On the relation between determinants and Green functions of elliptic operators with local boundary conditions. *

Horacio Alberto FALOMIR[1]   Ricardo Enrique GAMBOA SARAVI [1]
María Amelia MUSCHIETTI[2]   Eve Mariel SANTANGELO[1]
Jorge Eduardo SOLOMIN[2]

Facultad de Ciencias Exactas
Universidad Nacional de La Plata
[1]Departamento de Física
[2]Departamento de Matemática

Abstract

A formula relating quotients of determinants of elliptic differential operators sharing their principal symbol, with local boundary conditions, to the corresponding Green function is given.

Résumé

On établit une formule reliant quotients des déterminants des opérateurs différentiels elliptiques qui partagent leur symbol pricipal, avec des conditions au bord locales, et les fonctions de Green associées.

arXiv:hep-th/9608102v1 15 Aug 1996

1

1   Introduction

Functional determinants have wide applications in Quantum and Statistical Physics. A powerful tool to regularize such determinants in a gauge invariant way, the so called ζ-function method [1], is based on Seeley's construction of complex powers of elliptic differential operators.

This construction has been largely studied and applied in the case of boundaryless manifolds,(see, for instance, [2] and references therein).

For manifolds with boundary, the study of complex powers was performed in [3, 4] for the case of local boundary conditions, while for the case of nonlocal conditions, this task is still in progress (see, for example, [5].)

The aim of this paper is to establish a relationship between determinants of differential operators, under local elliptic boundary conditions, and the corresponding Green functions, which involves a finite number of Seeley's coefficients. (We have perfo rmed an application of such a relationship to a concrete physical situation, involving Dirac operators, in a previous paper [6], where no mathematical proofs were included.)

For the sake of simplicity, only first order operators will be considered in the following, although straightforward modifications would allow to generalize this result to elliptic boundary problems of any order.

2   Seeley's complex powers and regularized determinants for local elliptic boundary problems

Let D be a first order elliptic operator,

$$D : \mathcal{C}^\infty(M, E) \to \mathcal{C}^\infty(M, F), \tag{1}$$

where M is a bounded closed domain in $\mathbf{R}^\nu$ with smooth boundary ∂M, and E and F are k -dimensional complex vector bundles over M, with a local boundary condition $B : L^2(E/\partial M) \to L^2(G)$ being G a r-dimensional complex vector bundle over ∂M, $r < k$.

In a collar neighborhood of ∂M in M, we will take coordinates $\bar{x} = (x, t)$, with t the inward normal coordinate and x local coordinates for ∂M, and conjugated variables $\bar{\xi} = (\xi, \tau)$ in T^*M.

The elliptic boundary problem

$$\begin{cases} D\varphi = 0 \text{ in M} \\ \\ B\varphi = f \text{ on } \partial\text{M} \end{cases} \tag{2}$$

is said to admit a cone of Agmon's directions if there is a cone Λ in the λ complex plane such that
   1) $\forall \bar{x} \in M$, $\forall \bar{\xi} \neq 0$, Λ contains no eigenvalues of the matrix $\sigma_1(D)(\bar{x}, \bar{\xi})$.
   and
   2) $\forall \lambda \in \Lambda$, $\forall x \in \partial M$, $\forall g \in \mathbf{C}^r$, the initial value problem

$$\sigma_1(D)(x, 0; \xi, -i\partial_t) \ u(t) = \lambda \ u(t)$$

$$b(x) \ u(0) = g$$

has, for each $\xi \neq 0$, a unique solution satisfying $\lim\limits_{t \to \infty} u(t) = 0$, being $\sigma_1(D)$ the principal symbol of D and $b(x)$ such that $B(\phi)(x) = b(x)\phi(x)$.

Henceforth, we assume the existence of an Agmon's cone Λ. Moreover, we will consider only boundary conditions B giving rise to a discrete spectrum $sp(D_B)$, where D_B denotes the closure of D acting on the sections $\phi \in \mathcal{C}^\in fty(M, E)$ satisfying $B\phi = 0$ on ∂M. Note that, this is always the case for elliptic boundary problems unless $sp(D_B)$ is the whole complex plane. Now, for $|\lambda|$ large enough, $sp(D_B) \cap \Lambda$ is empty, since there is no λ in $sp(\sigma_1(D_B)) \cap \Lambda$. Then, $sp(D_B) \cap \Lambda$ is a finite set.

For $\lambda \in \Lambda$ not in $sp(D_B)$, and

$$\sigma(D - \lambda I) = a_0(x, t; \xi, \tau; \lambda) + a_1(x, t; \xi, \tau; \lambda), \tag{3}$$

with a_l homogeneous of degree l in $(\bar{\xi}, \lambda)$, an asymptotic expansion of the symbol of $R(\lambda) = (D_B - \lambda I)^{-1}$ can be explicitly given [3]:

$$\sigma(R(\lambda)) \sim \sum_{j=o}^{\infty} c_{-1-j} - \sum_{j=o}^{\infty} d_{-1-j} \tag{4}$$

where the *Seeley coefficients* c_{-1-j} and d_{-1-j} satisfy

$$\sum_{j=o}^{\infty} a_{1-j} \circ \sum_{j=0}^{\infty} c_{-1-j} = I, \tag{5}$$

$\circ$ denoting the usual composition of homogeneous symbols (see for instance [7],) and

$$\begin{cases} \sigma'(D - \lambda) \circ \sum\limits_{j=o}^{\infty} d_{-1-j} = 0 \\[2mm] \sigma(B) \circ \sum\limits_{j=o}^{\infty} d_{-1-j} = \sigma(B) \circ \sum\limits_{j=0}^{\infty} c_{-1-j} \quad \text{at} \ \ t = 0 \\[2mm] \lim\limits_{t \to \infty} d_{-1-j} = 0. \end{cases} \tag{6}$$

Here $\sigma'(D - \lambda I)$, the "partial symbol" of D at the boundary, is defined as follows:

$$\sigma'(D - \lambda I) = \sum_j a^{(j)}, \tag{7}$$

where

$$a^{(j)} = a^{(j)}(x, t, \xi, -i\partial_t, \lambda) = \sum_{l-k=j} \frac{t^k}{k!} a_l^{(k)}(x, 0, \xi, -i\partial_t, \lambda), \tag{8}$$

with $a_l^{(k)} = \partial_t^k a_l$ and a_l as in (3).

