Gauge Invariance and Finite Temperature Effective Actions of Chern-Simons Gauge Theories with Fermions

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We discuss the behavior of theories of fermions coupled to Chern-Simons gauge fields with a non-abelian gauge group in three dimensions and at finite temperature. Using non-perturbative arguments and gauge invariance, and in contradiction with perturbative results, we show that the coefficient of the Chern-Simons term of the effective actions for the gauge fields at finite temperature can be *at most* an integer function of the temperature. This is in a sense a generalized no-renormalization theorem. We also discuss the case of abelian theories and give indications that a similar condition should hold there too. We discuss consequences of our results to the thermodynamics of anyon superfluids and fractional quantum Hall systems.

PACS numbers: 11.10.Kk,11.10.Wx,11.15.-q,11.30.Er, 73.40.Hm,74.20.Kk

In the past decade a significant amount of effort has been devoted to study the behavior of field theories of matter coupled to Chern-Simons gauge fields. At zero temperature, pure Chern-Simons gauge theories are topological field theories which by now are well understood [1]. Much less is known about theories in which the gauge fields are coupled to matter. For the most part, these theories have been treated in perturbation theory or in cases in which the gauge fields are non-dynamical background fields. These results have shown that the fluctuations of a massive Fermi field induces a Chern-Simons term in the effective action of the gauge fields. This is the celebrated Parity Anomaly of Deser, Jackiw and Templeton [2]. The result of these studies is that the Chern-Simons coupling constant (often referred to as the topological mass) is equal to $\theta/4\pi$, where the parameter θ is an integer equal to the number of fermion species [3,2]. It was also found [4] that this one-loop result does not get renormalized by higher order loop corrections. For non-abelian pure Chern-Simons theories topological arguments have shown [2] that for a theory on a compact manifold the coefficient of the Chern-Simons action must be quantized, both at the classical and at the quantum level.

Non-relativistic versions of these theories have also been investigated. It has been established that these theories have superfluid ground states and have given a concrete model of anyon superconductivity. It was found that the superfluidity of the ground state is ensured by the exact cancellation of the coefficient of the one-loop induced Chern-Simons term against the bare Chern-Simons coupling constant which sets the fractional statistics [5,6]. Theories of non-relativistic matter coupled to Chern-Simons fields are commonly used in the study of the physics of the two dimensional electron gas in strong magnetic fields and provide a natural theoretical framework for the Fractional Quantum Hall effect [7–9]. Here too, the non-renormalization of the induced Chern-Simons terms in the effective action of the gauge fields is fundamental for the theory to be consistent with the requirements of Galilean invariance and for the Hall conductance to be determined by the filling fraction of the system.

Much less is understood at finite temperatures. Perturbative calculations, for both relativistic and non-relativistic theories, abelian and non-abelian, have in almost all cases, yielded induced actions with Chern-Simons coefficients which are smooth functions of the temperature [10]- [21]. However, Pisarski [13] has argued that the exact answer should be a constant, independent of the temperature. It should be emphasized that, in all these calculations, the Chern-Simons gauge fields were taken to be non-dynamical background fields.

There are significant physical reasons to suspect that an induced action with Chern-Simons coefficients which are smooth functions of the temperature cannot possibly be the right answer for the full theory. For non-abelian theories it is hard to believe that the topological arguments that lead to the exact quantization of the Chern-Simons coefficient at zero temperature could not be extended to finite temperature (even though the manifold is no longer a sphere). For abelian theories, the exact cancellation between the induced and bare Chern-Simons terms, required for anyon superfluidity to work, would be violated at finite temperature if this results would hold literally. In fact, several authors [22,23] have advocated a picture in which anyon superfluidity sort of “evaporates” at any non-zero temperature. A priori, on general grounds, one expects that an anyon superfluid should undergo a Kosterlitz-Thouless type transition at a non-zero critical temperature rather than an immediate destruction for all $T > 0$. A loophole in the perturbative approach is hinted by the following argument. The fermion excitations which renormalize the induced Chern-Simons coefficient in a temperature dependent fashion are gauge non-invariant states which cannot be part of the physical spectrum and, as such, they should not be included in the partition function. Furthermore, in the case of the anyon superfluid, these states have a logarithmically divergent self-energy. Thus, their weight in the partition function should vanish. Naturally, fermion-antifermion pairs are allowed finite energy excitations whose energy grows logarithmically with the pair size. In a sense, the fermion should be viewed as a vortex (as in the Kosterlitz-Thouless theory) and the unbinding of these pairs should trigger the actual phase transitions. However, the removal of these gauge non-invariant states from the partition function can only be achieved by including explicitly the fluctuations of the gauge fields and it is a non-perturbative effect. Thus, the mechanism which makes sure that the Chern-Simons coefficient is an integer is also responsible for a finite temperature phase transition.

