

Virtual photon effects in $D = 3$ Born-Infeld theory.

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Abstract

One loop effects due to virtual gauge field propagation in $2 + 1$ dimensional Born-Infeld theory are investigated. Although this field theory model is not power counting renormalizable, it can be consistently interpreted as an effective field theory. We derive the one-loop effective action in this framework.

Halpern's field strength's formulation is then applied to derive an effective description for the interaction between magnetically charged particles, when the gauge field dynamics is determined by a Born-Infeld action. We compare the results with those of the Maxwell theory.

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1 Introduction

QED_3 -like theories, consisting of charged matter fields in interaction with Abelian gauge fields in $2 + 1$ dimensions, have multiple applications in both Condensed Matter [1] and High Energy Physics [2]. Even after imposing the requisites of gauge invariance and renormalizability, the gauge field action is not uniquely determined, because in an odd number of space-time dimensions it is always possible to combine the standard Maxwell Lagrangian with a Chern-Simons term. This gives rise to interesting, distinctive phenomena.

A different, non quadratic Born-Infeld action [3]-[5] for the gauge field in $2 + 1$ dimensions has been recently considered [6], in order to study the effects its presence has on the properties of classical vortex configurations in a spontaneously broken gauge theory. Now, the Born-Infeld action induces non-renormalizable interactions so that a question that immediately arises is how to deal with the gauge field self-interactions, at the *quantum* level. The relevance of these self-interactions is already evident at the classical level: they provide a mechanism to avoid the divergence of the electric field produced by a point-like electric source, and it is certainly worth investigating what could these effects be at the quantum level. Namely, we would like to know, for example, to what extent the field distribution around a point-like charge is altered by quantum effects.

The study of loop effects due to non-linearities in a gauge field action is not a new subject. Indeed, in a precursor work by Halpern [7], a photon loop in the Euler-Heisenberg Lagrangian [8] for QED_4 was calculated in an effective field theory framework. That the theory is an *effective* theory is an important fact since, being quartic in the field strength, the Euler-Heisenberg Lagrangian has coupling constants with negative mass dimensions and hence is not renormalizable. There is however an intrinsic momentum cutoff, given by the mass of the charged particle that has been integrated out. The situation is somewhat similar to that taking place in chiral perturbation theory, where a non-renormalizable non-linear σ -model is used to calculate loops, yielding relevant information and having predictive power if only a few terms in the expansion are calculated.

In this paper, we shall consider the Born-Infeld action in $2 + 1$ dimensions (i.e. just the pure gauge sector of the model discussed in reference [6]) as a ‘classical’ effective theory for the gauge field A_μ , valid only below a certain momentum cutoff. This is consistent with the usual constraint in Born-

Infeld theory, which states that the field strength should be smaller than a maximum value called the “absolute field”. Indeed, following a suggestion by Born, there have been attempts [9] to interpret the Born-Infeld theory as arising from integrating out some unknown massive matter field, keeping only the terms containing $F_{\mu\nu}$, and no extra derivatives. We shall, in this paper, adopt this point of view, namely, to treat this kind of Lagrangian as if it came from a low-momentum expansion. It will then be reliable only up to a certain momentum cutoff, of the order of the mass of the original fields that have been integrated out.

We may call the Born-Infeld theory for A_μ “classical” since A_μ has not yet been quantized. The next step in an effective low momentum description amounts to adding to this “classical” effective theory the quantum effects due to virtual photons [7], calculating the corresponding one-loop “quantum effective action”. We will compute this one loop correction neglecting derivatives of $F_{\mu\nu}$, since they have already been disregarded at the ‘classical’ level, to avoid higher derivatives and non-causal propagation. This amounts to evaluating the effective one-loop action in the presence of a constant $F_{\mu\nu}$. This kind of calculation, for the case of a general quartic gauge field action in $2 + 1$ dimensions has been presented in [10].

A different approach is followed in the second part of this paper. We consider there the Born-Infeld theory in the so-called field-strength formalism. This formulation, proposed by Halpern in the seventies [16], amounts to a kind of dual first-order formulation. We shall use it in order to derive a ‘dual’ Born-Infeld action, appropriate to the description of the interaction between magnetically charged particles.

This paper is organized as follows: In section 2 the one-loop quantum correction to the classical Born-Infeld action for the gauge field is defined and calculated. In section 3 we derive an effective theory for the interaction between magnetically charged particles.

2 One-loop effective action

As we shall study here the quantum effects due to gauge field fluctuations, we consider the generating functional

$$\mathcal{Z}[J_\mu] = \int \mathcal{D}A_\mu \exp \left(iS_{BI}[A] - i \int d^3x J_\mu(x) A^\mu(x) \right) \quad (1)$$

where

$$S_{BI}[A] = -\beta^2 \int d^3x \left[R\left(\frac{F^2}{2\beta^2}\right) - 1 \right] \quad (2)$$

Here $R(x) = \sqrt{1+x}$, $F^2 \equiv F_{\mu\nu}F^{\mu\nu}$, β is a constant with mass dimensions $[\beta] = m^{3/2}$. It should be noted that the constraint

$$F^2 < 2\beta^2 \quad (3)$$

assures the reality of the classical action, a sensible physical constraint, if we want to have unitary time evolution. The constant β shall be used to fix the momentum cutoff, to a value $\beta^{2/3}$. Of course, the momentum cutoff could be not exactly equal to β , but must be of the same order ¹.

