] . On the other hand we have a theory such as Quantum Gravity, where a perturbative calculation leads to non-renormalizable divergences proportional to powers of the curvature tensor. There are cases in which starting with terms cuadratic in the curvature, one obtains divergences that can be removed by renormalization. These theories also give rise to fourth order equations.

Equation (6) implies n modes of propagation for the field φ (ref. [6]). It can also be written as:

$$\prod_{s=1}^{n} \left(\Box - e_s m^2 \right) \varphi = 0 \tag{7}$$

where

$$e_s = e^{\frac{2\pi i}{n}(s-1)} \qquad (s = 1, 2, ..., n) \tag{8}$$

Of course, $e_1 = 1$ for arbitrary n, so that in (7) we always have a Klein-Gordon factor.

For n=2l (l integer), $e_{l+1} = -1$ and we have a tachyon mode. For n=odd number no tachyon appears. The only real mass is otained for s=1. All other masses come in complex conjugate pairs.

Except for the s=1 state which is a normal bradyonic mode whose propagator is Feynman's causal function, none of the states with $s \neq 1$ can propagate as asymptotically free waves. The corresponding propagators are half advanced and half retarded (ref.[7]). This type of Green function was used in ref. [8] to describe the electromagnetic interaction of "perfect absorbers" I.e. when no asymptotic wave escapes the system.

In the next paragraphs we will analyze the behaviour of φ, when the electromagnetic field is introduced in eq.(6) by means of the gauge covariant derivative eq.(1). I.e.:

$$\Box \rightarrow \Box' \equiv \Box - 2ieA \cdot \partial - e^2 A^2 \quad , \quad (\partial_\mu A^\mu = 0) \tag{9}$$

With this substitution , eq.(6) is transformed into:

$$\left(\Box'^n - m^{2n} \right) \varphi = 0 \tag{10}$$

and of course, eq.(7) changes to:

$$\prod_{s=1}^{n} \left(\Box' - e_s m^2 \right) \varphi = 0 \tag{11}$$

2 Interaction terms and propagators

When we use the substitution given by eq.(9), the iterated D'Alambertian $\Box^n$ gives rise to the interaction terms of order n:

$$\Box^n \to \Box'^n = \left(\Box - 2ieA\cdot\partial - e^2A^2\right)^n \tag{12}$$

The development of (12) gives a polynomial in e of degree 2n:

$$\Box'^n = \Box^n + e\tilde{I}_1^{(n)} + e^2\tilde{I}_2^{(n)} + \cdots + e^{2n}\tilde{I}_{2n}^{(n)} \tag{13}$$

We are going to find the first few terms of (13). I will then be easy to guess the form of any term.

The first order term can only come from terms containing one factor $A\cdot\partial$ and (n-1) powers of the D'Alambertian.

$$\tilde{I}_1^{(n)} = -2i\left(\Box^{n-1}A\cdot\partial + \Box^{n-2}A\cdot\partial\Box + ...+\right.$$

$$\left.+\Box A\cdot\partial\Box^{n-2} + A\cdot\partial\Box^{n-1}\right) \tag{14}$$

When we take the Fourier transform of (14), the derivative operator $(-i\partial_\mu)$ is transformed into the momentum vector p_μ . The D'Alambertian is transformed into p^2 . The vector A_μ leaves its place to the polarization vector of the photon ϵ_μ .

$$I_1^{(n)} = (-1)^{n-1}2\left(p^{2(n-1)}\epsilon\cdot p + p^{2(n-2)}\epsilon\cdot pq^2 + \cdots + \epsilon\cdot pq^{2(n-1)}\right)$$

$$I_1^{(n)} = 2(-1)^{n-1}\epsilon\cdot pP^{n-1}\left(p^2,q^2\right); \quad q = p+k \quad \epsilon\cdot p = \epsilon\cdot q \tag{15}$$

$$P^t\left(p^2,q^2\right) = \sum_{s=0}^{s=t} p^{2(t-s)}q^{2s} \tag{16}$$

¿From now on we will write all interaction terms in momentum space. The second order term in e contains a part in A^2 (cf. eq.(12)), which is similar to (15) an another part in $\Box^a\ A\cdot\partial\ \Box^b\ A\cdot\partial\ \Box^c$, with a+b+c=n-2.

