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Non standard parametrizations and adjoint invariants of classical groups[1]

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Abstract

We obtain local parametrizations of classical non-compact Lie groups where adjoint invariants under maximal compact subgroups are manifest. Extension to non compact subgroups is straightforward. As a by-product parametrizations of the same type are obtained for compact groups. They are of physical interest in any theory gauge invariant under the adjoint action, typical examples being the two dimensional gauged Wess-Zumino-Witten-Novikov models where these coordinatizations become of extreme usefulness to get the background fields representing the vacuum expectation values of the massless modes of the associated (super) string theory.

1 Introduction

It is none to say what group theory meant (and means) for theoretical physics during this century, in particular the theory of continuous groups or Lie groups (see e.g. [1], [2] and references therein). The properties of their Lie algebras, easier to hand than the groups themselves, define them locally and in fact most part of the textbooks are dedicated to them [3]. However depending on the problem at hand, explicit parametrizations of the group manifold become necessary. Many of them are widely known, example of them the $SU(2)$ or more generally the Euler angles of orthogonal groups. Of course that we can always write locally a group element as a product of one-generator exponentials, or simply as the exponential of an arbitrary Lie algebra element. But in most cases theses obvious

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parametrizations are of few usefulness because they obscure the global properties of the group and generally lead to un-tractable computations.

An interesting parametrization is suggested by the Mackey theorem, the well-known coset decomposition: let G be a group and H a subgroup of it. Then for any $g \in G$,

$$g = k\ h, \quad k \in G/H\ , \quad h \in H \tag{1.1}$$

in a unique way. This leads to the theory of homogeneous spaces (or (left or right) coset spaces) G/H, good references on the subject being [4], [5]. Among others physical applications (1.1) is fundamental in the treatment of effective field theories with spontaneously broken symmetries [6].

But let us assume instead that we have a field theory including maps from “space-time” on a group manifold G among its degrees of freedom, and gauge invariant under the adjoint action of a subgroup H of G

$$g \ \rightarrow \ {}^{h}g = h\ g\ h^{-1} \quad , \quad h \in H \tag{1.2}$$

This means that effectively the theory depends on the invariants of the group under the adjoint action of the subgroup. That is, if we were able to write uniquely any element of G as

$$g \equiv h^{-1}\ \bar{g}\ h \quad , h \in H \tag{1.3}$$

then clearly making a gauge transformation (1.2) identifying the h's the theory will depend only on $\bar{g}$ that encloses the invariants mentioned above. We would like to remark that at difference of (1.1) there no exists any general theorem assuring the decomposition (1.3); in fact it is not difficult to find examples where it is not possible to write it.

It is the aim of this paper to get the class of local parametrizations of the type (1.3) in a whole set of cases of physical interest. Specific examples where they must be used (and were used in the lower dimensionality cases where parametrizations were available) are the two dimensional gauged Wess-Zumino-Witten-Novikov models [7], [8]. It is worth to say however that the results, being explicit parametrizations of classical groups, are valuable on their own right independently of the applications.

2 The orthogonal groups

We consider in this section the pseudo-orthogonal groups, $G \equiv SO(p,q)$. Its maximal compact subgroup is $H \equiv SO(p) \times SO(q)$. In order to get its decomposition (1.3) we need as a first step to get the

2.1 Reduction of $SO(p+1)$ under $SO(p)$.

We start by writing [5]

$$P_{p+1}(\vec{u}_p, P_p) = K_p(\vec{u}_p)\ H(P_p, 1) \tag{2.1}$$

where $\vec{u}_p$ is a p-dimensional real vector, $P_p \in SO(p)$ and generically we will mean

$$H(P,Q) = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} \tag{2.2}$$

In what follows the dimensionalities of matrices should be understood from the context when not stated explicitly. The right coset element in $SO(p+1)/SO(p) \sim S^p$ is given by

$$\begin{aligned} K_{p+1}(\vec{u}_p) &= \begin{pmatrix} 1-(1+u_{p+1})^{-1}\,\vec{u}_p\vec{u}_p^{\,t} & \vec{u}_p \\ -\vec{u}_p^{\,t} & u_{p+1} \end{pmatrix} \\ 1 &= \vec{u}_p^{\,t}\vec{u}_p + (u_{p+1})^2 \end{aligned} \tag{2.3}$$