Note that condition 2) implies the existence and unicity of the solution of (6).

Written in more detail, the first line in (6) becomes [3]

$$a^{(1)} d_{-1-j} + \sum_{\substack{l < j \\ k - |\alpha| - 1 - l = -j}} \frac{i^\alpha}{\alpha!} \frac{\partial^\alpha}{\partial \xi^\alpha} a^{(k)} \frac{\partial^\alpha}{\partial x^\alpha} d_{-1-l} = 0, \tag{9}$$

while the second one is

$$b_0 d_{-1-j} = b_0 c_{-1-j}|_{t=0}. \tag{10}$$

It is worth noticing that, although

$$\sigma(R(\lambda)) = \sum_{j=0}^{\infty} c_{-1-j} \ , \tag{11}$$

is an asymptotic expansion of $\sigma(R(\lambda))$, the fundamental solution of $(D_B - \lambda)$ obtained by Fourier transforming Eq.(11) does not in general satisfy the required boundary conditions. In fact, the coefficients d_{-1-j} are added to the expansion in order to correct this deficiency.

3

The coefficients $c_{-1-j}\,(x,t;\xi,\tau;\lambda)$ and $d_{-1-j}(x,t;\xi,\tau;\lambda)$ are meromorphic functions of λ with poles at those points where $\det[\sigma_1(D-\lambda)(x,t;\xi,\tau)]$ vanishes. The c_{-1-j}'s are homogeneous of degree $-1-j$ in (ξ,τ,λ); the d_{-1-j} 's are also homogeneous of degree $-1-j$, but in ($\frac{1}{t},\xi,\tau,\lambda$) [3].

From these coefficients we get an approximation to $(D_B-\lambda)^{-1}$, a parametrix constructed as in [3]

$$P_K(\lambda) = \sum_\varphi \psi \left[\sum_{j=0}^K Op(\theta_2\; c_{-1-j}) - \sum_{j=0}^K Op'(\theta_1\; \tilde{d}_{-1-j}) \right] \varphi, \tag{12}$$

where φ is a partition of the unity, $\psi \equiv 1$ in $Supp(\varphi)$, θ_1 and θ_2 cut-off functions for $|\xi|^2 + |\lambda|^2 \geq 1$ and $\chi(|\xi|^2 + |\tau|^2 + |\lambda|^2 \geq 1$ respectively, and

$$Op(\sigma)h(x,t) = \int \sigma(x,t;\xi,\tau)\; \hat{h}(\xi,\tau)\; e^{i(x\xi+t\tau)} \frac{d\xi}{(2\pi)^{\nu-1}}\; \frac{d\tau}{2\pi},$$

$$ \tag{13}$$

$$Op'(\sigma)h(x,t) = \int \int \tilde{\sigma}(x,t;\xi,s)\; \tilde{h}(\xi,s)\; e^{ix\xi} \frac{d\xi}{(2\pi)^{\nu-1}}\; \frac{ds}{2\pi},$$

where $\hat{h}(\xi,\tau)$ is defined in (25) and

$$\tilde{h}(\xi,s) = \int\; h(x,s)\; e^{-ix\xi}\; dx. \tag{14}$$

Moreover, it can be proved that, for $\lambda \in \Lambda$,

$$\| R(\lambda) \|_{L^2} \leq C|\lambda|^{-1} \tag{15}$$

with C a constant [3, 8].

The estimate (15) allows for expressing the complex powers of D_B as

$$D_B^z = \frac{i}{2\pi} \int_\Gamma \lambda^z\; R(\lambda)\; d\lambda \tag{16}$$

for $Re\; z < 0$, where Γ is a closed path lying in Λ, enclosing the spectrum of D_B [4]. Note that such a curve Γ always exists for $sp(D_B) \cap \Lambda$ finite.

For $Re\; z \geq 0$, one defines
$$D_B^z = D^l \circ D_B^{z-l} \;, \tag{17}$$

for l a positive integer such that $Re\; (z-l) < 0$.

If $Re(z) < -\nu$, the power D_B^z is an integral operator with continuous kernel $J_z(x,t;y,s)$ and, consequently, it is trace class (for an operator of order ω, this is true if $Re(z) < -\frac{\nu}{\omega}$). As a function of z, $Tr(D_B^z)$ can be extended to a meromorphic function in the whole complex plane $\mathbf{C}$, with only simple poles at $z = j - \nu$, $j = 0, 1, 2, ...$ and vanishing residues when $z = 0, 1, 2, ...$ (for an operator of order ω , there are only single poles at $z = \frac{j-\nu}{\omega}$, $j = 0, 1, 2, ...,$ with vanishing residues at $z = 0, 1, 2, ...)$ [4].

The function $Tr(D_B^z)$ is usually called $\zeta_{(D_B)}(-z)$ because of its similarity with the classical Riemann ζ-function: if $\{\lambda_j\}$ are the eigenvalues of D_B, $\{\lambda_j^z\}$ are the eigenvalues of D_B^z; so $Tr(D_B^z) = \sum \lambda_j^z$ when D_B^z is a trace class operator.

A regularized determinant of D_B can then be defined as

$$Det\; (D_B) = \exp[-\frac{d}{dz}\; Tr\; (D_B^z)]|_{z=0}. \tag{18}$$

Now, let $D(\alpha)$ be a family of elliptic differential operators on M sharing their principal symbol and analytically depending on α. Let B give rise to an elliptic boundary condition for all of them, in such

a way that $D(\alpha)_B$ is invertible and the boundary problems they define have a common Agmon's cone. Then, the variation of $Det\ D(\alpha)_B$ with respect to α is given by (see, for example, [9, 10])

$$\frac{d}{d\alpha} \ln\ Det\ D(\alpha)_B = \frac{d}{dz}\left[\ z\ Tr\{\frac{d}{d\alpha}\ (D(\alpha)_B)\ D(\alpha)_B^{z-1}\}\right]_{z=0}. \tag{19}$$

Note that, under the assumptions made, $\frac{d}{d\alpha}\ (D(\alpha)_B)$ is a multiplication operator.