$$I_2^{(n)} = 2(-1)^{n-1}\epsilon_1\cdot\epsilon_2P^{n-1}\left(p^2,q^2\right) + 4(-1)^{n-1}\epsilon_1\cdot p_1\epsilon_2\cdot q\cdot$$

$$\cdot P^{n-2}\left(p_1^2,q^2,p_2^2\right) + 4(-1)^{n-1}\epsilon_2\cdot p_1\epsilon_1\cdot pP^{n-2}\left(p_1^2,r^2,p_2^2\right).$$

$$(p_2 = p_1 + k_1 + k_2,\ q = p_1 + k_1,\ r = p_1 + k_2) \tag{17}$$

where

$$P^t\left(p^2, q^2, r^2\right) = \sum_{a+b+c=t} p^{2a} q^{2b} r^{2c} \tag{18}$$

For the third order we have to pick-up from (12), all terms containig three factors $A \cdot \partial$ or one factor $A \cdot \partial$ and a factor A^2. The corresponding D'Alembertians (as in (14)) give rise to the P-coefficients whose main properties are going to be specified in the next paragraph.The other interaction terms are found in a similar way.

The propagator for eq.(10) can be defined by:

$$\left(\Box^n - m^{2n}\right) \tilde{G}^{(n)} = i\delta \tag{19}$$

By Fourier transforming eq.(19) we get:

$$G^{(n)} = \frac{(-1)^n i}{p^{2n} - (-m^2)^n} \tag{20}$$

We now use the identity:

$$\frac{1}{x^n - a^n} = \frac{1}{na^{n-1}} \sum_{s=1}^{n} \frac{e_s}{x - e_s a} \tag{21}$$

where e_s is given by eq.(8).

With $x = p^2$ and $a = -m^2$, we get:

$$G^{(n)} = \frac{-i}{nm^{2(n-1)}} \sum_{s=1}^{n} \frac{e_s}{p^2 + e_s m^2} \tag{22}$$

The first term of (22) (s=1) represents the Klein-Gordon propagator. The other terms correspond to the other modes of propagation. The common factor $\left(nm^{2(n-1)}\right)^{-1}$ is the relative normalization of the wave function whose propagator is defined by (19), with respect to that of the usual second order equation. To obtain an n-independent normalization we have to divide each external line by the factor $\left(nm^{2(n-1)}\right)^{1/2}$.

The interaction resulting from (12) seems to be of the unrenormalizable type for $n > 1$. Compare for example $I_1^{(n)}$ (eq.(15) for $n > 1$ with $I_1^{(1)} = 2\,\epsilon \cdot p$.

However, the propagator (20) has (n-1) extra powers of p^2 in the denominator. So that, by power counting the theory turns out to be renormalizable. Furthermore, we are going to show that it is equivalent, for arbitrary n, to the usual Klein-Gordon theory for a charged scalar particle.

3 Vertex Factors

To determine the factors P^t we will now describe their properties.

Each $P^t(x_1, ..., x_s)$ is a sum over all monomials of degree t, formed with products of powers of its arguments.

$$P^t\left(x_1,...,x_s\right) = \sum_{a_1+...+a_s=t} x_1^{a_1} x_2^{a_2} ... x_s^{a_s} \tag{23}$$
$$a_l \geq 0 \left(l = 1, 2, ..., s\right)$$

We define

$$P^t = 0 \;\; for \;\; t < 0 \;\; and \;\; P^0 = 1 \tag{24}$$

¿From (23) we have

$$P^1\left(x_1,...,x_s\right) = \sum_{l=1}^{s} x_l \tag{25}$$

$$P^t\left(x\right) = x^t \tag{26}$$

$$P^t\left(x,y\right) = \sum_{l=0}^{t} x^{t-l} y^l \tag{27}$$

All P^t are symmetrical homogeneous functions of their arguments:

$$P^t\left(\alpha x_1, \alpha x_2, ..., \alpha x_s\right) = \alpha^t P^t\left(x_1,...,x_s\right) \tag{28}$$

