Bosonization of Vector and Axial-Vector Currents in $3+1$ Dimensions.

C.D. Fosco*
Centro Atómico Bariloche
8400 Bariloche, Argentina

F.A. Schaposnik†
Departamento de Física, Universidad Nacional de La Plata
C.C. 67, (1900) La Plata, Argentina

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Abstract

The bosonization of a massless fermionic field coupled to both vector and axial-vector external sources is developed, following a path-integral approach. The resulting bosonized theory contains two antisymmetric tensor fields whose actions consist of non-local Kalb-Ramond-like terms plus interactions. Exact bosonization rules that take the axial anomaly for the axial current into account are derived, and an approximated bosonized action is constructed.

*Investigador CONICET
†Investigador CICBA

Any physical system is suitable of many different mathematical descriptions, i.e., there is a lot of freedom in the choice of variables used to define its configurations. Different choices of variables are equivalent in the sense that they describe the same system. An extreme manifestation of this appears in some two-dimensional models, which can be described in terms of either fermionic or bosonic variables. The equivalence between these two formulations is made explicit by the so called bosonization rules, that map the fermionic to the bosonic variables.

There has been some progress in the program of extending, at least in an approximated way, the bosonization procedure to theories in more than two dimensions [1]-[9]. In this letter we are concerned with the problem of finding the bosonization rules for a simple system in four space-time dimensions, that of a massless fermionic field in the presence of vector and axial sources. This will allow us to find the bosonization rules for the vector and axial fermionic currents, this last one being consistent with the axial anomaly [10] in the presence of an external vector gauge field.

Our starting point is the generating functional for a massless Dirac field in $3+1$ (Euclidean) dimensions, coupled to Abelian vector (s_μ) and axial-vector (t_μ) external sources

$$
\begin{aligned}
\mathcal{Z}(s_\mu, t_\mu) \;&=\; \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \, \exp\left[-S(\bar{\psi}, \psi; s_\mu, t_\mu)\right] \\
S(\bar{\psi}, \psi; s_\mu, t_\mu) \;&=\; -i \int d^4x \, \bar{\psi}\, (i\not{\partial} - \not{s} - \gamma_5 \not{t})\, \psi \qquad (1)
\end{aligned}
$$

where we have adopted for the γ-matrices the following conventions:

$$
\gamma_\mu^\dagger \;=\; \gamma_\mu \;\;,\;\; \gamma_5^\dagger \;=\; \gamma_5 \;\;,\;\; \{\gamma_\mu \,,\, \gamma_\nu\} \;=\; 2\,\delta_{\mu\nu} \;. \qquad (2)
$$

The addition of the source s_μ is motivated by the reason that, in four dimensions, the vector and axial currents are *independent* fermionic bilinears. Thus not all the information provided by $\mathcal{Z}(s_\mu, t_\mu)$ can be obtained from, say, $\mathcal{Z}(s_\mu, 0)$. In two dimensions, because of the smaller number of generators for the Dirac algebra, these two current are related, and the bosonization rule for one of the currents also implies the proper rule for the other.

By a natural extension of the procedure followed to obtain the bosonized version of the generating functional with just a vector source (see for example

[5]-[9] for a detailed description of the approach) , we perform in (1) the change of variables

$$\psi(x) \;=\; e^{i\theta(x)-i\gamma_5\alpha(x)}\psi'(x)\;,\;\bar{\psi}(x)\;=\;\bar{\psi}'(x)e^{-i\theta(x)-i\gamma_5\alpha(x)}\;. \tag{3}$$

In terms of the new variables, the generating functional (1) reads

$$\mathcal{Z}(s_\mu,t_\mu)\;=\;\int\mathcal{D}\bar{\psi}\mathcal{D}\psi\;J(\alpha;s_\mu,t_\mu)\;\exp\left[-S(\bar{\psi},\psi\;;\;s_\mu+\partial_\mu\theta,t_\mu+\partial_\mu\alpha)\right] \tag{4}$$