For $F^2 < 2\beta^2$, an expansion of (3) yields

$$S_{BI}[A] = - \int d^3x \left[\frac{1}{4}F^2 - \frac{1}{32\beta^2}(F^2)^2 + \frac{1}{128\beta^4}(F^2)^3 + \mathcal{O}\left(\frac{(F^2)^4}{\beta^6}\right) \right] \quad (4)$$

where the fact that vertices with an arbitrary large number of derivatives arise is evident. Those vertices are responsible for the non-renormalizability.

To calculate the one-loop effective action for this model, we rotate to Euclidean space. The Euclidean version of (1) takes then the form

$$\mathcal{Z}[J_\mu] = \int \mathcal{D}A_\mu \exp\left(-S[A] + \int d^3x J_\mu(x)A_\mu(x)\right) \quad (5)$$

where

$$S[A] = +\beta^2 \int d^3x \left[R\left(\frac{F^2}{2\beta^2}\right) - 1 \right] \quad (6)$$

where now $F^2 \equiv F_{\mu\nu}F_{\mu\nu}$ is positive definite. The Euclidean effective action to one-loop order is, as usual, given by the following expression

$$\Gamma_{eff}[A] = S[A] + \Gamma^{(1)}[A] \quad (7)$$

where $S[A]$ is the classical Euclidean action (6), and

$$\Gamma^{(1)}[A] = \frac{1}{2} \text{Tr} \ln\left(\frac{\delta^2 S}{\delta A_\mu \delta A_\nu}\right) - \frac{1}{2} \text{Tr} \ln\left(\frac{\delta^2 S}{\delta A_\mu \delta A_\nu}\right) \Big|_{A=0}. \quad (8)$$

¹Note that β is the only dimensionful constant we have.

In computing the second functional derivative of S , we drop terms containing derivatives of $F_{\mu\nu}$ thus obtaining

$$\frac{\delta^2 S}{\delta A_\mu(v)\delta A_\nu(w)}\Big|_{F_{\mu\nu}=\text{const.}} = 2 \left\{ R' \left(\frac{F^2}{\beta^2} \right) (-\partial^2 \delta_{\mu\nu} + \partial_\mu \partial_\nu) - \frac{1}{\alpha} \partial_\mu \partial_\nu - \frac{2}{\beta^2} R'' \left(\frac{F^2}{\beta^2} \right) F_{\mu\alpha} F_{\nu\beta} \partial_\alpha \partial_\beta \right\} \delta(v-w), \quad (9)$$

where v, w are coordinates of two space-time points .

$\Gamma^{(1)} = \Gamma^{(1)}[F]$ denotes the part of the total effective action that includes one loop quantum effects coming from A_μ .

$$\Gamma^{(1)}[F] = \frac{1}{2} \text{Tr} \ln \left\{ R' \left(\frac{F^2}{2\beta^2} \right) (-\partial^2 \delta_{\mu\nu} + \partial_\mu \partial_\nu) - \frac{1}{\alpha} \partial_\mu \partial_\nu - \frac{2}{\beta^2} R'' \left(\frac{F^2}{2\beta^2} \right) F_{\mu\alpha} F_{\nu\beta} \partial_\alpha \partial_\beta \right\} - \frac{1}{2} \text{Tr} \ln \left\{ (-\partial^2 \delta_{\mu\nu} + \partial_\mu \partial_\nu) - \frac{1}{\alpha} \partial_\mu \partial_\nu \right\}, \quad (10)$$

where the usual field independent infinite constant has been subtracted. The symbol ‘Tr’ in (10) means both functional and discrete trace (over the indices μ, ν), and care should be taken of the fact that the log must be understood as the logarithm of a matrix function. Passing to momentum space, we obtain

$$\Gamma^{(1)}[F] = \int d^3x \int \frac{d^3k}{(2\pi)^3} \left\{ \text{tr} [R'(k^2 \delta_{\mu\nu} - k_\mu k_\nu) + \frac{1}{\alpha} k_\mu k_\nu + \frac{2R''}{\beta^2} F_{\mu\alpha} F_{\nu\beta} k_\alpha k_\beta] - \text{tr} [(k^2 \delta_{\mu\nu} - k_\mu k_\nu) + \frac{1}{\alpha} k_\mu k_\nu] \right\}. \quad (11)$$

To take the trace over the discrete indices we just calculate the eigenvalues of the corresponding three by three matrix. After doing that, and cancelling like terms between the two lines of (11), we get

$$\Gamma^{(1)}[F] = \frac{1}{2} \int d^3x \int \frac{d^3k}{(2\pi)^3} \left\{ 2 \ln R' + \ln \left[1 + \frac{2R''}{\beta^2 R'} F_{\mu\alpha} F_{\mu\beta} \frac{k_\alpha k_\beta}{k^2} \right] \right\}, \quad (12)$$

where we must keep in mind that the momenta must be integrated inside a ball of radius $\beta^{\frac{2}{3}}$, the momentum cutoff. This integration over momentum

yields, after some straightforward algebra,

$$\Gamma^{(1)}[F] = -\frac{\beta^2}{6\pi^2} \int d^3x \ln \sqrt{1 + \frac{F^2}{2\beta^2}} + \frac{\beta^2}{6\pi^2} \int d^3x \left[\frac{\arctan(\sqrt{\frac{F^2}{2\beta^2}})}{\sqrt{\frac{F^2}{2\beta^2}}} - 1 \right]. \quad (13)$$