In this paper we reexamine the properties of the effective action of the gauge fields at finite temperature and its consistency with the requirement of gauge invariance. We show that effective actions with smooth, temperature-dependent renormalizations of the Chern-Simons coupling constant are inconsistent with gauge invariance. We show that these coupling constants can be, at most, integer functions of the temperature. For the case of non-abelian theories, we show that the presence of configurations of gauge transformations with non-trivial winding number forces the quantization of the coefficients of the induced and bare Chern-Simons terms *at all temperatures*. Although, as they stand, our results apply directly only to non-abelian theories, we claim that the quantization of the induced term at finite temperature should apply for abelian theories as well.

We start from the three-dimensional (Euclidean) action

$$S = \int d^3x \bar{\psi}(i\partial\!\!\!/ + m + A\!\!\!/)\psi + \frac{\theta}{4\pi} S_{CS}[A] \tag{1}$$

where $S_{CS}[A]$ is the Chern-Simons action

$$S_{CS}[A] = i\epsilon_{\mu\nu\lambda}\mathrm{tr}\int d^3x (F_{\mu\nu}A_\lambda - \frac{2}{3}A_\mu A_\nu A_\lambda). \tag{2}$$

for a gauge field A_μ taking values in the Lie algebra of some gauge group G,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \tag{3}$$

As it is well known, the non-Abelian Chern-Simons action, is not gauge invariant but changes under large gauge transformations [2],

$$A_\mu \to A_\mu^g = g^{-1}A_\mu g + g^{-1}\partial g \tag{4}$$

$$S_{CS}[A] \to S_{CS}[A^g] = S_{CS}[A] + 8\pi^2 i w[g] \tag{5}$$

$$w[g] = \frac{1}{24\pi^2}\int d^3x \epsilon^{\mu\nu\alpha} g^{-1}\partial_\mu g g^{-1}\partial_\nu g g^{-1}\partial_\alpha g \tag{6}$$

Finite temperature calculations are carried out as usual by compactifying the (Euclidean) time variable into the range $0 \le \tau \le \beta = 1/T$ (in our units, $\hbar = c = k = 1$). Concerning bosonic (fermionic) fields, periodic (antiperiodic) boundary conditions (in time) have to be used. Moreover, we shall assume that the allowed gauge transformations are periodic in time. Compactifying space-time, the resulting manifold is $S^2 \times S^1$ and one can then prove [24] that for compact gauge groups, $w[g]$ is the winding number of g, $w[g] \in Z$. In writing eq.(5) we have dropped a surface term which vanishes provided A_μ vanishes rapidly enough at spatial infinity. From this analysis, we see that θ should be chosen to be an integer if $\exp(-S)$ is to be gauge invariant also for large gauge transformations [2].

The partition function at finite temperature is then defined as

$$\mathcal{Z} = \mathcal{N}(\beta)\int \mathcal{D}\bar{\psi}\mathcal{D}\psi DA_\mu \, \exp\left(-S_\beta\right), \tag{7}$$

where S_β is the action (1) at finite temperature, $\mathcal{N}(\beta)$ is a temperature dependent normalization constant and, as stated above, one integrates over gauge fields and fermion fields satisfying periodic and antiperiodic (in time) boundary conditions respectively. Integrating the fermions out we are left with

$$\mathcal{Z} = \mathcal{N}(\beta)\int DA_\mu \, \exp(-\frac{\theta}{4\pi}S_{CS}[A]) \times \det(i\partial\!\!\!/ + m + A\!\!\!/) \tag{8}$$