where the primed fermionic fields have been renamed as unprimed for the sake of simplicity, and J is the anomalous Jacobian corresponding to this fermionic change of variables, a well-known consequence of the chiral anomaly [11]. This Jacobian is evaluated by using the standard Fujikawa's recipe. To decide whether the consistent or covariant regularization is to be used in the calculation of this determinant, it is important to recall (see for example ref.[12]) that the consistent regularization, besides assuring the validity of Wess-Zumino consistency conditions, automatically enforces the conservation of the vector current. The covariant regularization, in turn, guarantees a gauge-covariant form for the anomaly, at the expense of generating an anomaly for the vector current as well. Regarding the case at hand, it is desirable, although not mandatory, to have a regularization that assures the conservation of the vector current, since one has in mind applications of the bosonization recipe to situations where interactions involve the vector current, which should then be non-anomalous, while the axial current can have an anomalous divergence without spoiling the consistency of the theory. This sets the natural choice of regularization in this case to be the consistent one, which we adopt. This justifies *a posteriori* the fact that we have written J in (4) as a function of α only, rather than depending also on θ. To obtain the Jacobian for this finite transformation, we shall use the techniques described in ref.[13]. The consistent regularization of the anomalous Jacobian may be based on the hermitean operator

$$\mathcal{D}\;=\;i\;\partial\!\!\!/-\;s\!\!\!/+i\gamma_5\;t\!\!\!/ \tag{5}$$

obtained by performing the analytic continuation $t_\mu\to -it_\mu$ in the operator appearing in the kinetic term of the fermionic action. One then introduces a parameter $r\in[0,1]$, to define the r-dependent transformation

$$\begin{aligned}\psi(x)\;&=\;e^{-ir\theta(x)-r\alpha(x)\gamma_5}\;{\psi'}_r(x)\\ \bar{\psi}(x)\;&=\;\bar{\psi}'_r(x)e^{ir\theta(x)-r\gamma_5\alpha(x)}\end{aligned} \tag{6}$$

which, of course, induces a change in the operator $\mathcal{D}$

$$\begin{aligned} \mathcal{D}_r &= e^{ir\theta(x) - r\gamma_5\alpha(x)} \mathcal{D} e^{-ir\theta(x) - r\alpha(x)\gamma_5} \\ &= i\,\partial\!\!\!/ - (\,s\!\!\!/ - r\,\partial\!\!\!/\theta) + i\gamma_5(\,t\!\!\!/ - r\,\partial\!\!\!/\alpha)\,. \end{aligned} \tag{7}$$

The Jacobian J for the finite transformation (3) corresponds to $r = 1$, and may be obtained by multiplying the Jacobians corresponding to infinitesimal steps dr, each one calculated with the appropriated regularization operator $\mathcal{D}_r$

$$J = \exp\left\{ 2\int_0^1 dr \lim_{t\to 0+} \mathrm{Tr}[\alpha\gamma_5 e^{-t\mathcal{D}_r^2}] \right\}\,. \tag{8}$$

The factor 2 in the exponent comes from multiplying the two Jacobians, associated to the changes in the measures $\mathcal{D}\psi$ and $\mathcal{D}\bar\psi$. Although each one of these Jacobians does depend on θ, like terms do cancel in the product. This is a consequence of the consistent regularization, which uses the same regularization operator for both $\mathcal{D}\psi$ and $\mathcal{D}\bar\psi$.

There just remains to insert into (8) the result for the functional trace, which can be read from the formula for the anomaly for an infinitesimal transformation

$$\lim_{t\to 0+} \mathrm{Tr}\left[\alpha\gamma_5 e^{-t\mathcal{D}_r^2}\right] = -\frac{1}{8\pi^2}\epsilon_{\mu\nu\rho\sigma}(-\partial_\mu s_\nu \partial_\rho s_\sigma + \frac{1}{3}\,\partial_\mu t_\nu \partial_\rho t_\sigma) \tag{9}$$

which, in this Abelian case, becomes r-independent, rendering the integration over r in (8) trivial. After undoing the analytic continuation on t, the anomalous Jacobian for the transformation (3) becomes

$$J(\alpha; s_\mu, t_\mu) = \exp\left[\frac{1}{4\pi^2}\int d^4x\, \alpha(x)\, \epsilon_{\mu\nu\rho\sigma}(\partial_\mu s_\nu \partial_\rho s_\sigma + \frac{1}{3}\partial_\mu t_\nu \partial_\rho t_\sigma)\right]\,. \tag{10}$$

