Quantal distribution functions in non-extensive statistics and an early universe test revisited

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Within the context of non-extensive thermostatistics, we use the factorization approximation to study a recently proposed early universe test. A very restrictive bound upon the non-extensive parameter is presented: $|q-1| < 4.01 \times 10^{-3}$.

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Although nonextensive formalisms are much in vogue in Physics, a very interesting one, the so-called Tsallis Thermostatistics (TT), seems to be one of the most actively studied. The proposed formalism basically relies upon two postulates [1,2]:

- A new definition of a nonextensive entropy,

$$S_q = k(1 - \sum_i^W p_i^q)/(q-1).$$

- A new definition of expectation value,

$$\langle \mathcal{O} \rangle_q = \sum_i p_i^q \mathcal{O}_i.$$

TT introduces a new parameter, $q \in \Re$, which is usually called the nonextensivity parameter or the Tsallis q-index, and it contains the standard, extensive Boltzmann-Gibbs statistics, as a special case where q is taken to be unity. Up to present days, TT has been found to admit generalizations of some of the important concepts of statistical physics [3], and to yield results which can explain some observational and experimental data where standard statistics is known to fail [4]. All these efforts accelerate new attempts to find the physical meaning of the nonextensivity parameter, a long standing puzzle which has only now started to be clarified. On one side, some works have been devoted to the study of dynamical and dissipative systems [5]. On the other, an entirely new fractal canonical ensemble was introduced in order to relate TT with a scale invariant thermodynamics [6]. Of course, an alternative way to search for the meaning of q is related with the estimation of bounds in measurable physical systems. We mention the study of the microwave background radiation [7,8], the Stefan-Boltzmann constant [9,10], the early Universe [11,12] and the primordial neutron to baryon ratio in a cosmological expanding background [13]. In these applications, except in Refs. [8,10], the quantal distribution functions of TT were obtained by an asymptotic approach of the kind $\beta(1-q) \to 0$, where β is the usual inverse temperature. However, quantal distribution functions have been previously generalized by Büyükkılıç et al. [14] using a rather straightforward procedure, referred to as Factorization Approximation. The formulae so obtained have proved to be simpler and more general than the ones derived with the common Tsallis et al. method [7]. Simpler, because the steps followed in the derivation are completely the same when compared with standard textbooks. More general, because the factorization approach does not need a value of $q \simeq 1$.

Quantal distribution functions within the factorization approximation were, however, regarded as a rather rough technique because of a work by Pennini et al. [15]. They considered fermion and boson systems with very small occupation numbers. However, very recently, Wang and Lé Méhauté [16] analysed the problem in detail and showed that there exist a temperature interval, a *forbidden zone*, where the deviation from the

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exact result is significant, but that outside this zone, the factorization approach results could be used with confidence. In addition, they verified that the magnitude of the *forbidden zone* remained constant with the increase of the number of particles, contrary to the result stated in Ref. [15]. This fact might motivate new efforts for the study of macroscopic systems (where the number of particles is $\sim 10^{23}$) within the simpler approach. The generalized distribution functions of the factorization approach could be used at temperatures up to about 10^{20} K for such a system [16].

All the above remarks led to recompute some bounds already obtained within the Tsallis et al. approach in this simpler framework, in order to get more reliable results and to check for consistency. This is what we briefly do below concerning the early universe test proposed in [11].

The test devised consist in compute a first order deviation in $(q-1)$ to the temperature of freezing out of the weak interctions in the early universe, T_f. This temperature is essential in the primordial nucleosynthesis scenario which, basically, is the competition between the rate of the expansion of the universe and that of the weak interactions which regulates the conversion of neutrons into protons and viceversa. When the expansion exceeds the rate of interactions they freeze out, and the final yields for the element production are roughly the ones we observe today. In particular, how a deviation in T_f cause a different prediction for the helium primordial yield, Y_p, was made by Casas et. al. [17]. The result is,

$$\delta Y_p = Y_p \left[\left(1 - \frac{Y_p}{2\lambda}\right) \ln\left(\frac{2\lambda}{Y_p} - 1\right) + \frac{-2t_f}{\tau_n} \right] \frac{\delta T_f}{T_f}. \quad (1)$$

Here, a radiation era relationship between time and temperature of the form ($T \propto t^{-\frac{1}{2}}$) is assumed [18] and one sets $\delta T_{nuc} = 0$, because it is fixed by the binding energy of the deuteron. $\lambda = \exp(-(t_{nuc} - t_f)/\tau)$ stands for the fraction of neutrons which decayed into protons between t_f and t_{nuc}, with t_f (t_{nuc}) the time of freeze out of the weak interactions (nucleosynthesis) and τ the neutron mean lifetime. Considering now, conservatively, $Y_p = Y_p^{obs} = 0.23$ and $|\delta Y_p| = 0.01$, which is the observational error for Y_p, and standard values for the times and the mean life of neutron –which in fact, is not modified at order $(q-1)$–, we must ask for

$$0.01 > 0.3766 \left| \frac{\delta T_f}{T_f} \right|. \quad (2)$$

A more detailed account of the processes that occurs in the early universe within this statistical framework is given elsewhere [13].