Although $J_z(x,t;x,t;\alpha)$, the kernel of $D(\alpha)_B^z$ evaluated at the diagonal, can be extended to the whole z-complex plane as a meromorphic function, the r.h.s. in (19) cannot be simply written as the integral over M of the finite part of

$$tr\{\frac{d}{d\alpha}\ (D(\alpha)_B)\ J_{z-1}(x,t;x,t;\alpha)\} \tag{20}$$

at $z = 0$ (where tr means matrix trace). In fact, $J_{z-1}(x,t;x,t;\alpha)$ is in general non integrable in the variable t near ∂M for $z \approx 0$.

Nevertheless, an integral expression for $\frac{d}{d\alpha} \ln\ Det\ D(\alpha)_B$ will be constructed in the next section from the integral expression for $Tr(D(\alpha)_B^{z-1})$, holding in a neighborhood of $z = 0$, obtained in the following way [4]:

if $T > 0$ is small enough, the function $j_z(x;\alpha)$ defined as

$$j_z(x;\alpha)\ =\ \int_0^T J_z(x,t;x,t;\alpha)\ dt \tag{21}$$

for $Re\ z < 1 - \nu$, admits a meromorphic extension to $\mathbf{C}$ as a function of z. So, if V is a neighborhood of ∂M defined by $t < \epsilon$, with ϵ small enough, $Tr(D(\alpha)_B^{z-1})$ can be written as the finite part of

$$\int_{M/V} tr\ J_{z-1}(x,t;x,t;\alpha)\ dxdt + \int_{\partial M} tr\ j_{z-1}(x;\alpha)\ dx\ , \tag{22}$$

where a suitable partition of the unity is understood.

3   Green functions and determinants

In this section, we will establish an expression for $\frac{d}{d\alpha} \ln Det[D(\alpha)_B]$ in terms of $G_B(x,t;y,s;\alpha)$, the Green function of $D(\alpha)_B$ (i.e., the kernel of the operator $[D(\alpha)_B]^{-1}$).

With the notation of the previous Section, (19) can be rewritten as:

$$\frac{d}{d\alpha} \ln\ Det\ D(\alpha)_B =\ \underset{z=0}{F.P.}\ \int_M tr\left[\frac{d}{d\alpha}\left(D(\alpha)_B\right)\ J_{-z-1}(x,t;x,t;\alpha)\right]\ d\bar{x}\ , \tag{23}$$

where the r.h.s. must be understood as the finite part of the meromorphic extension of the integral at $z = 0$.

The finite part of $J_{-z-1}(x,t;x,t;\alpha)$ at $z = 0$ does not coincide with the regular part of $G_B(x,t;y,s;\alpha)$ at the diagonal, since the former is defined through an analytic extension.

However, we will show that there exists a relation between them, involving a finite number of Seeley's coefficients. In fact, for boundaryless manifolds this problem has been studied in [11], by comparing the iterated limits $F.P. \underset{z\to-1}{\lim} \{\underset{\bar{y}\to\bar{x}}{\lim} J_z(x,t;y,s;\alpha)\}$ and $R.P. \underset{\bar{y}\to\bar{x}}{\lim}\{\underset{z\to-1}{\lim} J_z(x,t;y,s;\alpha)\} = R.P.\underset{\bar{y}\to\bar{x}}{\lim} G_B(x,t;y,s;\alpha)$.

In the case of manifolds with boundary, the situation is more involved owing to the fact that the finite part of the extension of $J_z(x,t;x,t;\alpha)$ at $z = -1$ is not integrable near ∂M . (A first approach to this problem appears in [12]). Nevertheless, as mentioned in Section 2, a meromorphic extension of $\int_0^T J_z(x,t;x,t;\alpha)dt$, with T small enough can be performed and its finite part at $z = -1$ turns to be integrable in the tangential variables. A similar result holds, *afortiori*, for $\int_0^T t^n J_z(x,t;x,t;\alpha)dt$, with $n = 1,2,3...$ Then, near the boundary, the Taylor expansion of the function $A_\alpha = \frac{d}{d\alpha}D(\alpha)_B$ will naturally appear, and the limits to be compared are $F.P. \underset{z\to-1}{\lim}\{\underset{\bar{y}\to\bar{x}}{\lim} \int_0^T t^n J_z(x,t;y,s;\alpha)dt\}$ and $R.P. \underset{\bar{y}\to\bar{x}}{\lim}\{\underset{z\to-1}{\lim} \int_0^T t^n J_z(x,t;y,s;\alpha)dt\} = R.P. \underset{\bar{y}\to\bar{x}}{\lim} \int_0^T t^n G_B(x,t;y,s;\alpha)dt$.

The starting point for this comparison will be to carry out asymptotic expansions and to analyze the terms for which the iterated limits do not coincide (or do not even exist).

An expansion of $G_B(x,t,y,s)$ in $M\backslash\partial M$ in homogeneous and logarithmic functions of $(\bar{x} - \bar{y})$ can be obtained from (4) for $\lambda = 0$ (see [7]):

$$G_B(x,t,y,s) = \sum_{j=1-\nu}^0 h_j(x,t,x-y,t-s) + M(x,t)\log|(x,t)-(y,s)| + R(x,t,y,s), \tag{24}$$

with h_j the Fourier transform $\mathcal{F}^{-1}(c_{-\nu-j})$ for $j > 0$ and $h_0 = \mathcal{F}^{-1}(c_{-\nu}) - M(x,t)\log|(x,t)-(y,s)|$. The function $M(x,t)$ will be explicitly computed below (see (33)). Our convention for the Fourier transform is

$$\mathcal{F}(f)(\bar{\xi}) = \hat{f}(\bar{\xi}) = \int f(\bar{x})\ e^{-i\bar{x}.\bar{\xi}}\ d\bar{x},$$

$$\mathcal{F}^{-1}(\hat{f})(\bar{x}) = f(\bar{x}) = \frac{1}{(2\pi)^\nu}\int \hat{f}(\bar{\xi})\ e^{i\bar{x}.\bar{\xi}}\ d\bar{\xi}. \tag{25}$$