We can also write (23) in the form:

$$P^t\left(x_1,...,x_s\right) = \sum_{a_1=0}^{t} x_1^{a_1} \sum_{a_2+...+a_s=t-a_1} x_2^{a_2} ... x_s^{a_s}$$

$$P^t\left(x_1,...,x_s\right) = \sum_{a_1=0}^{t} x_1^{a_1} P^{t-a_1}\left(x_2,...,x_s\right) \tag{29}$$

So that:

$$x_1 P^t(x_1,...,x_s) = \sum_{a_1=0}^{t} x_1^{a_1+1} P^{t-a_1}(x_2,...,x_s) =$$

$$= \sum_{b=1}^{t+1} x_1^b P^{t+1-b}(x_2,...,x_s) = \sum_{b=0}^{t+1} x_1^b P^{t+1-b}(x_2,...,x_s) - P^{t+1}(x_2,...,x_s)$$

$$x_1 P^t(x_1,...,x_s) = P^{t+1}(x_1,x_2,...x_s) - P^{t+1}(x_2,...,x_s) \tag{30}$$

Also:

$$x_2 P^t(x_1,...,x_s) = P^{t+1}(x_1,x_2,...,x_s) - P^{t+1}(x_1,x_3,...,x_s)$$

Then

$$(x_1 - x_2) P^t(x_1,...,x_s) = P^{t+1}(x_1,x_3,...,x_s) - P^{t+1}(x_2,x_3,...,x_s) \tag{31}$$

In particular

$$(x_1 - x_2) P^t(x_1,x_2) = P^{t+1}(x_1) - P^{t+1}(x_2) = x_1^{t+1} - x_2^{t+1} \tag{32}$$

If we choose $x_1 = p^2$ and $x_2 = -m^2$, we get from (32) :

$$\left(p^2 + m^2\right) P^{n-1}\left(p^2, -m^2\right) = p^{2n} - \left(-m^2\right)^n \tag{33}$$

And the denominator of the Green function (20), factorizes according to (33):

$$G^{(n)} = \frac{(-1)^n i}{(p^2 + m^2) P^{n-1}(p_1^2 - m^2)} \tag{34}$$

4 Compton effect

We are now ready to evaluate the cross section for any physical process, in a given perturbative order, for any higher order equation of the family (10).

We will take as an example the second order Compton effect. The initial and final momentum of the charged bradyon are p_1 and p_2. The incoming photon has a polarization ϵ_1 and a momentum k_1. The final photon has a polarization ϵ_2 and momentum k_2. We define $p = p_1 + k_1 = p_2 + k_2$, $q = p_1 - k_2 = p_2 - k_1$; $\epsilon_1 \cdot k_1 = 0$ $\epsilon_2 \cdot k_2 = 0$.

The matrix element corresponding to eq.(10), with the propagator (20) and the interaction vertices (15) and (17), is:

$$M^{(n)} =$$

$$\left[2i(-1)^{n-1}\epsilon_1 \cdot p_1 P^{n-1}\left(p_1^2, p^2\right)\right] \frac{(-1)^n i}{p^{2n} - (-m^2)^n} \left[2i(-1)^{n-1}\epsilon_2 \cdot p_2 P^{n-1}\left(p^2, p_2^2\right)\right] +$$

$$+ \left[2i(-1)^{n-1}\epsilon_2 \cdot p_1 P^{n-1}\left(p_1^2, q^2\right)\right] \frac{(-1)^n i}{q^{2n} - (-m^2)^n} \left[2i(-1)^{n}\epsilon_1 \cdot p_2 P^{n-1}\left(q^2, p_2^2\right)\right] +$$

$$i(-1)^n \{4\epsilon_1 \cdot p_1 \epsilon_2 \cdot p_2 P^{n-2}\left(p_1^2, p^2, p_2^2\right) + 4\epsilon_2 \cdot p_1 \epsilon_1 \cdot p_2 \cdot$$

