Loop Corrections and Bosonization Formulae.

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Abstract

We study the functional integrals that appear in a path integral bosonization procedure for more than two spacetime dimensions. Since they are not in general exactly solvable, their evaluation by a suitable loop expansion would be a natural procedure, even if the exact fermionic determinant were known. The outcome of our study is that we can consistently ignore loop corrections in the functional integral defining the bosonized action, if the same is done for the functional integral corresponding to the bosonic representation of the generating functional. If contributions up to some order l in the number of loops are included in both integrals, all but the lowest terms cancel out in the final result for the generating functional.

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I. INTRODUCTION.

The very useful property that the configurations of a given physical system can be equivalently described by different sets of variables has one of its more extreme manifestations in the bosonization procedure. This was originally created to deal with some two-dimensional models, which could be described in terms of either fermionic or bosonic variables. Bosonization turned out to be not just a curiosity but also a very useful tool indeed to understand and in some cases even to solve non-trivial interacting Quantum Field Theory models.

It is interesting to remark that there is no theoretical obstacle to the extension of this procedure to higher dimensions. Indeed, there has recently been some progress in the application, although in an approximated form, of the bosonization procedure to theories in more than two dimensions [1]- [10], dealing with both the Abelian and the non-Abelian cases.

The essential difficulty which makes this extended bosonization procedure non-exact is our inability to compute exactly a fermionic determinant in more than two spacetime dimensions. There is however another problem which seems to call for additional approximations, even if the fermionic determinant were exactly known. This is the fact that in order to obtain the bosonized action one has to calculate a functional Fourier transformation of the fermionic determinant. As the latter is in general a non-local and/or non-polynomial function, it is in general impossible to calculate that functional integral exactly, except for simple situations (like the quadratic approximation for the fermionic effective action). It is also possible to find the bosonized action for some non-trivial situations, like the non-Abelian case in three dimensions in a derivative expansion, by taking advantage of an underlying BRST symmetry. This symmetry however, is not powerful enough in the Abelian case as to allow us to obtain the bosonized action [11].

One would expect on intuitive grounds that, once the fermionic determinant is known, there should not be any physically relevant loop correction to perform. This is indeed what we will demonstrate below, namely, no loop corrections are necessary beyond the (one-loop) calculation involved in the fermionic determinant. This does not mean that one *must not*

calculate them, but rather that one can consistently ignore them in the integrals over the bosonic fields without affecting the exactness of the final result for the current correlation functions in the bosonic approach. Moreover, we shall show that if loops corrections are included, they do cancel in the final result.

The structure of this paper is as follows: In section 2 we explain the mechanism to evaluate the integrals in a 'minimal' way, understanding by that that the minimum number (i.e., zero) of loops has to be included in the path integrals over bosonic fields. We introduce the factors of $\hbar$ in order to combine what is of the same order, and to separate what is irrelevant to the physically meaningful results. Then we explain how do the loop corrections cancel (in non-minimal approaches) and extend our results to higher dimensional spaces. Section 3 presents the application of the previous results to a particular example, which consists of the Abelian case with the (log of the) fermionic determinant evaluated up to quartic order in the external field. We show explicitly the cancellation of one-loop diagrams. Section 4 contains an independent justification of the 'minimal' or 'classical' approach, and its application to the Abelian and non-Abelian cases.

II. CANCELLATION OF LOOP CORRECTIONS.

For the sake of simplicity, we shall be first concerned with the Abelian case in $2 + 1$ dimensions. Different cases will be considered afterwards. Our starting point is the bosonic form for $\mathcal{Z}(s)$, the generating functional of connected current correlation functions

$$\mathcal{Z}(s) \;=\; \int \mathcal{D}A_\mu \, \exp\left[-S_{bos}(A) - i\int d^3x \epsilon_{\mu\nu\lambda} s_\mu \partial_\nu A_\lambda\right] \tag{1}$$

where s_μ is an external source, A_μ is a bosonic gauge field, and $S_{bos}(A)$ is the 'bosonic action', a functional of A_μ that encodes all the fermionic current correlation functions in the bosonic description. It is defined by a sort of functional Fourier transformation of $Z(b) = \exp[-W(b)]$:

$$\exp[-S_{bos}(A)] \;=\; \int \mathcal{D}b_\mu \, \exp\left[-W(b) + i\int d^3x \epsilon_{\mu\nu\lambda} b_\mu \partial_\nu A_\lambda\right] \; . \tag{2}$$