Rotating back to Minkowski space-time, we obtain for the full one loop effective action

$$\begin{aligned} \Gamma_{eff}[F] &= -\beta^2 \int d^3x \left[R\left(\frac{F^2}{2\beta^2}\right) - 1 \right] \\ &+ \frac{\beta^2}{6\pi^2} \int d^3x \ln \sqrt{1 + \frac{F^2}{2\beta^2}} - \frac{\beta^2}{6\pi^2} \int d^3x \left[\frac{\arctan(\sqrt{\frac{F^2}{2\beta^2}})}{\sqrt{\frac{F^2}{2\beta^2}}} - 1 \right], \end{aligned} \quad (14)$$

or, in a more compact form,

$$\Gamma_{eff}[F] = -\beta^2 \int d^3x \left[R_{eff}\left(\frac{F^2}{2\beta^2}\right) - 1 \right] \quad (15)$$

where

$$R_{eff}(x) = R(x) - \frac{1}{6\pi^2} \ln \sqrt{1+x} + \frac{1}{6\pi^2} \left[\frac{\arctan(\sqrt{x})}{\sqrt{x}} - 1 \right], \quad (16)$$

Let us now discuss the behaviour of the extra terms in (14) that modify the Born-Infeld action. A large β expansion of the classical Born-Infeld action yields the Maxwell action, while for the quantum corrected version (14) we obtain

$$\Gamma_{eff}[F] = \int d^3x \left[-\left(\frac{1}{4} - \frac{5}{72\pi^2}\right)F^2 + \mathcal{O}\left(\left(\frac{F^2}{\beta^2}\right)^2\right) \right], \quad (17)$$

a very small modification indeed. In the opposite regime, we have instead

$$\Gamma_{eff}[F] = \beta^2 \int d^3x \left[-\sqrt{\frac{F^2}{2\beta^2}} + \frac{\ln(\frac{\sqrt{F^2}}{2\beta^2})}{6\pi^2} + \left(1 + \frac{1}{6\pi^2}\right) + \mathcal{O}\left(\sqrt{\frac{\beta^2}{F^2}}\right) \right], \quad (18)$$

where we see that the corrections grow at most logarithmically with $|F|$, while the classical action goes like $|F|$.

It is interesting to note at this point that Born original idea was not only to have a classical theory for charged particles without the infinite self-energy problem of Maxwell theory but also to compare the quantum answer of this last theory (i.e. the Euler Heisenberg effective action) with the classical ones arising from his non-linear electromagnetism which could account for the mass of the electron in terms of its electromagnetic energy. In this respect, Born [5] disregarded the discrepancy between the factors multiplying the quartic terms in the expansion of his non-linear classical Lagrangian and that of Euler-Heisenberg and just adjusted β in order to have an overall agreement. In this context, including one loop-effects in the Born-Infeld theory incorporates some quantum effects that should be comparable with those arising from a similar calculation in the Euler-Heisenberg effective theory (as done by Halter in the $d = 4$ case [7]).

The effective Lagrangian in Minkowski space is (see eq.(15))

$$L_{eff} = L_{BI} + L_q = -\beta^2 \left[R_{eff}\left(\frac{F^2}{2\beta^2}\right) - 1 \right] \quad (19)$$

with R_{eff} given by eq.(16)

$$R_{eff}(x) = \sqrt{1+x} - \frac{1}{6\pi^2} \ln \sqrt{1+x} + \frac{1}{6\pi^2} \left[\frac{\arctan(\sqrt{x})}{\sqrt{x}} - 1 \right] \quad (20)$$

As usual, we define

$$E^i = F^{0i} \quad B = \varepsilon_{ij} F^{ij} \quad (21)$$

so that

$$x = \frac{F^2}{2\beta^2} = \frac{1}{\beta^2} (B^2 - E^2) \quad (22)$$

with $E^2 = E^i E^i$.

It is interesting at this point to define electric and magnetic polarization vectors for the vacuum as

$$P_i \equiv \frac{\partial L_q}{\partial E^i} \quad (23)$$

$$M \equiv \frac{\partial L_q}{\partial B} \quad (24)$$

We obtain from (19)

$$P^i = \chi E^i \quad M = -\chi B \quad (25)$$

with the susceptibility χ taking the form

$$\chi(x) = \frac{1}{6\pi^2} \left[\frac{1-x}{x(1+x)} - \frac{\arctan(\sqrt{x})}{x^{3/2}} \right] \quad (26)$$

We thus see that the vacuum supports magnetic and electric polarization. As it also happens for the Euler-Heisenberg effective theory [17], these polarization vectors vanish whenever $E^2 = B^2$ since $\chi(0) = 0$. Moreover, this is in agreement with the conditions imposed long ago by Weisskopf [18] on the vacuum polarization arising in the study of Euler-Heisenberg effective actions, namely that the energy density and the susceptibility have to vanish in the absence of fields in order to have a consistent effective theory.