$$\cdot P^{n-2}\left(p_1^2, q^2, p_2^2\right) + 2\epsilon_1 \cdot \epsilon_2 P^{n-1}\left(p_1^2, p_2^2\right)\}$$

$$M^{(n)} = 4i(-1)^n \left\{\epsilon_1 \cdot p_1 \epsilon_2 \cdot p_2 \left[\frac{P^{n-1}\left(p_1^2, p^2\right) P^{n-1}\left(p^2, p_2^2\right)}{p^{2n} - (-m^2)^n} - P^{n-2}\left(p_1^2, p^2, p_2^2\right)\right] + \right.$$

$$\left. \epsilon_2 \cdot p_1 \epsilon_1 \cdot p_2 \left[\frac{P^{n-1}\left(p_1^2, q^2\right) P^{n-1}\left(q^2, p_2^2\right)}{q^{2n} - (-m^2)^n} - P^{n-2}\left(p_1^2, q^2, p_2^2\right)\right]\right\} +$$

$$+ 2i(-1)^n \epsilon_1 \cdot \epsilon_2 P^{n-1}\left(p_1^2, p_2^2\right) \tag{35}$$

In the last equation we use (33) and $p_1^2 = -m^2$, $p_2^2 = -m^2$:

$$M^{(n)} = 4i(-1)^n \left\{\epsilon_1 \cdot p_1 \epsilon_2 \cdot p_2 \left[\frac{P^{n-1}\left(-m^2, p^2\right)}{p^2 + m^2} - \frac{\left(p^2 + m^2\right) P^{n-2}\left(-m^2, p^2, -m^2\right)}{p^2 + m^2}\right] + \right.$$

$$\left. \epsilon_2 \cdot p_1 \epsilon_1 \cdot p_2 \left[\frac{P^{n-1}\left(-m^2, q^2\right)}{q^2 + m^2} - \frac{\left(q^2 + m^2\right) P^{n-2}\left(-m^2, p^2, -m^2\right)}{p^2 + m^2}\right]\right\} +$$

$$2i(-1)^n \epsilon_1 \cdot \epsilon_2 P^{n-1}\left(-m^2, -m^2\right) \tag{36}$$

But, according to (31):

$$\left(x + m^2\right) P^{n-2}\left(-m^2, x, -m^2\right) = P^{n-1}\left(x, -m^2\right) - P^{n-1}\left(-m^2, -m^2\right) \tag{37}$$

And, according to (27):

$$P^{n-1}(x,x) = nx^{n-1} \tag{38}$$

So, we finally get, for the normalized matrix element $\bar{M}^{(n)}$

$$\bar{M}^{(n)} = \left(nm^{2(n-1)}\right)^{-1} M^{(n)} =$$

$$= -4i\left(\epsilon_1\cdot p_1\epsilon_2\cdot p_2\frac{1}{p^2+m^2}+\epsilon_2\cdot p_1\epsilon_1\cdot p_2\frac{1}{q^2+m^2}\right) - 2i\epsilon_1\cdot\epsilon_2 \tag{39}$$

We can see from (39) the interesting fact that, no matter how high the order of the equation (10) is, we always end up with the matrix element corresponding to the second order Klein-Gordon equation coupled to the electromagnetic field.

The same fact is true when we consider, for example a "multiphoton" scattering, in which a collision of a photon with a charged bradyon produces any number of scattered photons. We do not intend to give here a direct proof, which involves some laborious algebraic manipulation of the matrix elements. In ref.[9] we show an explicit evaluation for the double photon scattering. Anyway, we are going to present a general proof in §5.

We can also answer the following question: what is the amplitude for the Compton effect to produce a change from the bradyon mode to any other mode of the field φ (solution of (10))?

To answer this question, we take again the matrix element $M^{(n)}$, eq.(35). But instead of taking $p_2^2 = -m^2$, we use $p_2^2 = -e_s m^2$; where e_s is given by eq.(8).