Under an adjoint transformation

$${}^hP_{p+1} = H(h,1)\;P_{p+1}\;H(h^t,1)\quad,\quad h \in SO(p) \tag{2.4}$$

the parameters of P_{p+1} get transformed as:

$$\begin{aligned} {}^h\vec{u}_p &= h\,\vec{u}_p \\ {}^hP_p &= h\;P_p\;h^t \end{aligned} \tag{2.5}$$

The procedure will be constructive. Les us pick an arbitrary matrix $V_p \in SO(p)$ decomposed this time as a left coset w.r.t. $SO(p-1)$

$$V_p = H(V_{p-1},1)\;K_p(\vec{v}_{p-1}) \tag{2.6}$$

and rewrite (2.1) for any such a V_p with the help of (2.5) as

$$\begin{aligned} P_{p+1} &= H(V_p^{\,t},1)\;\bar{P}_{p+1}\;H(V_p,1) \\ \bar{P}_{p+1} &= K_{p+1}(V_p\vec{u}_p)\;H(V_pP_pV_p^{\,t},1) \end{aligned} \tag{2.7}$$

The general idea to apply here and in the subsequent cases is to fix the whole matrix V_p ($\equiv \vec{v}_{p-1}, V_{p-1}$) in terms of variables of P_{p+1} ($\equiv \vec{u}_p, P_p$), leaving aside only precisely the invariants together with the matrix V_p as parameters of P_{p+1}. Evidently this procedure is equivalent to make a change of variables from the non-invariant parameters in P_{p+1} to V_p.

Equation (2.7) suggests to put $\vec{u}_p$ in some standard form by a specific choice of (a part of) V_p. In fact it is easy to show that the choice [2]

$$\begin{pmatrix} \vec{v}_{p-1} \\ -v_p \end{pmatrix} = -\frac{\vec{u}_p}{|\vec{u}_p|} \tag{2.8}$$

defines the rotation

$$K_p(\vec{v}_{p-1})\;\vec{u}_p = |\vec{u}_p|\;\;\check{e}_p \tag{2.9}$$

[2] As usual we will use the notation $(\check{e}_i)_j = \delta_{ij}, (E_{ij})_{kl} = \delta_{ik}\delta_{jl}, A_{ij} \equiv E_{ij} - E_{ji}, S_{ij} \equiv E_{ij} + E_{ji}$.

Note that we have changed the $(p-1)$ parameters from $\vec{u}_p$ indicating its direction in terms of the $(p-1)$ parameters in $\vec{v}_{p-1}$. With such a choice of $\vec{v}_{p-1}$ (V_{p-1} not fixed yet) we can write

$$\bar{P}_{p+1} = K_{p+1}(|\vec{u}_p|\ \check{e}_p)\ H(H(V_{p-1},1)\ P_p\ H(V_{p-1},1)^t,1) \tag{2.10}$$

where we have redefined $P_p \to K_p(\vec{v}_{p-1})^t\ P_p\ K_p(\vec{v}_{p-1})$. But according to (2.7) we can rewrite it as

$$\bar{P}_{p+1} = K_{p+1}(|\vec{u}_p|\ \check{e}_p)\ H(\bar{P}_p,1) \tag{2.11}$$

Inspection of this formula indicates an iterative process, the next step being to write the analogous expression for $\bar{P}_p$ and so on; after p steps we get

$$\bar{P}_{p+1} = \prod_{l=1}^{\overleftarrow{p}} H\left(K_{l+1}(|\vec{u}_l|\check{e}_l), 1_{p-l}\right) \tag{2.12}$$

It is convenient to introduce the angular variables

$$|\vec{u}_l| = \sin\theta_l\ \ ,\ \ 0 \le \theta_l \le \pi \tag{2.13}$$

and write the $SO(p+1)$ element in the final form

$$\begin{aligned} P_{p+1} &= H(V_p,1)^t\ \bar{P}_{p+1}\ H(V_p,1) \\ \bar{P}_{p+1} &= \prod_{l=1}^{\overleftarrow{p}} \exp\left(\theta_l\ A_{l,l+1}\right) \end{aligned} \tag{2.14}$$

which displays explicitly the p invariants $\{\theta_l, l = 1,\cdots,p\}$ under the adjoint action of $SO(p)$. Note that the number of parameters trivially matches: $\frac{p}{2}(p-1)+p = \frac{p}{2}(p+1)$, as should; this is the first of our results, to be used in the following.