Let us now analyse the electric field distribution around a point-like charge when one takes into account the contribution of $\Gamma^{(1)}$ to the Born-Infeld effective action. The Gauss law for the classical 2 + 1 Born-Infeld theory, for the case of a delta-like distribution of charge q , located at the origin yields,

$$\partial^j \left[\frac{E_j}{R(-\frac{E^2}{\beta^2})} \right] = q\delta(x) . \quad (27)$$

It is trivial to solve this equation, this leading to a solution corresponding to a radial electric field $E(r)$ of the form

$$\frac{E(r)}{\sqrt{1 - \frac{E^2(r)}{\beta^2}}} = \frac{q}{2\pi r} . \quad (28)$$

For the quantum corrected version the equivalent result is

$$2E(r)R'_{eff}\left(-\frac{E^2(r)}{\beta^2}\right) = \frac{q}{2\pi r} , \quad (29)$$

where

$$R'_{eff}(x) = \frac{1}{2\sqrt{1+x}} - \frac{1}{12\pi^2} \left(\frac{1}{1+x} - \frac{1}{x(1+x)} + \frac{\arctan(\sqrt{x})}{x^{3/2}} \right) \quad (30)$$

and x has to be replaced by $-E^2/\beta^2$. In order to plot this electric field as a function of r , it is convenient to measure it in units of β , namely, we define

$u(r) \equiv E(r)/\beta$. Hence, (29) implies

$$u(r) \left\{ \frac{1}{\sqrt{1-u^2(r)}} - \frac{1}{6\pi^2} \left[\frac{1+u^2(r)}{u^2(r)(1-u^2(r))} - \frac{\operatorname{arctanh}(|u(r)|)}{|u(r)|^3} \right] \right\} = \frac{q}{2\pi\beta r} \quad (31)$$

We note that in the small- u regime, (31) reduces to

$$u(r) \left[1 - \frac{5}{18\pi^2} + \mathcal{O}(u^2) \right] = \frac{q}{2\pi\beta r} \quad (32)$$

namely, so that it reduces to the usual behaviour, except for a constant renormalization of the charge. This renormalization constant could actually be read from (17). One can see in figure 1 that the electric fields for the point charge in the classical and quantum corrected versions of the BI theory qualitatively coincide.

Let us end this section by consider a formulation than allows to capture at least part of the non-perturbative dynamics of Born-Infeld theory. To this end, we shall study here a simplified situation. which corresponds to ignoring the magnetic field in (2), what yields a much simpler action S_{el} ,

$$S_{el}[A] = -\beta^2 \int d^3x \left[R\left(-\frac{\vec{E}^2}{\beta^2}\right) - 1 \right]. \quad (33)$$

It is already evident that this action decomposes into an integral of independent actions, one for each spatial point (there is no spatial propagation):

$$S_{el}[A] = m^2 \int d^2x S_x[A], \quad S_x[A] = -m \int_{t_1}^{t_2} dt \left[\sqrt{1 - \frac{(\vec{E}(x,t))^2}{m^3}} - 1 \right] \quad (34)$$

where we defined $m = \beta^{\frac{2}{3}}$, which has the units of a mass. A factor of m^2 was extracted in order to keep both S_{el} and S_x explicitly dimensionless, what simplifies the treatment.

We see from (34) that it is only necessary to understand S_x , regarding x as an index that labels independent fields. Now, if we use the timelike gauge $A_0 = 0$, this action becomes

$$S_x[A] = -m \int_{t_1}^{t_2} dt \left[\sqrt{1 - \frac{1}{m^3} \left(\frac{\partial}{\partial t} \vec{A}(x,t) \right)^2} - 1 \right]. \quad (35)$$

We realise that this action is equivalent to the one of a relativistic particle of mass m , with "spacetime coordinates"

$$X^\mu(x, t) = \left(t, \frac{1}{m^{\frac{3}{2}}} \vec{A}(x, t) \right), \quad (36)$$

where the factor of m was introduced in order to have the same dimensionality for all the components (mass^{-1}). The action S_x is written in a particular parametrization of the trajectory, that follows from using the time as a parameter labeling points along the path. A reparametrization invariant form can be obtained if one introduces a parameter τ , $t = t(\tau)$, such that the endpoints of the time interval are unchanged. Then $X^0 = X^0(\tau)$, and

$$S_x[A] = m(t_2 - t_1) - m \int_{t_1}^{t_2} d\tau \sqrt{\dot{X}_\mu(x, \tau) \dot{X}^\mu(x, \tau)}, \quad (37)$$

where contractions are performed with the usual Minkowskian metric in the space of the X^μ . (This action may also be written in quadratic form at the expense of introducing an einbein [19]). Using the known results about the quantum relativistic particle [19], we can, for example, calculate the amplitude corresponding to a transition from a given configuration X_1 to another one X_2 . Note that if the space coordinates x are different, that will introduce a δ -function. In terms of A , we can express the amplitude for going from a configuration $\vec{A}_1(x_1)$ at a time t_1 to another one $\vec{A}_2(x_2)$ at a time t_2 as follows

$$\begin{aligned} \langle \vec{A}_2(x_2, t_2) | \vec{A}_1(x_1, t_1) \rangle &= \delta(x_2 - x_1) \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2 - m^2 + i\epsilon} \times \\ &\exp\left(-ik^0(t_2 - t_1) + i\beta^{-3/2} \vec{k} \cdot (\vec{A}_2(x_2) - \vec{A}_1(x_1))\right) \end{aligned} \quad (38)$$