For $t > 0$, $R(x,t,y,s)$ is continuous even at the diagonal $(y,s) = (x,t)$. Nevertheless, $R(x,t,y,s)|_{(y,s)=(x,t)}$ is not integrable because of its singularities at $t = 0$. On the other hand, the functions $t^n R(x,t,y,t)$ are integrable with respect to the variable t for $y \neq x$ and $n = 0,1,2,....$ An expansion of $\int_0^\infty t^n R(x,t,y,t)dt$ in homogeneous and logarithmic functions of $(x - y)$ can also be obtained from (4):

$$\int_0^\infty t^n R(x,t,y,t)dt = \sum_{j=n+2-\nu}^0 g_{j,n}(x,x-y) + M_n(x)\log(|x-y|) + R_n(x,y) \tag{26}$$

where $R_n(x, y)$ is continuous even at $y = x$, and $g_{j,n}$ is the Fourier transform of the (homogeneous extension of) $\int_0^\infty t^n \tilde{d}_{-1-j}(x, t, \xi, t, 0) \, dt$, with

$$\tilde{d}_{-1-j}(x, t, \xi, s, \lambda) = -\int_{\Gamma^-} e^{-is\tau} \, d_{-1-j}(x, t, \xi, \tau, \lambda) \, d\tau \tag{27}$$

for Γ^- a closed path enclosing the poles of $d_{-1-j}(x, t, \xi, \tau, \lambda)$ lying in $\{Im \, \tau > 0\}$.

Since $\tilde{d}_{-1-j}$ is homogeneous of degree $-j$ in $(1/t, \, \xi, 1/s, \lambda)$, $g_{j,n}$ turns out to be homogeneous of degree j in $x - y$.

The following technical lemma will be used for the proof of our main result (Theorem 1):

Lemma 1: *Let $a(\xi)$ a function defined on $\mathbf{R}^\nu$, homogeneous of degree $-\nu$ for $|\xi| \geq 1$ and $a(\xi) = 0$ for $|\xi| < 1$. Then its Fourier transform can be written as*

$$\mathcal{F}^{-1}(a(\xi))(z) = h(z) + \; M \; \frac{\Omega_\nu}{(2\pi)^\nu} \; (\log|z|^{-1} + \mathcal{K}_\nu) + R(z), \tag{28}$$

where

a) $h(z)$ is a homogeneous function of degree 0, such that $\int_{|z|=1} h(z) \, d\sigma_z = 0$. It is given by

$$h(z) = \quad \mathcal{F}^{-1}(P.V.[a(\xi/|\xi|) - M \;]|\xi|^{-\nu})(z). \tag{29}$$

b)

$$M = \frac{1}{\Omega_\nu} \int_{|\xi|=1} a(\xi) \, d\sigma_\xi, \tag{30}$$

where $\Omega_\nu = Area(S^{\nu-1})$, and $\mathcal{K}_\nu = \ln 2 - \frac{1}{2}\gamma + \frac{1}{2}\frac{\Gamma'(\nu/2)}{\Gamma(\nu/2)}$ with γ the Euler's constant.

c) $R(z)$ is a function regular at $z = 0$ with $R(0) = 0$.

Proof: Writing $a(\xi) = \tilde{a}(\xi) + M|\xi|^{-\nu}\chi(\xi)$, with $\tilde{a}(\xi)$ having zero mean on $|\xi| = 1$, and χ the characteristic function of $|\xi| \geq 1$, this lemma follows from direct computations , by using the techniques for Fourier transforms of homogeneous functions (see for instance [7].)

Now, we introduce the main result of this paper.

Theorem 1: *Let M be a bounded closed domain in $\mathbf{R}^\nu$ with smooth boundary ∂M and E a k-dimensional complex vector bundle over M.*

Let $(D_\alpha)_B$ be a family of elliptic differential operators of first order, acting on the sections of E, with a fixed local boundary condition B on ∂M, and denote by $J_z(x, t; x, t; \alpha)$ the meromorphic extension of the evaluatio n at the diagonal of the kernel of $((D_\alpha)_B)^z$.

Let us assume that, for each α, $(D_\alpha)_B$ is invertible, the family is differentiable with respect to α, and $\frac{\partial}{\partial\alpha}(D_\alpha)_B f = A_\alpha f$, with A_α a differentiable function.

If V is a neighborhood of ∂M defined by $t < \epsilon$ and $T > 0$ small enough, then:
a)

$$\frac{\partial}{\partial\alpha} \ln \; Det(D_\alpha)_B = \; \underset{z=-1}{F.P.} \; \left[\int_{\partial M} \int_0^T tr\left\{ A_\alpha(x, t) \; J_z(x, t; x, t; \alpha) \right\} dt dx \right]$$

$$\tag{31}$$

$$+ \; \underset{z=-1}{F.P.} \; \left[\int_{M/V} tr\left\{ A_\alpha(\bar{x}) \; J_z(\bar{x}; \bar{x}; \alpha) \right\} d\bar{x} \right],$$

7

where a suitable partition of the unity is understood. (This expression must be understood as the finite part at $z = -1$ of the meromorphic extension).

b) For every α, the integral $\int_0^T A_\alpha(x,t) \ J_z(x,t;x,t;\alpha)dt$ is a meromorphic function of z, for each $x \in \partial M$, with a simple pole at $z = -1$. Its finite part (dropping, from now on, the index α for the sake of simplicity) is given by

$$
\underset{z=-1}{F.P.} \ \int_0^T A(x,t) \ J_z(x,t;x,t)dt = -\int_0^T A(x,t) \int_{|(\xi,\tau)|=1} \frac{i}{2\pi} \int_\Gamma \frac{\ln \lambda}{\lambda} \ c_{-\nu}(x,t;\xi,\tau;\lambda) \ d\lambda \ \frac{d\sigma_{\xi,\tau}}{(2\pi)^\nu} \ dt
$$

$$
+ \sum_{l=0}^{\nu-2} \frac{\partial_t^l A(x,0)}{l!} \int_{|\xi|=1} \int_0^\infty t^l \frac{i}{2\pi} \int_\Gamma \frac{\ln \lambda}{\lambda} \ \tilde{d}_{-(\nu-1)+l}(x,t;\xi,t;\lambda) \ d\lambda \ dt \ \frac{d\sigma_\xi}{(2\pi)^{\nu-1}}
$$

$$
+ \lim_{y \to x} \left\{ \int_0^T A(x,t) \left[G_B(x,t;y,t) - \sum_{l=1-\nu}^0 h_l(x,t;x-y,0) \ -M(x,t) \frac{\Omega_\nu}{(2\pi)^\nu} \left(\ln |x-y|^{-1} + \mathcal{K}_\nu \right) \right] dt \right.
$$