2.2 Reduction of $SO(p,q)$ under $SO(p)\times SO(q)$.

Our starting point is again the right coset parametrization [5] [3]

$$\Lambda_{p,q}(S,P_p,Q_q) = K_{p,q}(S)\ H(P_p,Q_q) \tag{2.15}$$

where $P_p(Q_q) \in (SO(p)(SO(q))$ and the coset element is given by

$$\begin{aligned} K_{p,q}(S) &= \exp\begin{pmatrix} 0 & N \\ N^t & 0 \end{pmatrix} = \begin{pmatrix} (1+SS^t)^{\frac{1}{2}} & S \\ S^t & (1+S^tS)^{\frac{1}{2}} \end{pmatrix} \\ S &= (NN^t)^{-\frac{1}{2}}\sinh(NN^t)^{\frac{1}{2}}N \quad \in \Re^{p\times q} \end{aligned} \tag{2.16}$$

Under an adjoint transformation

$$^h\Lambda_{p,q} = H(h_p,h_q)\ \Lambda_{p,q}\ H(h_p,h_q)^t \tag{2.17}$$

[3] An explicit derivation from the definition of pseudo unitary groups is given in Appendix A of [10].

with $h_p(h_q) \in SO(p)(SO(q))$, the parameters of $\Lambda_{p,q}$ transforms as

$$\begin{aligned} {}^h S &= h_p\ S\ {h_q}^t \\ {}^h P_p &= h_p\ P_p\ h_p^t \\ {}^h Q_q &= h_q\ Q_q\ h_q^t \end{aligned} \tag{2.18}$$

By following the strategy pursued in the past subsection we introduce two matrices V_p, V_q belonging to $SO(p), SO(q)$ respectively and rewrite (2.15)

$$\begin{aligned} \Lambda_{p,q} &= H(V_p, V_q)^t\ \bar{\Lambda}_{p,q}\ H(V_p, V_q) \\ \bar{\Lambda}_{p,q} &= exp \begin{pmatrix} 0 & V_p N {V_q}^t \\ (V_p N {V_q}^t)^t & 0 \end{pmatrix} H(V_p P_p {V_p}^t, V_q Q_q {V_q}^t) \end{aligned} \tag{2.19}$$

As in (2.6) we consider left coset parametrizations

$$\begin{aligned} V_p = H(V_{p-1}, 1)\ K_p(\vec{v}_{p-1}) \\ V_q = H(V_{q-1}, 1)\ K_q(\vec{v}_{q-1}) \end{aligned} \tag{2.20}$$

and try to totally fix $\vec{v}_{p-1}$ and $\vec{v}_{q-1}$ to put N in a standard form. It turns out that it is possible to choose these vectors in such a way that

$$V_p\ N\ {V_q}^t \equiv H(V_{p-1}, 1) K_p(\vec{v}_{p-1})\ N K_q(\vec{v}_{q-1})^t\ H({V_{q-1}}^t, 1) = \begin{pmatrix} N_r & 0 \\ 0 & n \end{pmatrix} \tag{2.21}$$

where $N_r \in \Re^{(p-1)\times(q-1)}$ and $n \in \Re$. In fact if we write

$$N = \begin{pmatrix} N'_r & \vec{n}_{p-1} \\ {\vec{n}_{q-1}}^t & n' \end{pmatrix} \tag{2.22}$$

then is straightforward to verify that the gauge fixing condition $\vec{n}_{p-1} = \vec{n}_{q-1} = \vec{0}$, i.e.