Using this technique, many other quantities can be calculated exactly (like for example the free energy at finite temperature).

It should be worthwhile at this point to comment on an aspect of renormalizability which has interesting implications. We have justified the use of a momentum cut-off in order to make sense from divergent quantities by considering the BI lagrangian as an effective one arising in the low-energy regime of some unknown theory. In fact, the renewed interest on BI theory stems from the fact it can be seen as arising as a part of an effective action derived from string theory [11]-[14].

Now, as already stressed in [9], the BI Lagrangian can be seen as a possible Euler-Heisenberg effective Lagrangian derived from a certain renormalizable supersymmetric Lagrangian. This, together with the original observation of Deser and Pusalowski [15] concerning the fact that the BI Lagrangian is among those non-polynomial lagrangians which admit supersymmetric extensions suggests that the analysis of supersymmetry in connection with BI models and renormalizability should be thoroughly considered. In the same sense points the remarkable fact that starting from a SUSY system of minimally coupled spin 1/2 and spin 0 particles one achieves agreement between the Euler-Heisenberg effective action and the Born-Infeld action up to and including terms of order 4 thus resolving the discrepancy, signaled above, between BI and the effective action for QED_4 . We hope to discuss this and other issues related to supersymmetry and Born-Infeld theory in a forthcoming work.

3 Field strength formulation and monopoles

We begin by reviewing Halpern's derivation, with some differences due to the fact that the action is of the Born-Infeld type rather than Maxwell. The generating functional for Euclidean Green's functions of the Abelian gauge field A_μ , with a classical Euclidean action S as in (6) is

$$\mathcal{Z}[J_\mu] = \int \mathcal{D}A_\mu \exp \left(-S[A] + \int d^3x J_\mu(x) A_\mu(x) \right) \quad (39)$$

where $S[A]$ of course satisfies $S[A + \partial\omega] = S[A]$, for any ω . We chose to work in terms of \tilde{F}_μ , the dual of $F_{\mu\nu}$, defined by $\tilde{F}_\mu = \epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda$. Thus

$$S[A] = I(\tilde{F}_\mu) \quad (40)$$

where I is the functional.

$$I(\tilde{F}_\mu) = \beta^2 \int d^3x \left[R \left(\frac{\tilde{F}_\mu^2}{\beta^2} \right) - 1 \right]. \quad (41)$$

We now include into (39) the gauge-fixing factor corresponding to the Landau gauge ($\partial \cdot A = 0$)

$$\mathcal{Z}[J_\mu] = \int \mathcal{D}A_\mu \delta(\partial \cdot A) \exp \left(-S(A) + \int d^3x J_\mu(x) A_\mu(x) \right) \quad (42)$$

where we have omitted the field-independent Faddeev-Popov factor $\det(-\partial^2)$, since in this case it can be absorbed into the normalization of the integration measure and has no effect on the Green's functions derived from (42). To obtain a formulation in terms of \tilde{F}_μ , we introduce in (42) a '1' written as follows:

$$1 = \int \mathcal{D}\tilde{F}_\mu \delta(\tilde{F}_\mu - \epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda) \delta(\partial \cdot \tilde{F}) . \quad (43)$$

Note the presence of a delta functional of the Bianchi identity, which is a consistency condition for the equation $\tilde{F}_\mu - \epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda = 0$, whose solutions are relevant to the first delta-function. The meaning of the inclusion of that factor can be made explicit by means of the following argument: Consider the rhs of Equation (43), but this time writing both delta-functionals in terms of functional Fourier transforms:

$$\begin{aligned} & \int \mathcal{D}\tilde{F}_\mu \delta(\tilde{F}_\mu - \epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda) \delta(\partial \cdot \tilde{F}) \\ &= \int \mathcal{D}\tilde{F}_\mu \mathcal{D}\lambda_\mu \mathcal{D}\theta \exp \left\{ i \int d^3x [\lambda_\mu (\tilde{F}_\mu - \epsilon_{\mu\nu\rho}\partial_\nu A_\rho) + \theta \partial_\mu \tilde{F}_\mu] \right\} \end{aligned} \quad (44)$$

where λ_μ and θ are Lagrange multipliers. Integrating out \tilde{F}_μ in (44) yields