$$
\left. + \sum_{j=0}^{\nu-2} \sum_{l=0}^{\nu-2-j} \frac{\partial_t^l A(x,0)}{l!} \ g_{j,l-(\nu-2-j)}(x,x-y) + \sum_{l=0}^{\nu-2} \frac{\partial_t^l A(x,0)}{l!} \ M_{\nu-2-l}(x) \frac{\Omega_{\nu-1}}{(2\pi)^{\nu-1}} \left(\ln |x-y|^{-1} + \mathcal{K}_{\nu-1} \right) \right\},
$$
(32)

with

$$
M(x,t) = \frac{1}{\Omega_\nu} \int_{|(\xi,\tau)|=1} c_{-\nu}(x,t;\xi,\tau;0) \ d\sigma_{\xi,\tau}
$$
(33)

$$
M_j(x) = \frac{1}{\Omega_{\nu-1}} \int_{|\xi|=1} \int_0^\infty t^{\nu-2-j} \ \tilde{d}_{-1-j}(x,t;\xi,t;0) \ dt \ d\sigma_\xi,
$$

where $\Omega_n = Area(S^{n-1})$, and where h_l and $g_{l,n}$ are related to the Green function G_B as in (24) and (26)

$$
h_{1-\nu+j}(x,t;w,u) = \mathcal{F}_{(\xi,\tau)}^{-1} \left[c_{-1-j}(x,t;(\xi,\tau)/|(\xi,\tau)|;0) \ |(\xi,\tau)|^{-1-j} \right] (w,u),
$$

$$
h_0(x,t;w,u) = \mathcal{F}_{(\xi,\tau)}^{-1} \left[P.V. \left\{ (c_{-\nu}(x,t;(\xi,\tau)/|(\xi,\tau)|;0) - M(x,t)) \ |(\xi,\tau)|^{-\nu} \right\} \right] (w,u),
$$
(34)

$$
g_{j,l}(x,w) = \mathcal{F}_\xi^{-1} \left[\int_0^\infty t^n \ \tilde{d}_{-1-j}(x,t;\xi/|\xi|,t;0) \ dt|\xi|^{-1-j-n} \right] (w),
$$

with $l = j + n - \nu + 2$ and

$$
g_{j,0}(x,w) = \mathcal{F}_\xi^{-1} \left[P.V. \left[\int_0^\infty t^{\nu-j-2} \ \tilde{d}_{-1-j}(x,t;\xi/|\xi|,t;0) \ dt - M_j(x) \right] |\xi|^{-(\nu-1)} \right] (w).
$$
(35)

c) The integral $\int_{M \setminus V} tr \ [A(\bar{x}) \ J_z(\bar{x};\bar{x})] \ d\bar{x}$ in the second term in the r.h.s. of (31) , is a meromorphic function of z with a simple pole at $z = -1$. Its finite part is given by

$$
\underset{z=-1}{F.P.} \ \int_{M \setminus V} tr \ [A(\bar{x}) \ J_z(\bar{x};\bar{x})] \ d\bar{x} = \int_{M \setminus V} A(\bar{x}) \int_{|\bar{\xi}|=1} \frac{i}{2\pi} \int \frac{\ln \lambda}{\lambda} \ c_{-\nu}(\bar{x},\bar{\xi};\lambda) \ d\lambda \frac{d\bar{\xi}}{(2\pi)^\nu}
$$
(36)

$$
+ \int_{M \setminus V} \lim_{\bar{y} \to \bar{x}} A(\bar{x})[G_B(\bar{x},\bar{y}) - \sum_{l=1-\nu}^0 h_l(\bar{x},\bar{x}-\bar{y}) - M(\bar{x}) \frac{\Omega_\nu}{(2\pi)^\nu} (\ln |\bar{x}-\bar{y}|^{-1} + \mathcal{K}_\nu)] \ d\bar{x}.
$$

8

Proof: Statement a) is a direct consequence of (19), (21) and (22).

In what follows, we will proof the assertion in (32).

We will use, as an approximation to $(D_B - \lambda)^{-1}$, the parametrix $P_K(\lambda)$ in (12).

Thus, we can approximate the kernel J_z of D_B^z by means of the kernel L_z^K of $\frac{i}{2\pi} \int_\Gamma \lambda^z P_K(\lambda) d\lambda$. We have

$$
\begin{aligned}
L_z^K(x,t;y,s) = \sum_\varphi \psi(x,t) \Bigg[&\sum_{j=0}^K \int_{\mathbf{R}^\nu} C_{-1-j}(x,t;\xi,\tau;z) \; e^{i[(x-y)\xi + (t-s)\tau]} \; \frac{d\xi}{(2\pi)^{\nu-1}} \; \frac{d\tau}{2\pi} \\
&- \sum_{j=0}^K \int_{\mathbf{R}^{\nu-1}} D_{-1-j}(x,t;\xi,\tau;z) \; e^{i(x-y)\xi} \; \frac{d\xi}{(2\pi)^{\nu-1}} \Bigg] \; \varphi(y,s)
\end{aligned}
\tag{37}
$$

with

$$
C_{-1-j}(x,t;\xi,\tau;z) = \frac{i}{2\pi} \int_\Gamma \lambda^z \; \theta_2(\xi,\tau;\lambda) \; c_{-1-j}(x,t;\xi,\tau;\lambda) \; d\lambda,
\tag{38}
$$

and

$$
D_{-1-j}(x,t;\xi,t;z) \equiv \frac{i}{2\pi} \int_\Gamma \lambda^z \theta_1(\xi,\lambda) \; \tilde{d}_{-1-j}(x,t;\xi,t;\lambda) \; d\lambda.
\tag{39}
$$

These expressions are, in fact, analytic functions of z for all complex z, since the singularities of $c_{-1-j}(\lambda)$ and $\tilde{d}_{-1-j}(\lambda)$ are in a compact set in the λ plane, for $(x,t;\xi,\tau)$ in a compact set.