$$K_p(\vec{v}_{p-1})\ N K_q(\vec{v}_{q-1})^t = \begin{pmatrix} N_r & 0 \\ 0 & n \end{pmatrix} \tag{2.23}$$

holds if we choose

$$\begin{aligned} \vec{v}_{p-1} &\equiv (1 + |\vec{t}_{p-1}|^2)^{-\frac{1}{2}}\ \vec{t}_{p-1} \\ \vec{v}_{q-1} &\equiv (1 + |\vec{t}_{q-1}|^2)^{-\frac{1}{2}}\ \vec{t}_{q-1} \end{aligned} \tag{2.24}$$

with $\vec{t}_{p-1}, \vec{t}_{q-1}$ satisfying [4]

$$\vec{t}_{p-1} = ({\vec{n}_{q-1}}^t \vec{t}_{q-1} - n)^{-1} \left(\vec{n}_{p-1} - N'_r \vec{t}_{q-1}\right)$$

[4] This set of equations can be reduced to a system of two (quadratic) equations with two unknowns; the important thing for us is that solutions exist and define the change of variables

$$(\vec{n}_{p-1}, \vec{n}_{q-1}) \to (\vec{v}_{p-1}, \vec{v}_{q-1})$$

$$\vec{t}_{q-1} \;=\; ({\vec{n}_{p-1}}^t \vec{t}_{p-1} - n)^{-1} \left(\vec{n}_{q-1} - N_r'^t \vec{t}_{p-1} \right) \tag{2.25}$$

A final redefinition $N_r \to {V_{p-1}}^t \; N_r \; V_{q-1}$ leads to (2.21). Finally (after reparametrizing $P_p \to K_p(\vec{v}_{p-1})^t \; P_p \; K_p(\vec{v}_{p-1})$, $Q_q \to K_q(\vec{v}_{q-1})^t \; Q_q \; K_q(\vec{v}_{q-1})$) we use (2.14) to fix V_{p-1} and V_{q-1}; the result can be recast in the form

$$\begin{aligned} \Lambda_{p,q} &= H(V_p, V_q)^t \; \bar{\Lambda}_{p,q} \; H(V_p, V_q) \\ \bar{\Lambda}_{p,q} &= \exp\left(\sum_{i=1}^{p-1} \sum_{j=1}^{q-1} N_{ij} \; S_{i,p+j} + n S_{p,p+q} \right) \prod_{k=1}^{\overleftarrow{p-1}} \exp(\theta_k A_{k,k+1}) \prod_{k=1}^{\overleftarrow{q-1}} \exp(\bar{\theta}_k A_{p+k,p+k+1}) \end{aligned} \tag{2.26}$$

Again we verify the matching in the number of parameters $V_p, V_q, N_{ij}, n, \theta_k, \bar{\theta}_k$,

$$\frac{p}{2}(p-1) + \frac{q}{2}(q-1) + (p-1)(q-1) + 1 + (p-1) + (q-1) = \frac{1}{2}(p+q)(p+q-1) \tag{2.27}$$

It is worth to say that the first term in (2.26) can be computed as in (2.16) with N as in the r.h.s. of (2.21); however this is a formal expression for which, to our knowledge, only the "minkowskian" cases $p = 1$ or $q = 1$ admit an explicit form.

3 The unitary groups

The treatment of these groups parallels that made in the case of the orthogonal ones, with some additional complications due to the complex character of them. As before we start considering

3.1 Reduction of $SU(p+1)$ under $U(p)$.

An element of $SU(p+1)$ can be written as

$$P_{p+1}(\vec{u}_p, U_p) = K_{p+1}(\vec{u}_p) \; H(U_p, u_p^*) = K_{p+1}(\vec{u}_p) \; H(P_p, 1) \; \exp\left(i \; \phi^p T_p\right) \tag{3.1}$$