The next step in the bosonization procedure follows from realizing that, as (3) is a change of variables, $\mathcal{Z}$ cannot depend on either θ or α. Thus θ and α can be integrated out without other effect than the introduction of an irrelevant constant factor in $\mathcal{Z}$, which we ignore. Integration over θ and α is equivalent to integration over two flat Abelian vector fields θ_μ and α_μ:

$$\begin{gathered} \theta_\mu = \partial_\mu\theta \quad , \quad \alpha_\mu = \partial_\mu\alpha \\ f_{\mu\nu}(\theta) \equiv \partial_\mu\theta_\nu - \partial_\nu\theta_\mu = 0 \;\; , \;\; f_{\mu\nu}(\alpha) \equiv \partial_\mu\alpha_\nu - \partial_\nu\alpha_\mu = 0\,. \end{gathered} \tag{11}$$

The fermionic action obviously depends on α and θ only through α_μ and θ_μ defined in (11). By an integration by parts, one sees that so does the Jacobian (10). Had the situation been different (i.e., through the dependence of J on α and not only on its gradient α_μ), it would not be possible to pass from a description in terms of α and θ to one in terms of α_μ and θ_μ. To make this explicit, it is convenient to define

$$J(\alpha_\mu; s_\mu, t_\mu) \;=\; \exp\left[-\frac{1}{4\pi^2}\int d^4x\, \alpha_\mu(x)\, \epsilon_{\mu\nu\rho\sigma}(s_\nu\partial_\rho s_\sigma \;+\; \frac{1}{3}t_\nu\partial_\rho t_\sigma)\right] \;. \tag{12}$$

Thus (4) becomes

$$\begin{aligned} \mathcal{Z} \;=\; \int \mathcal{D}\theta_\mu\, \mathcal{D}\alpha_\mu\, \mathcal{D}\bar{\psi}\, \mathcal{D}\psi\; \delta[f_{\mu\nu}(\theta)]\, \delta[f_{\mu\nu}(\alpha)]\; J(\alpha_\mu; s_\mu, t_\mu) \\ \exp\left[-S(\bar{\psi}, \psi; s_\mu + \theta_\mu, t_\mu + \alpha_\mu)\right] \;. \end{aligned} \tag{13}$$

Formally integrating out the fermionic fields and making the shift of variables

$$\theta_\mu \;\to\; \theta_\mu \,-\, s_\mu \quad , \quad \alpha_\mu \;\to\; \alpha_\mu \,-\, t_\mu \;, \tag{14}$$

(13) leads to

$$\begin{aligned} \mathcal{Z}(s_\mu, t_\mu) \;=\; \int \mathcal{D}\theta_\mu\, \mathcal{D}\alpha_\mu\, \delta[f_{\mu\nu}(\theta - s)]\delta[f_{\mu\nu}(\alpha - t)]\; J(\alpha_\mu - t_\mu; s_\mu, t_\mu) \\ \times \det(\not\partial + i \not\theta + i\gamma_5 \not\alpha) \;. \end{aligned} \tag{15}$$

By analogy with the bosonization procedure followed in [5]-[9], we exponentiate the two functional delta functions in (15) using two antisymmetric tensor fields $A_{\mu\nu}$ and $B_{\mu\nu}$ as Lagrange multipliers

$$\begin{aligned} \mathcal{Z}(s_\mu, t_\mu) \;=\; \int \mathcal{D}A_{\mu\nu}\, \mathcal{D}B_{\mu\nu}\, \mathcal{D}\theta_\mu\, \mathcal{D}\alpha_\mu\;\; J(\alpha_\mu - t_\mu; s_\mu, t_\mu) \\ \exp\left(i\int d^4x[\epsilon_{\mu\nu\rho\sigma}A_{\mu\nu}(\partial_\rho\theta_\sigma - \partial_\rho s_\sigma) \;+\; \epsilon_{\mu\nu\rho\sigma}B_{\mu\nu}(\partial_\rho\alpha_\sigma - \partial_\rho t_\sigma)]\right) \\ \times \;\; \det(\not\partial + i \not\theta + i\gamma_5 \not\alpha) \;. \end{aligned} \tag{16}$$