Since $(D_B - \lambda)^{-1} - P_K(\lambda)$ has a continuous kernel of $O(|\lambda|^{\nu - K - 1})$ for $\lambda \in \Lambda$ [3], it turns out that

$$
R(x,t;y,s;z) = J_z(x,t;y,s) - L_z^K(x,t;y,s)
\tag{40}
$$

is a continuous function of x, t, y, s and z, and analytic in z for $Re(z) < 0$, if $K \geq \nu$. Analyzing the last terms in L_z^K, we obtain that it is also true for $K = \nu - 1$. From now on, we call $L_z = L_z^{\nu - 1}$. Then

$$
\lim_{z \to -1} \left[\lim_{(y,s) \to (x,t)} (J_z - L_z) \right] = \lim_{(y,s) \to (x,t)} \left[\lim_{z \to -1} (J_z - L_z) \right].
\tag{41}
$$

Since

$$
J_{-1}(x,t;y,s) = G_B(x,t;y,s), \quad \text{for } (x,t) \neq (y,s),
\tag{42}
$$

we have

$$
\lim_{z \to -1} (J_z(x,t;x,t) - L_z(x,t;x,t)) = \lim_{(y,s) \to (x,t)} (G_B(x,t;y,s) - L_{-1}(x,t;y,s)).
\tag{43}
$$

One can cancel some terms in the equality (43) by studying the singularities of $L_z(x,t;x,t)$ at $z = -1$ and those of $L_{-1}(x,t;y,s)$ at $(x,t) = (y,s)$. More precisely:

Lemma 2 :

The following statement holds

$$\lim_{z \to -1} \left[J_z(x,t;x,t) + \frac{1}{(z+1)} \int_{|(\xi,\tau)|=1} c_{-\nu}(x,t;\xi,\tau;0) \, \frac{d\sigma_{\xi,\tau}}{(2\pi)^\nu} \right.$$

$$\left. + \int_{|(\xi,\tau)|=1} \frac{i}{2\pi} \int_\Gamma \frac{\ln \lambda}{\lambda} c_{-\nu}(x,t;\xi,\tau;\lambda) \, d\lambda \, \frac{d\sigma_{\xi,\tau}}{(2\pi)^\nu} + \sum_{j=0}^{\nu-1} \int_{\mathbf{R}^{\nu-1}} D_{-1-j}(x,t;\xi,t;z) \, \frac{d\xi}{(2\pi)^{\nu-1}} \right]$$

$$(44)$$

$$= \lim_{y \to x} \left[G_B(x,t;y,t) - \sum_{l=1-\nu}^{0} h_l(x,t;x-y,0) \right.$$

$$\left. - M(x,t) \frac{\Omega_\nu}{(2\pi)^\nu} (\ln |x-y|^{-1} + \mathcal{K}_\nu) + \sum_{j=0}^{\nu-1} \int_{\mathbf{R}^{\nu-1}} D_{-1-j}(x,t;\xi,t;-1) e^{i(x-y)\xi} \, \frac{d\xi}{(2\pi)^{\nu-1}} \right].$$

Proof: Eq.(44) is obtained from (43) in the following way:

In the l.h.s., the Fourier transform of the $C_{-1-j}(x,t;\xi,\tau;z)'s$ involved in $L_z(x,t;x,t)$ is written as $\int_{|(\xi,\tau)|\le 1} + \int_{|(\xi,\tau)|>1}$, taking into account their homogeneity for $|(\xi,\tau)| > 1$ and analyticity in z.

In the r.h.s., the Fourier transforms of $C_{-1-j}(x,t;(\xi,\tau)/|(\xi,\tau)|;-1)$ $|(\xi,\tau)|^{-1-j}$ is added and subtracted (this gives rise to the functions h_l for $l > 0$.) Next, compute the Fourier transform of C_ν by using Lemma 1.

Finally, repeated terms are canceled (Notice that, for $|(\xi,\tau)| \ge 1$, $C_{-\nu}(x,t;\xi,\tau;z = -1) = c_{-\nu}(x,t;\xi,\tau;\lambda = 0)$.)

The meromorphic extension of the terms involving the coefficients $C_{-1-j}(x,t;\xi,\tau;z)$ in $L_z(x,t;x,t)$ is a consequence of the previous arguments. Although $\sum_{j=0}^{\nu-1} \int_{\mathbf{R}^{\nu-1}} D_{-1-j}(x,t;\xi,t;z) \, \frac{d\xi}{(2\pi)^{\nu-1}}$ does not admit, in general, a meromorphic extension, such extension can be performed for

$$\int_0^T t^n \int_{\mathbf{R}^{\nu-1}} D_{-1-j}(x,t;\xi,\tau;z) \, \frac{d\xi}{(2\pi)^{\nu-1}} \, dt, \qquad (45)$$

for $n = 0, 1, ...$ and $j = 0, 1, 2, ...$ (see [4] and the next lemma.) Then, in order to prove b) in the Theorem, we first establish a technical result obtained from the fundamental estimate

$$|t^n \partial_\xi^\alpha \tilde{d}_{-1-j}(x,t,\xi,s;\lambda)| \le C e^{-c(t+s)(|\xi|+|\lambda|)} (|\xi| + |\lambda|)^{-j-n-|\alpha|}, \qquad (46)$$

for $t, s > 0$, $\lambda \in \Lambda$, due to R.T. Seeley [3]:

Lemma 3:
For $D_{-1-j}(x,t;\xi,t;z)$ as in (39) it holds:
i) If $r(x,t)$ is a function satisfying $|r(x,t)| \le Ct^n$ for $0 < t < T$, $n \in \mathbf{N}$, $T > 0$,

$$\int_0^T r(x,t) \int_{\mathbf{R}^{\nu-1}} D_{-1-j}(x,t;\xi,t;z) \, e^{i(x-y)\xi} \, \frac{d\xi}{(2\pi)^{\nu-1}} \, dt \qquad (47)$$

is an absolutely convergent integral for $Re(z) < j+n-\nu+1$. As a consequence, it is an analytic function of z in this region, and it is continuous in all the variables (x,y,z).

ii) If $x \ne y$, (47) is an absolutely convergent integral for all $z \in \mathbf{C}$, and so no analytic extension is needed out of the diagonal.