where $u_p \equiv \det U_p = \exp(ip\phi_p)$, $\vec{u}_p \in \mathbf{C}^p$, $U_p \in U(p), P_p \in SU(p)$, and we have introduced a convenient basis $\{T_k = \sum_{l=1}^k (E_{ll} - E_{k+1,k+1}), \; k = 1, \cdots, p\}$ in the Cartan subalgebra of $su(p+1)$. The right coset element in (3.1) belonging to $SU(p+1)/U(p) \sim CP^p$ is given by [5]

$$\begin{aligned} K_{p+1}(\vec{u}_p) &= \begin{pmatrix} 1 - (1 + u_{2p+1})^{-1} \; \vec{u}_p {\vec{u}_p}^\dagger & \vec{u}_p \\ -{\vec{u}_p}^\dagger & u_{2p+1} \end{pmatrix} \\ 1 &= {\vec{u}_p}^\dagger \vec{u}_p + (u_{2p+1})^2 \end{aligned} \tag{3.2}$$

The adjoint action under $H_p \in U(p)$ is ($h_p \equiv \det H_p$)

$$^H P_{p+1} = H(H_p, h_p^*) \; P_{p+1} \; H(H_p, h_p^*)^\dagger \;\longleftrightarrow\; \begin{cases} {}^V \vec{u}_p = h_p \; H_p \; \vec{u}_p \\ {}^H P_p = H_p \; P_p \; H_p^\dagger \\ {}^H \phi^p = \phi^p \end{cases} \tag{3.3}$$

As before we pick an arbitrary $V_p \in U(p)$ left coset parametrized

$$V_p = H(V_{p-1}, v_{p-1}^*)\; K_p(\vec{v}_{p-1})\; \exp(i\beta_p) \quad , \quad V_{p-1} \in U(p-1) \tag{3.4}$$

and write (3.1) with the help of (3.3) as

$$\begin{aligned} P_{p+1} &= H(V_p, v_p^*)^\dagger\; \bar{P}_{p+1}\; H(V_p, {v_p}^*) \\ \bar{P}_{p+1} &= K_{p+1}(v_p V_p \vec{u}_p)\; H(V_p P_p {V_p}^\dagger, 1)\; \exp(i\;\phi^p T_p) \end{aligned} \tag{3.5}$$

By choosing $\vec{v}_{p-1}$ and β_p (V_{p-1} free) as

$$\begin{aligned} \begin{pmatrix} \vec{v}_{p-1} \\ -v_{2p-1} \end{pmatrix} &= -\frac{(\vec{u}_p^*)^p}{|(\vec{u}_p)^p|}\,\frac{\vec{u}_p}{|\vec{u}_p|} \\ \exp(i\beta_p) &= \left(\frac{(\vec{u}_p^*)^p}{|(\vec{u}_p)^p|}\right)^{\frac{1}{p+1}} \end{aligned} \tag{3.6}$$

we have

$$v_p\; V_p\; \vec{u}_p = |\vec{u}_p|\; \check{e}_p \tag{3.7}$$

and identifying V_{p-1} as the $SU(p-1)$ matrix corresponding to the P_p decomposition in (3.5) (previous redefinition $P_p \to K_p\;(\vec{v}_{p-1})^\dagger P_p\; K_p(\vec{v}_{p-1})$) we get

$$\bar{P}_{p+1} = K_{p+1}(|\vec{u}_p|\check{e}_p)\; H(\bar{P}_p, 1)\; \exp(i\;\phi^p T_p) \tag{3.8}$$

By repeating the analysis for $\bar{P}_p$ and after p steps we arrive to the final result

$$\begin{aligned} P_{p+1} &= H(V_p, v_p^*)^\dagger\; \bar{P}_{p+1}\; H(V_p, v_p^*) \\ \bar{P}_{p+1} &= \prod_{l=1}^{\overleftarrow{p}} \exp\left(\theta_l\; A_{l,l+1}\right)\; C_{p+1}(\Phi) \end{aligned} \tag{3.9}$$

It differs from (2.14) from the unitary character of V_p and the comparison of the arbitrary Cartan element $C_{p+1}(\Phi) = \exp(i\sum_{l=1}^{p}\;\phi^l T_l)$ at right in $\bar{P}_{p+1}$.