The bosonized form of $\mathcal{Z}$ can then be obtained by integrating out θ_μ and α_μ in (16). This produces a generating functional with the tensor fields $A_{\mu\nu}$ and $B_{\mu\nu}$ as dynamical variables. This step requires the evaluation of the fermionic determinant, which in four dimensions is necessarily non-exact. Before embarking on such calculation, we derive the rules that map the vector and axial-vector currents into functions of the bosonic fields $A_{\mu\nu}$ and $B_{\mu\nu}$. This correspondence requires no approximation and may well be called 'exact'. These rules follow from elementary functional differentiation

$$j_\mu = \langle \bar{\psi}\gamma_\mu\psi\rangle = -i\frac{\delta}{\delta s_\mu}\log \mathcal{Z}|_{s_\mu=0} = -\epsilon_{\mu\nu\rho\sigma}\partial_\nu A_{\rho\sigma} \tag{17}$$

$$j_{5\mu} = \langle \bar{\psi}\gamma_5\gamma_\mu\psi\rangle = -i\frac{\delta}{\delta t_\mu}\log \mathcal{Z}|_{t_\mu=0} = -\epsilon_{\mu\nu\rho\sigma}\partial_\nu B_{\rho\sigma} - \frac{i}{4\pi^2}\,\epsilon_{\mu\nu\rho\sigma}s_\nu\partial_\rho s_\sigma\ . \tag{18}$$

From the antisymmetry of the tensors $A_{\mu\nu}$ and $B_{\mu\nu}$, we are entitled to derive the equations for the divergencies of the currents:

$$\begin{aligned} \partial_\mu j_\mu &= 0 \\ \partial_\mu j^5_\mu &= -\frac{i}{8\pi^2}\,\tilde{F}_{\mu\nu}(s)F_{\mu\nu}(s)\ . \end{aligned} \tag{19}$$

with $\tilde{F}_{\mu\nu} = (1/2)\epsilon_{\mu\nu\alpha\beta}F_{\alpha\beta}$. We then see that the bosonization rule (18) correctly reproduces the axial anomaly.

As stated above, although the bosonization recipe (17)-(18) for associating the fermionic currents with expressions written in terms of bosonic fields is exact, the bosonic action governing the boson field dynamics cannot be evaluated in an exact form in $d > 2$ dimensions. Different approximations for computing the fermionic determinant would yield alternative effective bosonic actions valid in different regimes. We shall describe here the evaluation of the fermionic determinant in (16) to second order in the fields θ_μ and α_μ. The use of this quadratic approximation can be motivated by the same kind of arguments (see in particular the 'quasi-theorem') used in ref. [14].

As usual, it is convenient to work in terms of W, the generating functional of connected Green's functions of the fermionic current

$$\det(\partial\!\!\!/ + i\,\theta\!\!\!/ + i\gamma_5\,\alpha\!\!\!/) \;=\; \exp\left[W(\theta_\mu,\alpha_\mu)\right]\ . \tag{20}$$

The *unregularized* form of the quadratic part of W in (20) becomes

$$W(\theta_\mu,\alpha_\mu) \;\simeq\; \frac{1}{2}\,\mathrm{Tr}\,\left[\partial\!\!\!/^{-1}(\theta\!\!\!/ + \gamma_5\,\alpha\!\!\!/)\;\partial\!\!\!/^{-1}(\theta\!\!\!/ + \gamma_5\,\alpha\!\!\!/)\right]\ . \tag{21}$$