10

iii)

$$\int_0^\infty t^n D_{-1-j}(x,t;\xi,t;z) \ dt \tag{48}$$

is an homogeneous function of ξ for $|\xi| \geq 1$, of degree $z - j - n$, analytic in z for $Re(z) < j + n$ and then

$$\int_{\mathbf{R}^{\nu-1}} \int_0^\infty t^n D_{-1-j}(x,t;\xi,t;z) \ dt \ \frac{d\xi}{(2\pi)^{\nu-1}} = \alpha_j^n(x;z) + \frac{1}{z-j-n+\nu-1} \beta_j^n(x;z) \tag{49}$$

with

$$\alpha_j^n(x;z) = \int_{|\xi|\leq 1} \int_0^\infty t^n D_{-1-j}(x,t;\xi,t;z) \ dt \ \frac{d\xi}{(2\pi)^{\nu-1}}$$

$$\tag{50}$$

$$\beta_j^n(x;z) = \int_{|\xi|=1} \int_0^\infty t^n D_{-1-j}(x,t;\xi,t;z) \ dt \ \frac{d\xi}{(2\pi)^{\nu-1}}$$

analytic functions of z for $Re(z) < j + n$.
iv)

$$\int_{\mathbf{R}^{\nu-1}} \int_T^\infty t^n D_{-1-j}(x,t;\xi,t;z) \ e^{i(x-y)\xi} \ dt \ \frac{d\xi}{(2\pi)^{\nu-1}} \tag{51}$$

is an entire function of z, continuous in (x,y,z).

Proof: It follows from the homogeneity and analyticity properties of the functions $\tilde{d}_{-1-j}$ and the estimate (46).

Now, to get part b) of the theorem, we study the limits $\lim_{z\to -1} \int_0^T A(x,t) \ R(x,t;z) \ dt$ and $\lim_{y\to x} \int_0^T A(x,t) \ S(x,t;y,t) \ dt$, where $R(x,t;z)$ and $S(x,t;y,t)$ denote the expressions appearing in the limits on the l.h.s. and r.h.s. of (44) respectively. By considering the expansion of $A(x,t)$ in powers of t, we obtain:

Lemma 4: *If $A(x,t)$ has $\nu - 1 - j$ continuous derivatives in the variable t, $t \geq 0$, then*
i) For $\nu - 1 - j > 0$,

$$\int_0^T A(x,t) \int D_{-1-j}(x,t;\xi,t;z) \ \frac{d\xi}{(2\pi)^{\nu-1}} \ dt = \psi_j(x,z)$$

$$- \frac{1}{z+1} \frac{\partial_t^{\nu-j-2} A(x,0)}{(\nu-j-2)!} \int_{|\xi|=1} \int_0^\infty t^{\nu-j-2} D_{-1-j}(x,t;\xi,t;-1) \ dt \ \frac{d\sigma_\xi}{(2\pi)^{\nu-1}} \tag{52}$$

$$- \frac{\partial_t^{\nu-j-2} A(x,0)}{(\nu-j-2)!} \int_{|\xi|=1} \int_0^\infty t^{\nu-j-2} \partial_z D_{-1-j}(x,t;\xi,t;-1) \ dt \frac{d\sigma_\xi}{(2\pi)^{\nu-1}},$$

with $\psi_j(x,z)$ an analytic function of z for $Re(z) < 0$.
Moreover,

$$\int_0^T A(x,t) \int D_{-1-j}(x,t;\xi,t;-1) \ e^{i(x-y)\xi} \ \frac{d\xi}{(2\pi)^{\nu-1}} \ dt = \varphi_j(x,y)$$

$$\tag{53}$$

$$+ \sum_{\substack{n=0 \\ l=j+n-\nu+2}}^{\nu-j-2} \frac{\partial_t^n A(x,0)}{n!} \ g_{j,l}(x,x-y) \ + \frac{\partial_t^{\nu-j-2} A(x,0)}{(\nu-j-2)!} \ M_j(x) \ \frac{\Omega_{\nu-1}}{(2\pi)^{\nu-1}} (\ln |x-y|^{-1} + \mathcal{K}_{\nu-1}),$$

where $\varphi_j(x,y)$ is a continuous function.

11

ii) For $\nu - 1 - j = 0$,

$$\int_0^T A(x,t) \int D_{-1-j}(x,t;\xi,t;z) \, \frac{d\xi}{(2\pi)^{\nu-1}} \, dt = \psi_j(x,z) \tag{54}$$

is an analytic function of z for $Re(z) < 0$, and

$$\int_0^T A(x,t) \int D_{-1-j}(x,t;\xi,t;-1) \, e^{i(x-y)\xi} \, \frac{d\xi}{(2\pi)^{\nu-1}} \, dt = \varphi_j(x,y) \tag{55}$$

is a continuous function.
   iii) For all j

$$\lim_{z\to-1} \psi_j(x,z) = \lim_{y\to x} \varphi_j(x,y) \tag{56}$$

Proof: For analyzing the expression $\int_0^T A(x,t) \int D_{-1-j}(x,t;\xi,t;z) \, d\xi \, dt$ for $z \to -1$, we develop A in powers of t and apply Lemma 3.

For the integral $\int_0^T A(x,t) \int D_{-1-j}(x,t;\xi,t;-1) \, e^{i(x-y)\xi} \, d\xi \, dt$, we expand A, use Lemma 3 and evaluate the Fourier transforms with the same technique as in Lemma 2, finding the singular terms given by the functions $g_{j,l}$ and $\ln|x-y|$.

Finally, in order to get part b) of Theorem 1 we write the equality in Lemma 2 as

$$\lim_{z\to-1} R(x,t;z) = \lim_{y\to x} S(x,y,t) \tag{57}$$

and evaluate the integrals $\int_0^T A(x,t) \, R(x,t;z) \, dt$ and $\int_0^T A(x,t) \, S(x,y,t) \, dt$.
   For the first one, we have