It is important to stress that the gauge field integration in (8) ranges over all (periodic) gauge field configurations. One can make this explicit by means of the Faddeev-Popov procedure [25] which consists in writing the measure DA_μ in the form

$$DA_\mu = DA_\mu \delta[F[A^g]]\Delta_{FP}[A]Dg \tag{9}$$

where Δ_{FP} is the Faddeev-Popov determinant associated with the $F[A] = 0$ condition and Dg is the usual Haar measure giving the volume element on the group of gauge transformations. After inserting (9) in (8) and changing $A^g \to A$, $g \to g^{-1}$, one gets

$$\mathcal{Z} = \mathcal{N}(\beta)\int DA_\mu \delta[F[A]]\Delta_{FP}[A]Dg \; \exp(-\frac{\theta}{4\pi}S_{CS}[A^g]) \times \det(i\partial\!\!\!/ + m + A\!\!\!/^g) \tag{10}$$

It will be useful to define an effective action $S_{eff}[A]$ in the form

$$\exp(-S_{eff}[A]) = \int Dg \ \exp(- \frac{\theta}{4\pi} S_{CS}[A^g]) \times \det(i\partial\!\!\!/ + m + A\!\!\!/^g) \quad (11)$$

so that

$$\mathcal{Z} = \mathcal{N}(\beta) \int DA_\mu \delta[F[A]] \Delta_{FP}[A] \exp(-S_{eff}[A]) \quad (12)$$

If the Chern-Simons action and the fermion determinant were gauge invariant, the group integration in eq. (11) would be trivial and the effective action would be the one that is usually expected.

One can easily check that the effective action (11) is gauge invariant. Indeed,

$$\exp(-S_{eff}[A^h]) = \int Dg \ \exp(- \frac{\theta}{4\pi} S_{CS}[A^{hg}]) \times \det(i\partial\!\!\!/ + m + A\!\!\!/^{hg}) \quad (13)$$

or, after making the change of variables $g \to h^{-1}g$

$$\exp(-S_{eff}[A^h]) = \exp(-S_{eff}[A]) \quad (14)$$

This, of course, follows from the fact that the (properly regularized) fermion determinant satisfies a natural condition of consistency with the group property of the gauge transformations. Indeed, defining

$$\exp(i\alpha[A,g]) = \det(i\partial\!\!\!/ + m + A\!\!\!/^g)/\det(i\partial\!\!\!/ + m + A\!\!\!/) \quad (15)$$

one has the 1-cocycle condition

$$\delta\alpha = \alpha[A^g, h] - \alpha[A, gh] + \alpha[A, g] = 0 \quad (16)$$

This property is at the root of consistent quantization of anomalous gauge theories since it ensures the gauge invariance of the effective action [26]- [28]. Moreover, similar arguments about gauge invariance can be used for proving that monopole contributions are wiped out, at zero temperature, from the QED_3 generating functional [29]- [30].

The arguments presented above ensure also in the present case that the *exact* effective action is gauge invariant. The exact expression for the fermion determinant in three space-time dimensions is not known. However, there is an extensive literature on the approximate form of this determinant at finite temperature. Perturbative arguments suggest that the anomalous part of the fermion determinant has the form [10]- [21]

$$\det(i\partial\!\!\!/ + m + A\!\!\!/) = \exp(-\frac{1}{4\pi} F[T] S_{CS}[A]) \quad (17)$$

Almost without exception, the results on this subject, based in perturbation theory, yield a function $F[T]$ which is a function of the temperature

$$F[T] = \frac{1}{2} \tanh(\frac{m\beta}{2}) \quad (18)$$

This result holds for both abelian and non-abelian theories. In contrast, Pisarski [13], using a non-perturbative formal argument claimed that, for a non-abelian theory, this coefficient is actually temperature independent and equal to its value at zero temperature.