$W(b)$ is the generating functional of *connected* current correlation functions. On the other hand, in the fermionic representation $\mathcal{Z}(s)$ is expressed by the functional integral

$$\mathcal{Z}(s) \;=\; \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \, \exp\left[-\int d^3x \bar{\psi}(\slashed{\partial} + M + i\,\slashed{s})\psi \right] \;. \tag{3}$$

The connected correlation functions of the current $j_\mu(x) = \bar{\psi}(x)\gamma_\mu\psi(x)$ in the presence of the external source s_μ are

$$\langle j_{\mu_1}(x_1)\cdots j_{\mu_n}(x_n)\rangle_{conn} \;=\; \frac{\delta}{i\delta s_{\mu_1}(x_1)} \cdots \frac{\delta}{i\delta s_{\mu_n}(x_n)} W(s) \;. \tag{4}$$

In more than two spacetime dimensions, we do not know the exact expression for $\mathcal{Z}(b)$ (and hence for $W(b)$). Even in the hypothetical case of knowing the exact form of the fermionic determinant, we should then have to confront the (perhaps more cumbersome) task of calculating the functional integral over the auxiliary field b_μ in (2). And having thus obtained (either exactly, or in some approximation) that functional integral, one should then use the resulting bosonic action S_{bos} in (1) in order to calculate correlation functions in the bosonic version of the theory. This is again in general a non-trivial functional integral, where there seems to be no hope for an exact evaluation.

It is the purpose of this section to clarify some issues related to the evaluation of the functional integrals over the fields b_μ and A_μ, appearing in equations (2) and (1), respectively. In particular, by keeping track of the dependence on $\hbar$, we show that there is a consistency requisite for the approximations done in the evaluation of those integrals, namely, that both should be evaluated up to the same order in the number of loops. This admits the 'minimal' solution of using for both integrals the 'tree' approximation.

If we reintroduce the dependence on $\hbar$ in the fermionic version (3) of the generating functional of the full current correlation functions, we see that it should be rewritten as

$$\mathcal{Z}(s) \;=\; \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \, \exp\left[-\frac{1}{\hbar} \int d^3x \bar{\psi}(\slashed{\partial} + M + i\,\slashed{s})\psi \right] \;. \tag{5}$$

This means that, for a given diagram, each fermionic line will have a factor $\hbar$ attached, while each vertex will introduce a factor of $\hbar^{-1}$. As the functional integral (5) only contains

one-loop diagrams, with an equal number of fermion lines and vertices, all the factors of $\hbar$ cancel out. Thus $\mathscr{Z}$ is independent of $\hbar$. When $\hbar$ is reintroduced into the game, one also modifies the relation between W and $\mathscr{Z}$

$$\frac{1}{\hbar}W(b) = -\log Z(b) , \tag{6}$$

so that W has the dimensions of an action. As, by counting powers of $\hbar$, we have seen that $\mathscr{Z}$ is independent of $\hbar$, we can use (6) to make the statement that $W(b)$ is of order $\hbar$. Namely,

$$W(b) = \hbar \, \mathcal{W}(b) \tag{7}$$

where $\mathcal{W}$ is independent of $\hbar$. Of course we are not saying more than the well-known fact that a 1PI diagram with L loops carries a factor of $\hbar^{L-1}$ (and $L = 1$ in our case).