$$
\begin{aligned}
&\int_0^T A(x,t) \left[J_z(x,t;x,t) + \frac{1}{z+1} \int_{|(\xi,\tau)|=1} c_{-\nu}(x,t;\xi,\tau;0) \frac{d\sigma_{\xi,\tau}}{(2\pi)^\nu} \right. \\[2mm]
&\left. + \int_{|(\xi,\tau)|=1} \frac{i}{2\pi} \int \frac{\ln\lambda}{\lambda} c_{-\nu}(x,t;\xi,\tau;\lambda) \, d\lambda \, \frac{d\sigma_{\xi,\tau}}{(2\pi)^\nu} \right] \, dt \\[3mm]
&= -\sum_{j=0}^{\nu-1} \int_0^T A(x,t) \int D_{-1-j}(x,t;\xi,t;z) \frac{d\xi}{(2\pi)^{\nu-1}} \, dt + \int_0^T A(x,t) \, R(x,t;z) \, dt \\[3mm]
&= \sum_{j=0}^{\nu-2} \frac{1}{z+1} \frac{\partial_t^{\nu-j-2} A(x,0)}{(\nu-j-2)!} \int_{|\xi|=1} \int_0^\infty t^{\nu-j-2} D_{-1-j}(x,t;\xi,t;-1) \, dt \, \frac{d\sigma_\xi}{(2\pi)^{\nu-1}} \\[3mm]
&+ \sum_{j=0}^{\nu-2} \frac{\partial_t^{\nu-j-2} A(x,0)}{(\nu-j-2)!} \int_{|\xi|=1} \int_0^\infty t^{\nu-j-2} \partial_z D_{-1-j}(x,t;\xi,t;-1) \, dt \, \frac{d\sigma_\xi}{(2\pi)^{\nu-1}} \\[3mm]
&- \sum_{j=0}^{\nu-1} \psi_j(x,z) + \int_0^T A(x,t) \, R(x,t;z) \, dt.
\end{aligned}
\tag{58}
$$

12

For the integral involving $S(x, y, t)$, we have

$$\int_0^T A(x,t)\left[G_B(x,t;y,t) - \sum_{l=-(\nu-1)}^{0} h_l(x,t;x-y;0) - M(x,t)\frac{\Omega_\nu}{(2\pi)^\nu}\left(\ln|x-y|^{-1} + \mathcal{K}_\nu\right)\right]dt$$

$$= -\sum_{j=0}^{\nu-1}\int_0^T A(x,t)\int D_{-1-j}(x,t;\xi,t;-1)\, e^{i(x-y)\xi}\,\frac{d\xi}{(2\pi)^{\nu-1}}\,dt + \int_0^T A(x,t)\ S(x,y,t)\,dt$$

$$= -\sum_{j=0}^{\nu-2}\sum_{n=0}^{\nu-j-2}\frac{\partial_t^n A(x,0)}{n!}g_{j,j+n+\nu-2}(x,x-y)$$

$$-\sum_{j=0}^{\nu-2}\frac{\partial_t^{\nu-j-2}A(x,0)}{(\nu-j-2)!}\ M_j(x)\frac{\Omega_{\nu-1}}{(2\pi)^{\nu-1}}\left(\ln|x-y|^{-1} + \mathcal{K}_{\nu-1}\right) - \sum_{j=0}^{\nu-1}\varphi_j(x,y) + \int_0^T A(x,t)\ S(x,y,t)\,dt.$$

$$(59)$$

Then, taking into account that the last terms in (58) and (59) satisfy

$$\lim_{z\to-1}\left(-\sum_{j=0}^{\nu-1}\psi_j(x,z) + \int_0^T A(x,t)\ R(x,t;z)\,dt\right) = \lim_{y\to x}\left(-\sum_{j=0}^{\nu-1}\varphi_j(x,y) + \int_0^T A(x,t)\ S(x,y,t)\,dt\right),$$

$$(60)$$

we obtain part b) of Theorem 1.

The proof of c) is similar to the one of b), and even simpler because in this case the parametrix in (12) does not include terms of the form Op'$(\theta_1, \tilde{d}_{-1-j})$. ∎

Eq.(32) looks cumbersome, but it is not so complicated. In fact, all terms can be systematically evaluated. Moreover, the terms containing h_l subtract the singular part of the Green function in the interior of the manifold (see (24)) and can, thus, be easily identified from the knowledge of G_B. $R(x,t,y,t)$, the regular part so obtained, is still nonintegrable near the boundary. Those terms containing $g_{j,l}$ subtract the singular part of the integrals $\int_0^T t^n R(x,t,y,t)\,dt$ (see (26)). Finally, the terms containing $c_{-\nu}$ and $\tilde{d}_{-\nu+1}$ arise as a consequence of having replaced an analytic regularization by a *point splitting* one.

Even though Seeley's coefficients c and $\tilde{d}$ are to be obtained through an iterative procedure, which can make their evaluation a tedious task, in some cases of physical interest only the few first of them are needed. An example of this fact is the calculation we performed in [6]. There, we considered the determinant of the Dirac operator $D = i\partial\!\!\!/ + A\!\!\!/$ acting on Dirac fermions defined on a two dimensional disk, under rather general local elliptic boundary conditions. In that computation we needed only two Seeley's coefficients.

References

[1] S. W. Hawking, Comm. Math. Phys. **55**, 133 (1977).

[2] R.E. Gamboa Saraví, M.A. Muschietti, F.A. Schaposnik and J.E. Solomin, Ann. Phys. **157**, 360 (1984).

[3] R.T. Seeley, Am. J. Math. **91**,889 (1969).

[4] R.T. Seeley, Am. J. Math. **91**, 963 (1969).

[5] G. Grubb and R.T. Seeley, C.R. Acad. Sci. Paris **317**, serie I, 1123 (1993);
Zeta and eta fuctions for Atiyah-Patodi-Singer operators, Copenh. Univ. Math. Dept. Preprint Ser., #11 (1993);
Weakly parametric pseudodifferentials operators and Atiyah-Patodi-Singer boundary problems, Copenh. Univ. Math. Dept. Preprint Ser. #5 (1993).

[6] *Determinants of Dirac operators with local boundary conditions,* H. Falomir, R. E. Gamboa Saraví, M. A. Muschietti, E. M. Santangelo y J. Solomin, *Journal of Mathematical Physics, in press.*

[7] A.P.Calderón, *Lectures notes on pseudodifferential operators and elliptic boundary value problems, I* (Publicaciones del I.A.M., Buenos Aires, 1976).

[8] P. B. Gilkey and L. Smith, Communications on Pure and Applied Mathematics, Vol. XXXVI, 85 (1983).

[9] M. F. Atiyah, V.K. Patodi and I. M. Singer, Math. Proc. Camb. Phil. Soc. **77**, 43 (1975); **78**, 405 (1975); **79**, 71 (1976).

[10] R. Forman, Invent. math. **88**, 447 (1987).

[11] R.E. Gamboa Saraví, M.A. Muschietti, F.A. Schaposnik and J.E. Solomin, J. Math. Phys. **26**, 2045 (1985).

[12] H. Falomir, M. A. Muschietti and E.M. Santangelo, J. Math. Phys. **31**, 989 (1990).