It is easy to follow the procedure that leads to eqs.(36) to (39). Now the matrix element is proportional to (cf. eq.(27)) :

$$P^{n-1}\left(-m^2, -e_s m^2\right) = \sum_{l=0}^{n-1}\left(-m^2\right)^{n-1-l}\left(-m^2 e_s\right)^l =$$

$$= \left(-m^2\right)^{n-1}\sum_{l=0}^{n-1} e_s^l = \left(-m^2\right)^{n-1}\frac{1-e_s^n}{1-e_s}$$

For any s, $e_s^n = 1$ (cf. eq.(8)). So that when $e_s \neq 1$,

$$P^{n-1}\left(-m^2, -e_s m^2\right) = 0 \quad , \quad s \neq 1 \tag{40}$$

Eq. (40) tells us that the probability amplitude for a change from a bradyon mode to any other (different) mode is exactly zero.

5 General proof

We will use functional methods to establish the equivalence of the general order equation (10) and the second order one.

The Lagrangian corresponding to the field φ interacting minimally with the electromagnetic field is:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\varphi} \left(\Box'^{n} - m^{2n} \right) \varphi \tag{41}$$

where $\Box'$ is defined by eq.(19).

Our generating functional is:

$$\mathcal{Z}\left(\mathcal{J}, \bar{\mathcal{J}}, \mathcal{K}\right) = \int [\mathcal{D}A]\,[\mathcal{D}\varphi]\,[\mathcal{D}\bar{\varphi}]\, e^{i \int dx \left(\mathcal{L} + \mathcal{J}\varphi + \bar{\mathcal{J}}\bar{\varphi} + \mathcal{K}A \right)} \tag{42}$$

where $\mathcal{J}$, $\bar{\mathcal{J}}$ and $\mathcal{K}$ are external sources. To assure the Lorentz gauge for A_μ , a term $\xi(\partial_\mu A^\mu)^2$ should be added to the Lagrangian (41).

The exponent in (42) is a quadratic function of the scalar field φ . We can then use the general gaussian formula [10]:

$$\int [\mathcal{D}\varphi]\,[\mathcal{D}\bar{\varphi}]\, e^{i \int dx \left(\bar{\varphi}\mathcal{Q}\varphi + \mathcal{J}\varphi + \bar{\mathcal{J}}\bar{\varphi} \right)} =$$

$$N' \mid \mathcal{Q} \mid^{-2} e^{-i \int dx \mathcal{J} \mathcal{Q}^{-1} \bar{\mathcal{J}}} \tag{43}$$

where $\mid \mathcal{Q} \mid$ is the functional determinant of the operator $\mathcal{Q}$. For our case we introduce the notations:

$$\mathcal{P} = \Box'^{n} - m^{2n} \tag{44}$$

$$\mathcal{P}_s = \Box' - e_s m^2 \quad , \quad e_s = e^{\frac{2\pi i}{n}(s-1)} \quad (s = 1, ..., n) \tag{45}$$

The factorization of eq.(10), eq.(11), means in our notation, that:

$$\mathcal{P} = \prod_{s=1}^{n} \mathcal{P}_s \tag{46}$$

Note also that:

$$[\mathcal{P}_a, \mathcal{P}_b] = 0 \quad , \quad (a, b = 1, ..., n) \tag{47}$$

The functional determinant of P, also factorizes according to (46):

$$| \mathcal{P} |= \prod_{s=1}^{n} | \mathcal{P}_s | \tag{48}$$

¿From (46) and (47) we get:

$$\mathcal{P}^{-1} = \prod_{s=1}^{n} \mathcal{P}^{-1}$$

For the inverse of the P-operators to be properly defined, we must take into account the boundary conditions imposed on the Green functions. $\mathcal{P}_s^{-1}$ $(s \neq 1)$ correspond to the half advanced and half retarded Wheeler's function. However, for our purpose here, we do not need to be more specific. It suffices to give our formulae a symbolic character.