$W(\theta_\mu, \alpha_\mu)$ will consist of three parts

$$W(\theta_\mu, \alpha_\mu) \;=\; W_{\theta\theta}(\theta_\mu, \theta_\mu) \;+\; W_{\alpha\alpha}(\alpha_\mu, \alpha_\mu) \;+\; W_{\theta\alpha}(\theta_\mu, \alpha_\mu) \tag{22}$$

corresponding to the three terms in the quadratic part of W. Before extracting the three parts of W, a regularization should be introduced, since, at it will become clear next, it does have a non-trivial effect. Our choice of a regularization is dictated by the requisite of compatibility with the regularization of the anomalous Jacobian, which we decided to be consistent. A form of assuring consistency is to use Pauli-Villars regularization in a way that treats the vector and axial-vector vertices symmetrically. This amounts to defining the regulated W by

$$W_{reg}(\theta_\mu, \alpha_\mu) \;=\; \frac{1}{2}\sum_{s=0}^{2} C_s \,\mathrm{Tr}\,\left[(\not{\partial} + M_s)^{-1}\,(\not{\theta} + \gamma_5\ \not{\alpha})(\not{\partial} + M_s)^{-1}\,(\not{\theta} + \gamma_5\ \not{\alpha})\right] \tag{23}$$

where

$$\begin{array}{c} C_0 \;=\; 1 \quad C_1 \;=\; 1 \quad C_2 \;=\; -2 \\ M_0^2 \;=\; \mu^2 \quad M_1^2 \;=\; 2\Lambda^2 - \mu^2 \quad M_2^2 \;=\; \Lambda^2 \end{array} \tag{24}$$

where Λ is the cutoff and we have introduced an IR regulator μ that gives the gauge field a small mass inside the loop. The regulated version of (22) becomes

$$W_{reg}(\theta_\mu, \alpha_\mu) \;=\; W_{reg}^{\theta\theta}(\theta_\mu, \theta_\mu) \;+\; W_{reg}^{\alpha\alpha}(\alpha_\mu, \alpha_\mu) \;+\; W_{reg}^{\theta\alpha}(\theta_\mu, \alpha_\mu) \tag{25}$$

with

$$W_{reg}^{\theta\theta}(\theta_\mu, \theta_\mu) \;=\; \frac{1}{2}\sum_{s=0}^{2} C_s \mathrm{Tr}\,\left[(\not{\partial} + M_s)^{-1}\ \not{\theta}\,(\not{\partial} + M_s)^{-1}\ \not{\theta}\right] \tag{26}$$

$$W_{reg}^{\alpha\alpha}(\alpha_\mu, \alpha_\mu) \;=\; \frac{1}{2}\sum_{s=0}^{2} C_s \mathrm{Tr}\,\left[(\not{\partial} + M_s)^{-1}\ \not{\alpha}\,(\not{\partial} - M_s)^{-1}\ \not{\alpha}\right] \tag{27}$$

for the first two terms. Knowledge of the mixed term involving θ_μ and α_μ is not necessary, since because of parity violation and Lorentz invariance its form is restricted to be

$$W_{reg}^{\theta\alpha}(\theta_\mu, \alpha_\mu) \;=\; \int d^4x \int d^4y\; \epsilon_{\mu\nu\rho\sigma}\, \partial_\mu\theta_\nu(x) H(x-y) \partial_\rho\alpha_\sigma(y) \tag{28}$$

which is a total derivative, and can thus be safely ignored (at least in this Abelian case we are dealing with). We then consider the terms (26) and (27). The first one is evidently equivalent to the quadratic part of the effective action for a Pauli-Villars regulated fermionic field in the presence of an external vector field θ_μ. Setting the renormalization conditions at zero momentum, we have for this renormalized two-point function

$$W^{\theta\theta}_{reg}(\theta_\mu, \theta_\mu) \;=\; \frac{1}{2}\int d^4x d^4y\, \theta_\mu(x)\delta^{\perp}_{\mu\nu}F(x-y)\theta_\nu(y) \tag{29}$$

where $\delta^{\perp}_{\mu\nu}$ is the transverse δ

$$\delta^{\perp}_{\mu\nu} \;=\; \delta_{\mu\nu} \;-\; \partial_\mu\partial^{-2}\partial_\nu \tag{30}$$

and

$$F(x-y) \;=\; \int \frac{d^4k}{(2\pi)^4}\, e^{ik\cdot(x-y)}\tilde{F}(k)$$

$$\tilde{F}(k) \;=\; \frac{1}{2\pi^2}\int_0^1 dx\, x(1-x)\, \log\left[1 + x(1-x)\frac{k^2}{\mu^2}\right] \;. \tag{31}$$

As it will become evident at the end, to impose a renormalization condition on $W^{\theta\theta}$ is equivalent to set *the same* renormalization condition on the (one-particle irreducible) current-current correlation function. Of course, the current is the vector one for θ_μ and the axial-vector one for α.