$$\begin{aligned} & \int \mathcal{D}\tilde{F}_\mu \delta(\tilde{F}_\mu - \epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda) \delta(\partial \cdot \tilde{F}) = \\ & \int \mathcal{D}\lambda \mathcal{D}\theta \delta(\lambda_\mu - \partial_\mu \theta) \exp \left(-i \int d^3x \lambda_\mu \epsilon_{\mu\nu\rho} \partial_\nu A_\rho \right) = \\ & \int \mathcal{D}\theta \exp \left(-i \int d^3x \partial_\mu \theta \epsilon_{\mu\nu\rho} \partial_\nu A_\rho \right) = \int \mathcal{D}\theta \exp \left(i \int d^3x \theta \epsilon_{\mu\nu\rho} \partial_\mu \partial_\nu A_\rho \right) \end{aligned} \quad (45)$$

where the vanishing of $F_{\mu\nu}$ at infinity was used on the last line, in order to ignore the surface contribution. We conclude, after integrating out θ in (45) that

$$\begin{aligned} & \int \mathcal{D}\tilde{F}_\mu \delta(\tilde{F}_\mu - \epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda) \delta(\partial \cdot \tilde{F}) \\ &= \delta(\epsilon_{\mu\nu\rho} \partial_\mu \partial_\nu A_\rho) . \end{aligned} \quad (46)$$

Thus the '1' behaves as a constant factor when inserted into a functional integration over A_μ fields whose second partial derivatives commute.

After insertion of the ‘1’, the generating functional becomes

$$\mathcal{Z}[J_\mu] = \int \mathcal{D}A_\mu \mathcal{D}\tilde{F}_\mu \delta(\partial \cdot A) \delta(\partial \cdot \tilde{F}) \delta(\tilde{F}_\mu - \epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda) e^{-I(\tilde{F}_\mu) + \int d^3x J_\mu(x) A_\mu(x)} . \quad (47)$$

Now we observe that, introducing the two delta-functionals $\delta(\partial \cdot A)$ and $\delta(\tilde{F}_\mu - \epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda)$, A_μ can be written in terms of \tilde{F}_μ :

$$A_\mu = -\epsilon_{\mu\nu\lambda} \frac{1}{\partial^2} \partial_\nu \tilde{F}_\lambda , \quad (48)$$

and the dependence on A_μ (only from the source term) can be completely erased by replacing it by its expression (48) in terms of \tilde{F}_μ . The A_μ field is thus integrated out, yielding for \mathcal{Z} the expression:

$$\mathcal{Z}[J_\mu] = \int \mathcal{D}\tilde{F}_\mu \delta(\partial \cdot \tilde{F}) \exp \left(-I(\tilde{F}_\mu) - \int d^3x J_\mu \epsilon_{\mu\nu\lambda} \partial_\nu \partial^{-2} \tilde{F}_\lambda \right) , \quad (49)$$

which contains only \tilde{F}_μ as a dynamical variable, and may be thought of as the generating functional for a theory describing the dynamics of a pseudovector field \tilde{F}_μ , with the constraint $\partial \cdot \tilde{F} = 0$. We note that, because of the form of the source term in (49), there is a simple relation between Green’s functions for \tilde{F}_μ and the ones for A_μ :

$$\begin{aligned} \langle A_{\mu_1}(x_1) A_{\mu_2}(x_2) \cdots A_{\mu_n}(x_n) \rangle &= \epsilon_{\mu_1 \nu_1 \lambda_1} \partial_{\nu_1}^{x_1} \partial_{x_1}^{-2} \epsilon_{\mu_2 \nu_2 \lambda_2} \partial_{\nu_2}^{x_2} \partial_{x_2}^{-2} \cdots \\ &\quad \cdots \epsilon_{\mu_n \nu_n \lambda_n} \partial_{\nu_n}^{x_n} \partial_{x_n}^{-2} \langle F_{\lambda_1}(x_1) F_{\lambda_2}(x_2) \cdots F_{\lambda_n}(x_n) \rangle . \end{aligned} \quad (50)$$

Although a naive look at (49) may suggest that it is tantamount to a gauge fixed version for some gauge-invariant theory, this is not necessarily the case, as the form of the ‘action’ I for the pseudovector field is not ‘gauge invariant’ in this sense.

In order to do actual calculations with the theory defined in terms of \tilde{F}_μ , a set of Feynman rules should be defined. It is convenient to introduce a Lagrange multiplier field θ in order to deal with the delta-functional $\partial \cdot \tilde{F}$, and also to add a source term for θ , since \tilde{F}_μ and θ are coupled. We add a source term for \tilde{F}_μ (not to be confused with the source for A_μ), since the Green’s functions for A may be obtained by applying (50) to the \tilde{F}_μ ’s Green’s functions.