¿From the identity (21) we deduce:

$$\frac{1}{\Box'^n - m^{2n}} = \frac{1}{nm^{2(n-1)}} \sum_{s=1}^{n} \frac{e_s}{\Box' - e_s m^2}$$

Or, using the notations (44), (45):

$$\mathcal{P}^{-1} = \frac{1}{nm^{2(n-1)}} \sum_{s=1}^{n} e_s \mathcal{P}_s^{-1} \tag{49}$$

With (48) and (49) we obtain:

$$(43) = N' \prod_{s=1}^{n} | \mathcal{P}_s |^{-2} e^{-i \int \frac{dx}{nm^{2(n-1)}} \sum_{t=1}^{n} \mathcal{J} e_t \mathcal{P}_t^{-1} \bar{\mathcal{J}}}$$

$$(43) = N' \prod_{s=1}^{n} | \mathcal{P}_s |^{-2} e^{-i \int \frac{dx}{nm^{2(n-1)}} \mathcal{J} e_s \mathcal{P}_s^{-1} \bar{\mathcal{J}}} \tag{50}$$

We now introduce n scalar fields φ_s $(s = 1, ..., n)$ and we use again the functional gaussian formula (43).

$$\int [\mathcal{D}\varphi_s] [\mathcal{D}\bar{\varphi}_s] e^{i \int dx \{\bar{\varphi}_s \bar{e}_s \mathcal{P}_s \varphi_s + \frac{1}{\sqrt{n} m^{(n-1)}} (\mathcal{J} \varphi_s + \bar{\mathcal{J}} \bar{\varphi}_s)\}} =$$

$$= N_s | \mathcal{P}_s |^{-2} e^{-i \int \frac{dx}{nm^{2(n-1)}} \mathcal{J} e_s \mathcal{P}_s^{-1} \bar{\mathcal{J}}} \tag{51}$$

After a renormalization of the scalar field:

$$\varphi_s \to \sqrt{n} m^{(n-1)} \varphi_s$$

we can introduce (51) in (50) to write:

$$\int [\mathcal{D}\varphi] [\mathcal{D}\bar{\varphi}] \, e^{i \int dx \left(\bar{\varphi} \mathcal{P} \varphi + \mathcal{J} \varphi \bar{\mathcal{J}} \bar{\varphi} \right)} =$$

$$N \int \prod_s [\mathcal{D}\varphi_s] [\mathcal{D}\bar{\varphi}_s] \cdot$$

$$e^{i \int dx \sum_s \left(n m^{2(n-1)} \bar{\varphi}_s \bar{e}_s \mathcal{P}_s \varphi_s + \mathcal{J} \varphi_s + \bar{\mathcal{J}} \bar{\varphi}_s \right)} \tag{52}$$

To obtain again the generating functional (42), we multiply both members of (52) with:

$$e^{i \int dx \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \mathcal{K} \cdot A \right)}$$

and perform a functional integration over A_μ .

$$\mathcal{Z}\left(\mathcal{J}, \bar{\mathcal{J}}, \mathcal{K}\right) = N \int [\mathcal{D}A] \prod_s [\mathcal{D}\varphi_s] [\mathcal{D}\bar{\varphi}_s]$$

$$e^{i \int dx \left(\tilde{\mathcal{L}} + \sum_t \left(\mathcal{J} \varphi_t + \bar{\mathcal{J}} \bar{\varphi}_t \right) + \mathcal{K} A \right)} \tag{53}$$

Or,

$$\mathcal{Z}\left(\mathcal{J}, \bar{\mathcal{J}}, \mathcal{K}\right) = \tilde{\mathcal{Z}}\left(\mathcal{J}, \bar{\mathcal{J}}, \mathcal{K}\right) \tag{54}$$

In (53) we have introduced the definition:

$$\tilde{\mathcal{L}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \sum_{s=1}^{n} n m^{2(n-1)} \bar{\varphi}_s \bar{e}_s \mathcal{P}_s \varphi_s \tag{55}$$

The equivalence of the generating functionals $\mathcal{Z}$ and $\tilde{\mathcal{Z}}$, expressed by eqs.(53) and (54), implies the equivalence of the lagrangians (41) and (55).