Regarding the term (27), we may rearrange it in such a way that it shows a part which is functionally identical to (26), but with α_μ as argument, plus a remainder $\Delta W^{\alpha\alpha}_{reg}$

$$W^{\alpha\alpha}_{reg}(\alpha_\mu, \alpha_\mu) \;=\; W^{\theta\theta}_{reg}(\alpha_\mu, \alpha_\mu) \;+\; \Delta W^{\alpha\alpha}_{reg} \tag{32}$$

where

$$\Delta W^{\alpha\alpha}_{reg} \;=\; \sum_{s=0}^{s=2} C_s M_s^2\, \mathrm{Tr}\left[(-\partial^2 + M_s^2)^{-1}\, \not{\alpha}(-\partial^2 + M_s^2)^{-1}\, \not{\alpha}\right] \;. \tag{33}$$

Of course, the renormalized term $W^{\theta\theta}(\alpha_\mu, \alpha_\mu)$ will be given by

$$W^{\theta\theta}(\alpha_\mu, \alpha_\mu) \;=\; \frac{1}{2}\int d^4x d^4y\, \alpha_\mu(x)\delta^{\perp}_{\mu\nu}F(x-y)\alpha_\nu(y) \tag{34}$$

with F as in (31).

A straightforward evaluation shows that $\Delta W^{\alpha\alpha}_{reg}$ is given by

$$\Delta W^{\alpha\alpha}_{reg} \;=\; -\frac{\Lambda^2}{(2\pi)^2} \int d^4x \, \alpha_\mu(x)\alpha_\mu(x) \; . \tag{35}$$

The meaning of this quadratic divergence is that the consistent regularization violates axial gauge invariance, as expected. The renormalization of this divergence requires the introduction of a mass counterterm, and also the choice of renormalization conditions for $W^{\alpha\alpha}$ which should say which is the value of the renormalized mass for α. Thus the renormalized $\Delta W^{\alpha\alpha}$ will correspond to a finite mass term:

$$\Delta W^{\alpha\alpha} \;=\; -\frac{m^2}{2} \int d^4x \, \alpha_\mu(x)\alpha_\mu(x)$$

$$= \; -\frac{m^2}{2} \int d^4x \left[\alpha_\mu(x)\delta^{\perp}_{\mu\nu}\alpha_\mu(x) \,+\, \alpha_\mu(x)\delta^{\parallel}_{\mu\nu}\alpha_\mu(x) \right] \tag{36}$$

where $\delta^{\parallel}_{\mu\nu} = \partial_\mu \partial^{-2} \partial_\nu$, and m is the renormalized mass. Of course we can set the value of the renormalized mass to zero to this order. If α_μ is absolutely non-dynamical, it is possible to set the renormalized mass equal to zero to all orders, since the only correction to the mass term is produced by the term we are considering. If the field were dynamical, the anomalous behaviour of the axial symmetry would have spoiled the masslessness of α to higher orders. From the previous results it follows that we can write the renormalized functional W as follows:

$$W(\theta_\mu, \alpha_\mu) \;=\; -\frac{1}{2} \int d^4x d^4y \, \Big[\theta_\mu(x)\delta^{\perp}_{\mu\nu}F(x-y)\theta_\nu(y)$$

$$+ \, \alpha_\mu(x)\delta^{\perp}_{\mu\nu}G(x-y)\alpha_\nu(y) \,+\, m^2 \, \alpha_\mu(x)\delta^{\parallel}_{\mu\nu}\delta(x-y)\alpha_\nu(y) \Big] \tag{37}$$

where

$$G(x-y) \;=\; F(x-y) \,+\, m^2 \delta(x-y) \; . \tag{38}$$

Inserting (37) into (16), we see that the functional integral is Gaussian with respect to θ_μ and α_μ:

$$\mathcal{Z}(s_\mu, t_\mu) \;=\; \int \mathcal{D}A_{\mu\nu} \, \mathcal{D}B_{\mu\nu} \, \mathcal{D}\theta_\mu \, \mathcal{D}\alpha_\mu$$