3.2 Reduction of $SU(p,q)$ under $S(U(p)\times U(q))$.

In order not to be repetitive we will skip some steps in what follows. An arbitrary element in $SU(p,q)$ can be left coset decomposed under $S(U(p)\times U(q))$ as

$$\begin{aligned} \Lambda_{p,q}(S, U_p, U_q) &= K_{p,q}(S)\; H(U_p, U_q) \quad , \quad u_p = {u_q}^* \\ K_{p,q}(S) &= \begin{pmatrix} (1+SS^\dagger)^{\frac{1}{2}} & S \\ S^\dagger & (1+S^\dagger S)^{\frac{1}{2}} \end{pmatrix} = \exp\begin{pmatrix} 0 & N \\ N^\dagger & 0 \end{pmatrix} \\ S &= (NN^\dagger)^{-\frac{1}{2}} \sinh(NN^\dagger)^{\frac{1}{2}} N \quad , \quad S, N \in \mathcal{C}^{p\times q} \end{aligned} \tag{3.10}$$

The adjoint action under $H(h_p, h_q) \in S(U(p)\times U(q))$ is

$${}^H\Lambda_{p,q} = H(h_p, h_q)\; \Lambda_{p,q} H(h_p, h_q)^\dagger \longleftrightarrow \begin{cases} {}^H S = h_p\; S\; {h_q}^\dagger \\ {}^H U_p = h_p\; U_p\; h_p^\dagger \\ {}^H U_q = h_q\; U_q\; {h_q}^\dagger \end{cases} \tag{3.11}$$

Two matrices V_p, V_q belonging to $U(p), U(q)$ respectively with $v_p v_q = 1$ are introduced and following similar steps as in Subsection 2.2 we find that it is possible to fix N in the way

$$N = \begin{pmatrix} N_r & 0 \\ 0 & n \end{pmatrix} \tag{3.12}$$

where now $N_r \in \mathcal{C}^{(p-1)\times(q-1)}$ and $n \in \Re$. Then by using the results of the past subsection we get the final result for the parametrization

$$\begin{aligned} \Lambda_{p,q} &= H(V_p,V_q)^\dagger \,\bar{\Lambda}_{p,q}\, H(V_p,V_q) \\ \bar{\Lambda}_{p,q} &= \exp\begin{pmatrix} 0 & N \\ N^\dagger & 0 \end{pmatrix} \overleftarrow{\prod_{l=1}^{p-1}} \exp\left(\theta_l\, A_{l,l+1}\right) \overleftarrow{\prod_{l=1}^{q-1}} \exp\left(\bar{\theta}_l\, A_{p+l,p+l+1}\right)\, C(\Phi) \end{aligned} \tag{3.13}$$

where $H(V_p, V_q) \in S(U(p) \times U(q))$ and we denote by $C(\Phi)$ an arbitrary element in the Cartan subalgebra of the Lie algebra of $S(U(p) \times U(q))$.

3.3 Decomposition under the maximal torus

Some times is useful to have the adjoint decomposition of unitary groups under the Cartan subalgebra. We will work out for definiteness the case of $SU(n+1)$; the non compact versions differ as usual by signs in the coset elements and Wick rotations of some compact generators.

To this end we begin by searching for the coset decomposition of $SU(n+1)$ under $C(SU(n+1))$; from (3.1)

$$\begin{aligned} P_{n+1}(\vec{u}_n, U_n) &= K_{n+1}(\vec{u}_n)\; H(U_n, u_n^*) \\ K_{n+1}(\vec{u}_n) &= \exp\theta_n \begin{pmatrix} 0 & \check{r}_n \\ -\check{r}_n{}^\dagger & 0 \end{pmatrix} = \begin{pmatrix} 1-(1+u_{2n+1})^{-1}\,\vec{u}_n\vec{u}_n{}^\dagger & \vec{u}_n \\ -\vec{u}_n{}^\dagger & u_{2n+1} \end{pmatrix} \\ |\vec{u}_n|^2 + (u_{2n+1})^2 &= 1 \quad , \quad \vec{u}_n = \sin\theta_n\, \check{r}_n \quad , \quad \check{r}_n{}^\dagger \check{r}_n = 1 \quad , \quad \theta_n \in [0,\pi] \end{aligned} \tag{3.14}$$