Now we deal with the functional integral over b_μ in (2). From the previous review, we see that after reintroducing $\hbar$, the proper expression for that integral is

$$\exp[-\frac{1}{\hbar}S_{bos}(A)] = \int \mathcal{D}b_\mu \, \exp\left[-\frac{1}{\hbar}W(b) + i \int d^3x \, \epsilon_{\mu\nu\lambda} b_\mu \partial_\nu A_\lambda\right] . \tag{8}$$

We now note that we have in (8) a functional integral where $W(b)$ plays the role of an 'action' for the field b_μ. By contrast with the usual situation, the 'action' $W(b)$ is of order $\hbar$. At this point, we should decide up to which order in $\hbar$ we will work. We shall first develop what we call the 'minimal' solution, namely, the lowest approximation which however yields an exact result for the correlation functions. Non minimal solutions will be discussed afterwards.

In this approximation, we want to know the result of the functional integral (8) up to order $\hbar$ (which is indeed the order of W). As $W(b)$ is already of order $\hbar$, we only need to use the 'classical' approximation, namely,

$$\exp[-\frac{1}{\hbar}S_{bos}(A)] \simeq \exp\left\{-\frac{1}{\hbar}W[\hat{b}(A)] + i \int d^3x \, \epsilon_{\mu\nu\lambda} \hat{b}_\mu(A)\partial_\nu A_\lambda\right\} \equiv \exp[-\frac{1}{\hbar}S_{bos}^{cl}(A)] , \tag{9}$$

where $\hat{b}_\mu(A)$ is the solution to the equation

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$$\frac{\delta}{\delta b_\mu(x)} W(\hat{b}) - i\epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda(x) = 0 . \tag{10}$$

It is clear that the 'classical' bosonic action S_{bos}^{cl} is of order $\hbar$. One should however not confuse our procedure with a saddle point evaluation of the integral (which wouldn't be in place here). It is more simple than that: In the functional integral (8), W plays the role of an 'action' for the field b_μ, $i\epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda$ is a 'source' for b, and S_{bos} is the would be generating functional of connected correlation functions of b (except for a trivial operator acting on each leg). We know on general grounds that S_{bos} is related to the corresponding effective action by a simple Legendre transformation. To order $\hbar$, this effective action is just the 'action' $W(b)$, and the Legendre transformation is accomplished by equations (9) and (10).

Inserting (10) into (1),

$$\mathcal{Z}(s) \simeq \int \mathcal{D}A_\mu \, \exp\left[-\frac{1}{\hbar}S_{bos}^{cl}(A) - i\int d^3x\epsilon_{\mu\nu\lambda}s_\mu\partial_\nu A_\lambda\right] \tag{11}$$

where it becomes evident that, as far as order $\hbar$ results are concerned, we can once more calculate the integral in the tree level approximation. This is tantamount to evaluating the exponent at its extreme value

$$\mathcal{Z}(s) \simeq \exp\left[-\frac{1}{\hbar}S_{bos}^{cl}(\hat{A}(s)) - i\int d^3x\epsilon_{\mu\nu\lambda}s_\mu\partial_\nu \hat{A}_\lambda(s)\right] \tag{12}$$

where $\hat{A}(s)$ satisfies

$$\frac{\delta}{\delta A_\mu(x)} S_{bos}^{cl}(\hat{A}(s)) + i\epsilon_{\mu\nu\lambda}\partial_\nu s_\lambda = 0 . \tag{13}$$

On the other hand, we can now check whether the two-step approximation we have done, consisting of deriving first a bosonized action valid to order $\hbar$ and then using this action to calculate $\mathcal{Z}(s)$ to the same order makes sense. We recall that $\mathcal{Z}(s)$ in (12) can be written as $\mathcal{Z}(s) = \exp[-\frac{1}{\hbar}W(s)]$, where W is of order $\hbar$. But we have calculated the same object to the same order. Thus (12) must be exact. Indeed, this is so by virtue of the simple fact that what we have done in this two-step procedure is nothing more than an iterated Legendre transformation on W, which is exactly involutive, and thus comes back to W after