$$\exp\left[-\frac{1}{4\pi^2}\int d^4x\,(\alpha_\mu - t_\mu)\,\epsilon_{\mu\nu\rho\sigma}(s_\nu\partial_\rho s_\sigma \;+\; \frac{1}{3}t_\nu\partial_\rho t_\sigma)\right]$$
$$\exp\left\{i\int d^4x[\epsilon_{\mu\nu\rho\sigma}A_{\mu\nu}(\partial_\rho\theta_\sigma - \partial_\rho s_\sigma) \;+\; \epsilon_{\mu\nu\rho\sigma}B_{\mu\nu}(\partial_\rho\alpha_\sigma - \partial_\rho t_\sigma)]\right\}$$
$$\exp\left\{-\frac{1}{2}\int d^4xd^4y\,[\theta_\mu(x)\delta^{\perp}_{\mu\nu}F(x-y)\theta_\nu(y)\right.$$
$$\left. + \alpha_\mu(x)\delta^{\perp}_{\mu\nu}G(x-y)\alpha_\nu(y) \;+\; m^2\,\alpha_\mu(x)\delta^{\parallel}_{\mu\nu}\delta(x-y)\alpha_\nu(y)]\right\}\;. \qquad (39)$$

Performing the Gaussian integration over θ_μ and α_μ in (39) we get

$$\mathcal{Z}(s_\mu, t_\mu) \;=\; \exp[\mathcal{C}(s_\mu, t_\mu)]\int \mathcal{D}A_{\mu\nu}\,\mathcal{D}B_{\mu\nu}\,\times$$
$$\exp\left\{-i\int d^4x[s_\mu\epsilon_{\mu\nu\rho\sigma}\partial_\nu A_{\rho\sigma} + t_\mu(\epsilon_{\mu\nu\rho\sigma}\partial_\nu B_{\rho\sigma} + \frac{i}{4\pi^2}\epsilon_{\mu\nu\rho\sigma}s_\nu\partial_\rho s_\sigma)]\right\}\times$$
$$\exp\left\{-\frac{1}{3}\int d^4xd^4y[A_{\mu\nu\rho}(x)F^{-1}(x-y)A_{\mu\nu\rho}(y)+\right.$$
$$\left.B_{\mu\nu\rho}(x)G^{-1}(x-y)B_{\mu\nu\rho}(y)]\right\}\times\exp\left\{-\frac{i}{4\pi^2}\int d^4xd^4y\partial_\mu B_{\nu\rho}(x)\times\right.$$
$$\left.G^{-1}(x-y)\delta_{\mu\nu\rho,\alpha\beta\gamma}(s_\alpha\partial_\beta s_\gamma + \frac{1}{3}t_\alpha\partial_\beta t_\gamma)\right\} \qquad (40)$$

where

$$\begin{aligned} A_{\mu\nu\rho} &= \partial_\mu A_{\nu\rho} + \partial_\nu A_{\rho\mu} + \partial_\rho A_{\mu\nu} \\ B_{\mu\nu\rho} &= \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu} \\ \delta_{\mu\nu\rho,\alpha\beta\gamma} &= \det\begin{pmatrix} \delta_{\mu\alpha} & \delta_{\mu\beta} & \delta_{\mu\gamma} \\ \delta_{\nu\alpha} & \delta_{\nu\beta} & \delta_{\nu\gamma} \\ \delta_{\rho\alpha} & \delta_{\rho\beta} & \delta_{\rho\gamma} \end{pmatrix} \end{aligned} \qquad (41)$$

and

$$\mathcal{C}(s_\mu, t_\mu) \;=\; \frac{1}{2(2\pi)^4}\int d^4xd^4y\,\left\{[s_\mu(x)\partial_\nu s_\lambda(x) \;+\; \frac{1}{3}t_\mu(x)\partial_\nu t_\lambda(x)]\right.$$
$$\delta_{\mu\nu\rho,\alpha\beta\gamma}G^{-1}(x-y)[s_\alpha(y)\partial_\beta s_\gamma(y) \;+\; \frac{1}{3}t_\alpha(y)\partial_\beta t_\gamma(y)]$$
$$+\frac{1}{2(2\pi)^4}\int d^4xd^4y\,\mathcal{G}(x)\partial^{-2}G^{-1}(x-y)\mathcal{G}(y)$$
$$\left.+\frac{1}{2m^2(2\pi)^4}\int d^4xd^4y\,\mathcal{G}(x)\partial^{-2}(x-y)\mathcal{G}(y)\right\} \qquad (42)$$