To compute the $(q-1)$ corrections to T_f we recall the output of the factorization approach. The quantal distribution functions are given by,

$$n_{q\,[bosons]} = \frac{1}{e^x - 1} - \frac{(1-q)}{2} \frac{x^2 e^x}{(e^x - 1)^2}, \quad (3)$$

$$n_{q\,[fermions]} = \frac{1}{e^x + 1} - \frac{(1-q)}{2} \frac{x^2 e^x}{(e^x + 1)^2}. \quad (4)$$

There are two main corrections acting upon T_f. The first comes from the computation of the energy density of the universe. When the particles are higly relativistic, $T \gg m$, and non-degenerate $T \gg \mu$, we get

$$\rho_{bosons} = \frac{g_b}{2\pi^2} \int_0^\infty dE E^3 n_{q\,[bosons]}, \quad (5)$$

$$\rho_{fermions} = \frac{g_f}{2\pi^2} \int_0^\infty dE E^3 n_{q\,[fermions]}, \quad (6)$$

$g_{b,f}$ stands for the degeneracy factor of each one of the species involved. Using (3,4), we finally obtain

$$\rho_{total} = \rho_{bosons} + \rho_{fermions} = \frac{\pi^2}{30} g T^4 + 35.85\, T^4 (q-1), \quad (7)$$

where $g = g_b + 7/8 g_f$. At high enough temperatures, the energy density of the universe is essentially dominated by e^-, e^+, ν and $\hat{\nu}$ and so $g_b = 2$ and $g_f = 2 + 2 + 2 \times 3 = 10$.

The second correction comes from the computation of the weak interaction rates. We shall denote by $\lambda_{pn}(T)$ the rate for the weak processes to convert protons into neutrons and by $\lambda_{np}(T)$ the rate for the associated, reverse ones. Within the standard approximations applicable in the early universe regime [11,19], it is possible to see that the weak interaction rate Λ is given by $\Lambda \simeq 4\lambda_{\nu+n\to p+e^-}$. This last rate for the particular reaction quoted must be computed using,

$$\lambda_{\nu+n\to p+e^-} = A\int_0^\infty dp_\nu p_\nu^2 p_e E_e (1- < \hat{n}_e >) n_q(\nu) \tag{8}$$

where A is a constant fixed by the experimental value of the neutron lifetime. Using (3,4), we get

$$\lambda_{\nu+n\to p+e^-} = \lambda^{standard}_{\nu+n\to p+e^-} + 354.8 T^5\, A(q-1), \tag{9}$$

as the biggest correction. Multiplying this result by 4, we obtain,

$$\frac{\delta\Lambda}{A} = 1419.2\, T^5\, (1-q). \tag{10}$$

Having in hands the main corrections that non-extensive quantal distribution functions provide, we move onwards to get the final bound upon q. To do so, we first compute the first order correction to T_f, which is defined as the temperature where the equality

$$\Lambda \simeq \left(\frac{\dot{a}}{a}\right) = \sqrt{\frac{8\pi G}{3}\rho_{total}} \tag{11}$$

holds. The result is

$$\frac{\delta T_f}{T_f^{st}} = 6.61(q-1). \tag{12}$$

Using, finally, Eq. (2) we get the following bound,

$$|q-1| < 4.01\times 10^{-3}. \tag{13}$$

The previous bound is, as one should expect, more restrictive than that obtained within the Tsallis et al. approach. [1] This conclude the objective of this brief letter.

Summing up, in this work we have revisited the recently proposed early Universe test of TT and computed the related bound on q using the generalized distribution functions within the factorization approximation. Our main result is written in Eq. (13): 1 to 100 seconds after the Big Bang, q must be that close to 1 in order to be able to reproduce observational results on helium abundance. As it is expected, the bound is found to be consistent with those of Tsallis et al. approach, a result which was also obtained in Refs. [8,10] . Although all these efforts clarify the fact that the factorization approach results can be used with confidence for physical systems, our belief is that it will be used much more in the near future especially for the applications with q values far from unity, where Tsallis et al. approach is unapplicable. Therefore new attempts on this line would be highly welcomed.[2]

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[1] Recall the Errata in [11]. The final bound turns out to be $|1-q| < 3.4\times 10^{-3}$.

[2] After submission of this work we became aware of the work by Rajagopal et al. [Phys. Rev. Lett. 80, 3907 (1998)] where exact results for quantal distribution functions are given. We have explored how these results match with Tsallis et al.'s and factorization approaches and we shall report on it in a future comunication.

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