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performing it twice. The first Legendre transformation starts from $W(b)$ and goes to $S^{cl}_{bos}(A)$ (where A is simply related to the derivative of W with respect to b); and the second one starts from $S^{cl}_{bos}(A)$ and goes to a function of s (related to the derivative of S with respect to A), which because of the involutive property is

$$S^{cl}_{bos}(\hat{A}(s)) + i \int d^3x \epsilon_{\mu\nu\lambda} s_\mu \partial_\nu \hat{A}_\lambda(s) = W(s) . \tag{14}$$

It is worth remarking that neither the functional integral over b_μ, nor the one over A_μ has been exactly calculated, but we can approximate them in a synchronized way as to preserve the result for the fermionic determinant. Of course one might try to evaluate both integrals exactly, but in the end no improvement upon the previous results would be obtained. Moreover, our 'minimal' approach assures that the only true 'quantum' corrections come from the fermionic determinant, and no extra loops have to be computed if that object has already been calculated. Note also that we avoid in this way a potentially dangerous situation: Assume that we have for the exact fermionic determinant vertices which are non-renormalizable. If loops have to be calculated one should face the problem of making sense of a non-renormalizable theory, whereas this problem does not arise in our approach. Of course there are also practical advantages, since we just have to calculate tree diagrams in order to obtain the bosonized action, and since the latter has to be used in the tree approximation, *this is in fact the effective (1PI) action* .

We shall now elaborate upon the problem of including loops and how do they cancel when both integrals are calculated.

It should come as no surprise that some cancellation between the loops corresponding to the two integrals should occur, if one realizes that the input of the bosonization procedure is $W(b)$, a one-loop object, and the outcome is a bosonic representation of the same object: $W(s)$.

It is straightforward to put that cancellation in a more evident fashion. We start from the integral over b_μ of equation (2), where we perform a shift from b_μ to β_μ:

$$b_\mu = \hat{b}_\mu(A) + \beta_\mu \tag{15}$$

where $\hat{b}(A)$ depends on A in a generally complicated way since it is given by (10). Then we can write

$$\exp[-S_{bos}(A)] = \exp[-\hat{S}_{bos}(A) - \sigma(A)] \tag{16}$$

where

$$\exp[-\sigma(A)] = \int \mathcal{D}\beta \, \exp\left\{-[W(\hat{b}_A + \beta) - W(\hat{b}_A)] + i \int d^3x \beta_\mu \epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda\right\} \tag{17}$$

where $\hat{S}(A) \equiv S_{bos}^{cl}(A)$, as given by (9). We note that there is no linear term in β if the exponent on the rhs of (16) is expanded in powers of β. Analogously, we shift now the integration variable A_μ in the integral yielding the generating functional in the bosonic representation. Now the shift is defined by

$$A = \hat{A}_s + \alpha \tag{18}$$

where $\hat{A}_s$ depends on the source s_μ, and is determined by (13). This leads to the exact relation

$$e^{-W(s)} = exp\left[-\hat{S}_{bos}(\hat{A}_s) - i \int d^3x s_\mu \epsilon_{\mu\nu\lambda} \partial_\nu \hat{A}_\lambda\right]$$

$$\int \mathcal{D}\alpha \, \exp\left\{-[\hat{S}_{bos}(\hat{A}_s + \alpha) - \hat{S}_{bos}(\hat{A}_s)] - \sigma(\hat{A}_s + \alpha) - i \int d^3x \alpha_\mu \epsilon_{\mu\nu\lambda} \partial_\nu s_\lambda\right\} . \tag{19}$$

On the other hand, we know by the involution of the Legendre transformation that

$$exp\left[-\hat{S}_{bos}(\hat{A}_s) - i \int d^3x s_\mu \epsilon_{\mu\nu\lambda} \partial_\nu \hat{A}_\lambda\right] = \exp[-W(s)] , \tag{20}$$

thus we conclude that the integrals over the fluctuations α and β satisfy [1] the relation

$$\int \mathcal{D}\alpha \, \exp\left\{-[\hat{S}_{bos}(\hat{A}_s + \alpha) - \hat{S}_{bos}(\hat{A}_s)] - \sigma(\hat{A}_s + \alpha) - i \int d^3x \alpha_\mu \epsilon_{\mu\nu\lambda} \partial_\nu s_\lambda\right\} = 1 , \tag{21}$$

and this is the identity which, if expanded in loops, shows the order by order cancellation.