where $\mathcal{G} = \epsilon_{\mu\nu\rho\lambda}(\partial_\mu s_\nu\partial_\rho s_\lambda \;+\; \frac{1}{3}\partial_\mu t_\nu\partial_\rho t_\lambda)$.

Summary and Conclusions

We have applied the bosonization technique developed in ref's [5-9] to the case of massless Dirac fermions in four dimensions in the presence of both vector and axial-vector sources. This has allowed us to find the bosonization rules for both fermionic currents, eqs.(17)-(18), in terms of Kalb-Ramond bosonic fields. While the bosonization rule for the vector current can be written in a natural and compact form, reminiscent of the well-known two-dimensional bosonization rule,

$$\bar{\psi}\gamma_\mu\psi \to -\epsilon_{\mu\nu\rho\sigma}\partial_\nu A_{\rho\sigma}\ , \tag{43}$$

the result for the axial current is more involved and includes the vector source

$$\bar{\psi}\gamma_5\gamma_\mu\psi \to -\epsilon_{\mu\nu\rho\sigma}\partial_\nu B_{\rho\sigma} - \frac{i}{4\pi^2}\,\epsilon_{\mu\nu\rho\sigma}s_\nu\partial_\rho s_\sigma\ . \tag{44}$$

In our approach, this is a consequence of the anomalous behaviour of the fermionic measure under axial gauge transformations and in this way the bosonic form of the axial current correctly yields its anomalous divergence. We also mention the possibility of considering the particular case of a purely chiral external source ($s_\mu \equiv \pm t_\mu$), and obtaining a bosonized version for this model. The Kalb-Ramond field then corresponds to a particular 'chiral' combination of A and B, namely $A_{\mu\nu} \pm B_{\mu\nu}$.

As stressed above, recipes (43) and (44) can be considered exact apart from the fact that if one is to work in the bosonic version one has to use an approximate expression for the bosonic action. The one we proposed is based in a quadratic approximation and leads to the bosonic generating functional presented in eqs.(40)-(42).

It should be noted that, besides playing an important role in the bosonization rule for the axial current, the chiral Jacobian also affects the actual form of the bosonized action, being the cause of the existence of non-quadratic terms in the currents, and of the coupling between the field strength for the Kalb-Ramond field B and the currents. This situation may be contrasted with the one of having just a vector current, where all the non-quadratic terms disappear if the fermionic determinant is evaluated up to second order in the fields, as we did. These non-quadratic terms are a signal of the anomalous Ward identity linking the divergence of the axial current and two vector currents. In spite of the fact that we haven't included the triangle diagram in our approximate evaluation of the fermionic determinant, partial information from it has shown up from the Jacobian whose calculation implies the knowledge of the exact axial anomaly. It is interesting to note that, as it happens for the complete $d = 2$ bosonization recipe, the axial anomaly determines basic properties of $d = 4$ bosonization (For odd dimensional spaces it is the parity anomaly which seems to play a similar role [5]-[7]).

Finally, a comment about the choice of the actual value of the renormalized 'mass term' for the axial source s_μ: If the source is not dynamical, there is no propagator associated to it and, needless to say, its natural renormalized value is zero, since this value will not be modified by any other higher order correction (they would require diagrams with internal s_μ lines). When the source is dynamical, the actual value of the renormalized mass is arbitrary and becomes a new quantity to be 'measured'. A model with dynamics for s_μ is not necessarily anomalous, one may consider a system of many fermionic species, with their charges adjusted in order to cancel non-trivially the anomaly. It should be emphasized that if the vector source is not dynamical, the regularization can be chosen differently since the criterion of preserving the conservation of the vector current at the quantum level no longer applies.

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