By introducing

$$\begin{aligned} U_n &= P_n \begin{pmatrix} 1_{n-1} & 0 \\ 0 & u_n \end{pmatrix} \quad , \quad P_n \in SU(n) \\ u_n &\equiv \det U_n = \exp(i\varphi^n) \end{aligned} \tag{3.15}$$

we can rewrite (3.14) as

$$P_{n+1}(\vec{u}_n, U_n) = K_{n+1}(\vec{u}_n)\; H(P_n, 1)\; \exp(i\,\varphi^n H_n) \tag{3.16}$$

where this time is convenient to introduce the basis $\{H_k = E_{kk} - E_{k+1,k+1}, k = 1, \cdots, n\}$ in the Cartan subalgebra of $su(n+1)$. By repeating with P_n and iterating we get

$$P_{n+1} = \overleftarrow{\prod_{l=1}^{n}} \begin{pmatrix} K_{l+1}(\vec{u}_l) & 0 \\ 0 & 1_{n-l} \end{pmatrix} \exp(i\,\vec{\varphi}\cdot\vec{H}) \tag{3.17}$$

Now let us pick an element $C_{n+1}(\vec{\alpha}) \equiv \exp(\alpha^i H_i) \in C(SU(n+1))$ and write as usual

$$P_{n+1} = C_{n+1}(\vec{\alpha})^\dagger \overleftarrow{\prod_{l=1}^{n}} \left(C_{n+1}(\vec{\alpha}) \begin{pmatrix} K_{l+1}(\vec{u}_l) & 0 \\ 0 & 1_{n-l} \end{pmatrix} C_{n+1}(\vec{\alpha})^\dagger \right) C_{n+1}(\vec{\varphi})\, C_{n+1}(\vec{\alpha}) \tag{3.18}$$

We see that the (φ^l) variables are invariant; we must choose the α^i's to kill parameters in the productory. It is easy to show that the versors $\check{r}_l$ get transformed as

$$^{\alpha}\check{r}_l = \exp(-i\tilde{\alpha}_{l+1})\ \exp(i \sum_{i=1}^{l} \tilde{\alpha}_i E_{ii})\ \check{r}_n \quad , \quad l = 1, \cdots, n \tag{3.19}$$

where

$$\tilde{\alpha}_k = \begin{cases} \alpha_1 & if \quad k = 1 \\ -\alpha_{k-1} + \alpha_k & if \quad k = 2, \ldots, n \\ -\alpha_n & if \quad k = n+1 \end{cases} \tag{3.20}$$

Then we can put the phases of the $(^{\alpha}\check{r}_l)^l$ components, $l = 1, \cdots, n$, to zero by choosing the $\alpha' s$ such that [5]

$$\begin{aligned} 2\alpha_1 - \alpha_2 &= -phase(\check{r}_1)^1 \\ -\alpha_1 + 2\alpha_2 - \alpha_3 &= -phase(\check{r}_2)^2 \\ \vdots \quad & \quad \vdots \\ -\alpha_{n-1} + 2\alpha_n &= -phase(\check{r}_n)^n \end{aligned} \tag{3.21}$$

In other words, from (3.18) we get the final result

$$\begin{aligned} P_{n+1} &= C_{n+1}(\vec{\alpha})^\dagger\ \bar{P}_{n+1}\ C_{n+1}(\vec{\alpha}) \\ \bar{P}_{n+1} &= \overleftarrow{\prod_{l=1}^{n}} \begin{pmatrix} K_{l+1}(\vec{u}_l) & 0 \\ 0 & 1_{n-l} \end{pmatrix} C_{n+1}(\vec{\varphi}) \end{aligned} \tag{3.22}$$

with the constraints implied by (3.21): $(\check{r}_l)^l \in \Re\ ,\ l = 1, \ldots, n.$