We shall now discuss the consistency of the ansatz (17) with the requirement of gauge invariance. Thus, we will assume that eq.(17) holds and write

$$\exp(-S_{eff}[A]) = \int Dg \exp\left(-\frac{1}{4\pi}(\theta + F[T]) S_{CS}[A^g]\right) \quad (19)$$

or, after (5)

$$\exp(-S_{eff}[A]) = \exp\left(-\frac{1}{4\pi}(\theta + F[T]) S_{CS}[A]\right) \times \int Dg \exp\left(-2\pi i(\theta + F[T]) w[g]\right) \quad (20)$$

We can now split up the integral over g in sectors $g^{(n)}$ according to $w[g^{(n)}] = n$

$$\exp(-S_{eff}[A]) = \exp\left(-\frac{1}{4\pi}(\theta + F[T]) S_{CS}[A]\right) \times \sum_{n=-\infty}^{\infty} \int Dg^{(n)} \exp\left(2\pi i n(\theta + F[T])\right) \quad (21)$$

One can easily see that the Haar measure $Dg^{(n)}$ is the same for all topological sectors, labelled by the winding number n, so that the integral for each n gives the same factor (the volume of the gauge group $V[G]$). Hence one ends up with

$$\exp(-S_{eff}[A]) = V[G] \exp\left(-\frac{1}{4\pi}(\theta + F[T]) S_{CS}[A]\right) \times \sum_{n=-\infty}^{\infty} \exp\left(2\pi i n(\theta + F[T])\right) \quad (22)$$

We recognize in the last factor a representation of the (periodic) delta function

$$\exp(-S_{eff}[A]) = \exp\left(-\frac{1}{4\pi}(\theta + F[T]) S_{CS}[A]\right) \times \sum_{k=-\infty}^{\infty} \delta\left(\theta + F[T] - k\right) \quad (23)$$

Hence, the partition function vanishes unless the following constraint is satisfied

$$F[T] + \theta = 0 \ \ [\text{mod} \ \ k] \quad (24)$$

This is the main result in our work. It states that $F[T]$, the coefficient of the Chern-Simons term induced by the fermion integration, must be an integer valued function of the temperature since the model should be consistent even for $\theta = 0$. This, in turn, implies that in general θ should be, as it is the case at $T = 0$, an integer.

A similar line of argument can be followed for the case of an *abelian* gauge group. However, in this case there is no topological invariant analogous to $w[g]$. Thus, this line of reasoning does not yield any directly useful information for abelian theories. However, as we will see below, the non-abelian result has potentially far reaching implications even for the abelian case.

It should be noted that perturbative approaches leading to the temperature dependence described by eq.(18) (both in the Abelian and non-Abelian cases) are all based on a $\not{p}/m$ expansion. Now, in the $m \to \infty$ limit, the coefficent of the Chern-Simons term as given by (18) tends to a step function, up to exponentially small corrections, which precisely satisfies the necessary condition (24).

A number of important conclusions can be drawn from our results. They imply that the effective action for *dynamical* gauge fields at finite temperature *cannot* contain a Chern-Simons term with a coefficient which is a smooth function of the temperature. For the case of non-abelian theories the Chern-Simons coefficient is quantized at zero temperature. Our results show that it must also be an integer at any non-zero temperature. Notice that our argument does not exclude the possibility of a temperature dependence of this coefficient. It states that *at most* it must be an integer-valued function of the temperature. Consequently, it can change only by integers at different ranges of temperature. Thus, up to integer shifts, the Chern-Simons coupling constant remains unrenormalized. Therefore we conclude that the perturbative approaches which yield Chern-Simons coefficients with a smooth temperature dependence, are *inconsistent* with the requirement of gauge invariance. Since the same perturbative approaches yield the same result also for abelian theories, we strongly suspect that they must also fail in the abelian case in spite of the fact that our arguments do not give any useful information for abelian theories. Thus our results give a strong hint that, quite generally, the Chern-Simons coefficient does not get smoothly renormalized at non-zero temperature.

DC and GLR are members of CONICET (Argentina) and FAS is an Investigador CICBA (Argentina). This work was supported in part by the National Science Foundation through the grant NSF DMR-94-24511 at the University of Illinois at Urbana Champaign (EF), by CICBA and CONICET (FAS), by the NSF-CONICET International Cooperation Program through the grant NSF-INT-8902032 and by Fundación Antorchas through the grant A-13218/1-153. F. S. would like to thank C. Fosco for useful comments. E. F. thanks the Universidad de La Plata for its kind hospitality.

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