The lagrangian (41) describes the gauge invariant electromagnetic interaction of a scalar field obeying a n-th order equation (cf. eq.(10)). On the other hand, eq.(55) refers to the electromagnetic interaction of n independent scalar fields obeying second order equations:

$$\mathcal{P}_s\varphi_s \equiv \left(\Box' - e_s m^2\right)\varphi_s = 0 \;\;,\;\; (s = 1, ..., n) \tag{56}$$

For s=1, eq.(56) is a normal Klein-Gordon equation for a charged scalar particle. For $s \neq 1$, φ_s is a "virtual field". It can only exist associated with closed loops attached to photon lines.

The equivalence shown by eq.(54) also tells us about the unitarity of the higher order theory. In fact, if we take any closed loop corresponding to one of the eqs.(56), with $s \neq 1$, and note that the propagators are half advanced and half retarded, we see that the imaginary part of the loop can only come from the imaginary part of the mass parameter. Then, the absortive part of this diagram cancels with another similar one for which the internal lines are related to the complex conjugate mass parameter (See also ref.[11]). In other words, any absortive part coming from a loop corresponding to $e_s m^2$ ($s \neq 1$) is exactly canceled by another contribution coming from a similar loop corresponding to $\bar{e}_s m^2$.

6 Discussion

Starting with a n-th order equation of motion for a scalar field φ, we introduce the electromagnetic interaction by means of the gauge invariant minimal coupling procedure. The field φ has n modes of propagation. Its evolution can be followed perturbatively with the Feynman's diagrams techniques. The Green function has n poles corresponding to n modes of propagation. The probability amplitude and the cross-section for any physical process can be determined without ambiguities.

We have shown that it is possible to reduce algebraically, each matrix element corresponding to the n-th order equation, to simple matrix elements in which the electromagnetic interaction and the propagation are described by second order equations.

The general proof of §5 shows that φ, with its n modes of propagation, behaves like n independent scalar fields φ_s , each obeying a simple second order equation.

The equivalence also shows the interesting fact that, no matter how high the order n is, the theory is unitary and renormalizable. Thus, we have the equivalence of two different points of view. One of them is the usual theory for a normal charged Klein-Gordon particle. The other is the theory for a field that obeys a higher order equation, minimally coupled to electromagnetism. The algebraic reduction of the matrix elements for the latter theory to those of the former one, presented in §4, appears to be rather mysterious. The simplification seems to be the result of a fortuitous coincidence. However, the functional proof (§5), sheds light on the nature of the equivalence.

It is clear that the proof is based on two fundamental properties of the equation of motion:

a) **Factorabizability** (cf. eqs. (10) and (11)). I.e.:

$$\mathcal{P}\varphi = 0$$

$$\mathcal{P} = \prod_{s=1}^{n} \mathcal{P}_s \quad , \quad [\mathcal{P}_a, \mathcal{P}_b] = 0$$

b) **Separability** (cf. eq. (22)). I.e., the general propagator can be expressed as a linear combination of individual propagators.

$$\mathcal{P}^{-1} = \sum_{s=1}^{n} \alpha_s \mathcal{P}_s^{-1}$$

Any operator which is both, factorizable and separable, gives rise to an equivalence theorem.

As a matter of facts, any arbitrary higher order equation can be factorized (we take $c_n = 1$):

$$\mathcal{P} \equiv \sum_{t=0}^{n} c_t \Box^t = \prod_{s=1}^{n} \left(\Box - m_s^2 \right)$$

$$\prod_{s=1}^{n} \mathcal{P}_s$$

where the "masses" m_s^2 are the n roots of $\mathcal{P}$ =0. Also the propagator $\mathcal{P}^{-1}$ can be expressed as:

$$\mathcal{P}^{-1} = \frac{1}{\prod_{s=1}^{n} (\Box - m_s^2)} = \sum_{s=1}^{n} \frac{d_s}{\Box - m_s^2}$$

where d_s (s=1,...,n) are appropriate constants.

The distinctive feature of eq.(10) is the fact that the masses have the particular form $m_s^2 = e_s m^2$, where $m^2 > 0$ and e_s are phase factors given by eq.(8).

The form of eq.(6) is dictated by its physical origin, based on supersymmetry in higher dimensions [5].

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