4 The decomposition of $Sl(n)$ under $SU(n)$.

This is one of the two irreducible riemannian cases [6] in the sense that the coset element is not of the form $\exp \begin{pmatrix} 0 & N \\ \pm N^\dagger & 0 \end{pmatrix}$ for some matrix N (off-diagonal cosets). We start from the well-known coset decomposition under $SO(n)$ of any unimodular real $n \times n$ matrix

$$g_n = S_n\ P_n \ ,\ S_n{}^t = S_n \quad , \quad P_n \in SO(n) \tag{4.1}$$

[5] Equations (3.21) can be formally solved by $\alpha^i = -{K^{-1}}^i{}_j\ phase(\check{r}_j)^j$ where K is the Killing-Cartan matrix of the A_n algebra; it is probable that this fact does not be an accident but occurs in other cases $G/C(G)$.

[6] The other one is $SU^*(2n)/USp(2n)$ and will not be considered here.

Also S_n is positive definite and $\det S_n = 1$. But we know from elementary linear algebra that any such a matrix is diagonalizable by an orthogonal one Q_n completely determined

$$\begin{aligned} S_n &= {Q_n}^t \, Diag({\lambda_1}^2, \ldots, {\lambda_n}^2) \, Q_n \\ \prod_{i=1}^{n} \lambda_i &= 1 \quad , \quad \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0 \end{aligned} \tag{4.2}$$

from where after a redefinition $P_n \to {Q_n}^t P_n Q_n$ we get

$$\begin{aligned} g_n &= {Q_n}^t \, \bar{g}_n \, Q_n \quad , Q_n \in SO(n) \\ \bar{g}_n &= Diag({\lambda_1}^2, \ldots, {\lambda_n}^2) \, P_n \end{aligned} \tag{4.3}$$

Analogous steps using well known results yield the complexification of (4.3), namely the decomposition of $Sl(n, \mathcal{C})$ under $SU(n, \mathcal{C})$ that we quote without proof

$$\begin{aligned} g_n &= {V_n}^\dagger \, \bar{g}_n \, V_n \quad , \quad V_n \in SU(n, \mathcal{C}) \\ \bar{g}_n &= C_n(\vec{\alpha}) \prod_{l=1}^{\overleftarrow{n-1}} \begin{pmatrix} K_{l+1}(\vec{u}_l) & 0 \\ 0 & 1_{n-1-l} \end{pmatrix} C_n(\vec{\beta}) \end{aligned} \tag{4.4}$$

where $C_n \in C(SU(n, \mathcal{C}))$ and the vectors $\vec{u}_l$ are constrained by $(\check{r}_l)^l \in \Re$ as in Section 3.3.

5 Conclusions

We have obtained in this paper adjoint parametrizations defined by (1.3) w.r.t. maximally compact subgroups for a large set of non-compact groups, the classically riemannian cosets. The constructive procedure used allows to extend them straightforwardly to non riemannian decompositions, for example $SO(p+n, q)$ under $SO(n, q)$. Few words about symplectic groups: these groups can be treated in the same way as made here; the fact that $USp(2p, 2q) \sim U(2p, 2q; \mathcal{C}) \bigcap Sp(2p+2q; \mathcal{C})$ seems to suggest that the corresponding decomposition under $USp(2p) \times USp(2q)$ could follow from replacing in (3.13) E by Z generators [7] and respective Cartan subalgebras, but we have not checked this.

We remark the locality of the parametrizations obtained; the changes of variables needed to carry out the job are singular in some points of the group manifold as can be seen by direct inspection of (2.8) , (3.6) for example.

Possible applications in physical problems of these parametrizations are in the context of GWZW models as models of strings moving on background fields. Equation (2.26) can be used to treat systematically all the models in [9] where the lowest dimensional cases were considered. Also the measure on the group can be computed straightforwardly through the Maurer-Cartan forms without introducing Fadeed-Popov ghost due to constraints because they were solved once for all. For $p+q=2$ (A_1 algebras) parametrization (3.13) is widely known; for $p=2, q=1$ was introduced in [10]; it allows to extend the study of coset models in the search of physically relevant string backgrounds represented by exact conformal field theories.

[7] See reference [5], chapter 5 for definitions.

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