Thus the generating functional we define is

$$\mathcal{Z} = \int \mathcal{D}\tilde{F}_\mu \mathcal{D}\theta \exp \left\{ -I(\tilde{F}_\mu) + \int d^3x [i\theta \partial \cdot \tilde{F} + J_\mu \tilde{F}_\mu + j_\theta \theta] \right\} \quad (51)$$

and Euclidean correlation functions are simply obtained by functional differentiation. Free propagators are obtained from evaluation of the Gaussian integral corresponding to a quadratic action,

$$I(\tilde{F}_\mu) \equiv I_0(\tilde{F}_\mu) = \int d^3x \frac{1}{2} \tilde{F}_\mu \tilde{F}_\mu . \quad (52)$$

It is immediate to extract the free propagators that follow from (51) with the action (52)

$$\begin{aligned} \langle \tilde{F}_\mu \tilde{F}_\nu \rangle &= (\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}) \\ \langle \theta \theta \rangle &= \frac{1}{k^2} \\ \langle \tilde{F}_\mu \theta \rangle &= \frac{k_\mu}{k^2} \end{aligned} \quad (53)$$

The field strength formulation in the Abelian case can be applied to the study of the interaction between magnetic monopoles [16]. This can be achieved by replacing in (49) the delta functional of the Bianchi identity by a delta functional that enforces $\partial_\mu \tilde{F}_\mu$ to equal a monopole density ρ :

$$\mathcal{Z}[J_\mu] = \int \mathcal{D}\tilde{F}_\mu \delta(\partial \cdot \tilde{F} - \rho(x)) e^{-I(\tilde{F}_\mu) - \int d^3x J_\mu \epsilon_{\mu\nu\lambda} \partial_\nu \partial^{-2} \tilde{F}_\lambda} . \quad (54)$$

An argument similar to the one used at the beginning of this section shows that the ‘1’ in this case reduces to a constant factor when put inside a functional integral where the partial derivatives of A_μ don’t commute, but rather satisfy

$$\epsilon_{\mu\nu\lambda} \partial_\mu \partial_\nu A_\lambda = \rho. \quad (55)$$

It should be stressed at this point that actual monopoles are in fact static configurations in Minkowski 3 + 1 dimensional space-time while the present discussion corresponds to 3-dimensional Euclidean space-time. Then, it should be more appropriate to consider these configurations as instantons (i.e. points

in space time) rather than as particles (monopoles). The fact that the magnetic field resulting from (55) has the typical monopole behavior justifies however our terminology.

The interaction action for the monopoles is then deduced by setting $J_\mu = 0$, and integrating out over \tilde{F}_μ . This yields an effective description for the interaction between monopoles, in terms of the scalar field θ and the density ρ . Of course this will represent the interaction *due to the gauge field*, as we are not giving rho any dynamics. It could, of course, be introduced at the end, and this will not affect the following discussion.

Of course, in the Born-Infeld case the integral over \tilde{F}_μ is not Gaussian, but, as there are no derivatives of \tilde{F}_μ , we see that the corresponding action is of the ‘ultralocal’ kind, since the integral decomposes into an infinite product of uncoupled integrals (one for each space-time point). The result of performing the functional integral will then be an infinite product (one factor for each spacetime point), of normal, three dimensional integrals. Denoting by $\mathcal{Z}(\rho)$ the functional integral for $J = 0$, we have

$$\mathcal{Z}(\rho) = \int \mathcal{D}\theta \exp(i \int d^3x \theta \rho) \Pi_x \left\{ \int d\tilde{F}_\mu \exp[-\beta^2 (R(\frac{\tilde{F}_\mu^2}{\beta^2}) - 1) + i\tilde{F}_\mu \partial_\mu \theta(x)] \right\} \quad (56)$$

where \tilde{F}_μ is understood as x independent, and the product runs over all the spacetime points. To obtain the integral over \tilde{F}_μ we just need to know the x -dependent, normal (rather than functional) integral

$$\exp[-W(\theta(x))] \equiv \int d\tilde{F}_\mu \exp[-\beta^2 (R(\frac{\tilde{F}_\mu^2}{\beta^2}) - 1) + i\tilde{F}_\mu \partial_\mu \theta(x)]. \quad (57)$$

The range of integration over \tilde{F}_μ should be restricted to be inside a sphere of radius β , if that is the value of the absolute field. We assume that kind of cutoff, if present, has been included into the function R . Then,

$$\mathcal{Z}(\rho) = \int \mathcal{D}\theta \exp(i \int d^3x \theta \rho) \Pi_x \exp[-W(\theta(x))]. \quad (58)$$

The integral (57) can of course be rewritten in spherical coordinates:

$$e^{-W(\theta(x))} = 2\pi \int_0^\infty d|\tilde{F}_\mu| |\tilde{F}_\mu|^2 \int_0^\pi d\omega \sin \omega \exp[-\beta^2 (R(\frac{\tilde{F}_\mu^2}{\beta^2}) - 1) + i|\tilde{F}_\mu| |\partial_\mu \theta(x)| \cos \omega], \quad (59)$$

where $|\tilde{F}_\mu| = \sqrt{\tilde{F}^2}$ and $|\partial_\mu\theta(x)| = \sqrt{(\partial \cdot \theta)^2}$. Performing the angular integration yields

$$e^{-W(\theta(x))} = 4\pi \int_0^\infty d|\tilde{F}_\mu| |\tilde{F}_\mu| \frac{\sin[\sqrt{\tilde{F}^2} \sqrt{(\partial \cdot \theta)^2}]}{\sqrt{(\partial \cdot \theta)^2}} \exp[-\beta^2 (R(\frac{\tilde{F}^2}{\beta^2}) - 1)] \quad (60)$$

For the sake of comparison, we evaluate first the integral (60) in the Maxwell case ($R(x) = x$), what yields for the $\mathcal{Z}(\rho)$ the expression

$$\mathcal{Z}(\rho) = \int \mathcal{D}\theta \exp(i \int d^3x \theta \rho) \exp \left\{ -\frac{1}{4} \int d^3x \partial_\mu \theta \partial_\mu \theta \right\} . \quad (61)$$

This shows that, for the Maxwell case, the monopoles shall interact through $1/r$ propagators, the three dimensional equivalent of the usual logarithmic interaction between vortices for the $O(2)$ model on the plane. Unfortunately, the integral (60) cannot be evaluated exactly for the Born-Infeld case. However, to understand at least qualitatively the kind of modification introduced by that kind of action on the interaction between monopoles, we shall use a function R which allows us to perform the integral exactly,

$$R(x) = \sqrt{x} . \quad (62)$$

Of course, this function differs from the one corresponding to the Born Infeld theory but it has in common with it that, for large fields, the action grows linearly in contrast with what happens in the Maxwell case. The result for \mathcal{Z}_ρ is,

$$\mathcal{Z}(\rho) = \int \mathcal{D}\theta \exp(i \int d^3x \theta \rho) \exp \left\{ - \int d^3x \beta^2 \ln[\partial_\mu \theta \partial_\mu \theta + \beta^2]^2 \right\} \quad (63)$$

so that one can see that the interaction between monopoles is strongly different to the one induced for the usual, Maxwell case.

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References

- [1] See for example E. Fradkin, *Field Theories of Condensed Matter Systems*, Addison Wesley, Reading, 1991.
- [2] See for example R.Jackiw, *Topics in Planar Physics*, Banff NATO ASI 1989, pp.191-240.
- [3] M. Born, Proc. R. Soc. London **A143** (1934) 410.
- [4] M. Born and M. Infeld, Proc. R. Soc. London **A144** (1934) 425.
- [5] *For a review see:* M. Born, *Theorie non-linéaire du champ électromagnétique*, Ann. Inst. Poincaré **7** (1939) 155.
- [6] E. Moreno, C. Núñez and F.A. Schaposnik; LA-PLATA-TH-04-98, e-Print Archive: hep-th/9802175.
- [7] J. Halter; Phys. Lett. **B316**, 155 (1993).
- [8] W. Heisenberg and H. Euler; Z. Phys. 98, 714 (1936).
- [9] T. Hagiwara; Nucl. Phys. **B189**, 135 (1981).
- [10] I. J. R. Aitchison, C. D. Fosco and F. D. Mazzitelli, Phys. Rev. D **54**, 4059 (1996).
- [11] E.S. Fradkin and A.A. Tseytlin, Phys. Lett. **163B** (1985) 123; A.A. Tseytlin, Nucl. Phys. **B276** (1986) 391.
- [12] R.G. Leigh, Mod. Phys. Lett. **A4** (1989) 2767 and 2073.
- [13] J.Polchinski, *Tasi Lectures on D-Branes*, hep-th/9611050.
- [14] G. Gibbons, Nucl.Phys. **B514** (1998) 603.
- [15] S. Deser and R. Puzalowski, *J. Phys.* **A13** (1980) 2501.
- [16] M. B. Halpern, Phys. Rev. D **16**, 1798 (1977).
M. B. Halpern, Phys. Rev. D **16**, 3515 (1977).
M. B. Halpern, Nuc. Phys. B **139**, 477 (1978).
M. B. Halpern, Phys. Rev. D **19**, 517 (1979).

- [17] See for example L. Lifchitz and L.Pitayevski, *Théorie Quantique Relativiste* vol II, sect.127. Ed. Mir, Moscow, 1973.
- [18] V. Weisskopf, *Kgl. Danske. Viden. Selskab.Math.phys.* **14** (1936) 6.
- [19] A. Cohen, G. Moore, P. Nelson and J. Polchinski; *Nuc. Phys.* **B267**, 143 (1986).

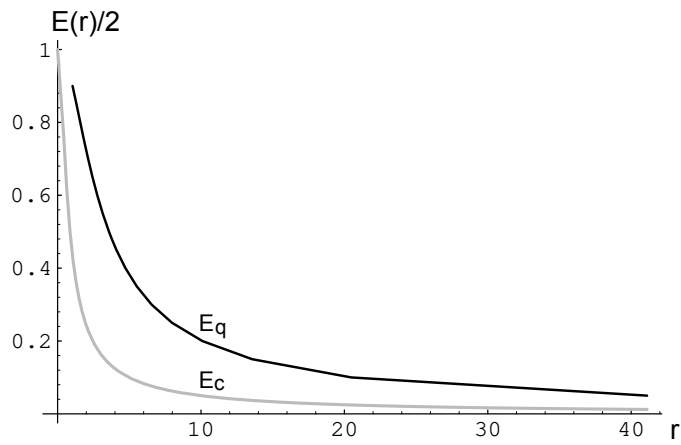


Fig.1: The electric field of a point charge for the classical (E_c) and for the quantum corrected (E_q) Born-Infeld theory. (We have chosen $\beta=2$ and $q = 2\pi$)