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[1]Note that σ is defined through an integration over β in (17).

Let us consider an extension of this three dimensional Abelian case to higher dimensional spaces. In the d-dimensional case, the Lagrange multiplier is a Kalb-Ramond field with $d-2$ indices, the partition functional reads,

$$Z[s] = \int DA_{\mu_1\ldots\mu_{d-2}} \, Db_\mu e^{(-W[b]+\frac{i}{2}\int d^3x A(f[b]-f[s]))} \tag{22}$$

where we have used the notation

$$A(f[b]-f[s]) = \epsilon_{\mu_1\ldots\mu_d}(f_{\mu_1\mu_2}[b] - f_{\mu_1\mu_2}[s])A_{\mu_3\ldots\mu_d} \tag{23}$$

As we have done before, we shift

$$b_\mu = \hat{b}_\mu[A] + \beta_\mu \tag{24}$$

with $\hat{b}_\mu[A]$ determined from the analogous of equation (10).

$$\frac{\delta}{\delta b_{\mu_1}(x)}W(\hat{b}) - i\epsilon_{\mu_1\ldots\mu_d}\partial_{\mu_2}A_{\mu_3\ldots\mu_d}(x) = 0 \, . \tag{25}$$

and then change the variables in the Kalb-Ramond field,

$$A_{\mu_3\ldots\mu_d} = \hat{A}_{\mu_3\ldots\mu_d}[s] + \alpha_{\mu_3\ldots\mu_d} \tag{26}$$

with $\hat{A}[s]$ obtained from

$$\frac{\delta}{\delta A_{\mu_3\ldots\mu_d}(x)}S_{cl}^{bos}(\hat{A}) - i\epsilon_{\mu_1\ldots\mu_d}\partial_{\mu_1}s_{\mu_2}(x) = 0 \, . \tag{27}$$

It is straightforward to obtain analogous expressions to eqs.(20)-(21), which shows the order by order cancellation. We show explicitly in the example of the next section how does this cancellation (in $d=3$) works at the one-loop order.

III. APPLICATION TO THE ABELIAN CASE IN THE QUARTIC APPROXIMATION.

We shall apply here the minimal approach to the construction of the bosonized action $S_{bos}(A)$ for the case of a massive fermionic field in 3 Euclidean dimensions, with the assumption that $W(b) \; (= -\log\det(\not{\partial}+i\not{b}+M))$ has been evaluated up to order 4 in the external

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field b_μ. This is a non-trivial addition to the already studied case of the quadratic [9] approximation, where the problem we are now dealing with was absent, since the integrals were Gaussian. Moreover, the results we will obtain may be relevant not only to the bosonization of the four-current correlation function, but also for the case of the two-point function in the presence of an external source. It also shows clearly the interplay between the approximation of retaining terms with up to four $b's$ and the minimal approximation for the integral over that field.

To begin with, we note that the most general form (in coordinate space) for the functional $W(b)$ in the case at hand is

$$W(b) = \frac{1}{2} \int d^3x_1 d^3x_2 \, W^{(2)}_{\mu_1\mu_2}(x_1, x_2) b_{\mu_1}(x_1) b_{\mu_2}(x_2)$$

$$+ \frac{1}{4!} \int d^3x_1 d^3x_2 d^3x_3 d^3x_4 \, W^{(4)}_{\mu_1\mu_2\mu_3\mu_4}(x_1, x_2, x_3, x_4) b_{\mu_1}(x_1) b_{\mu_2}(x_2) b_{\mu_3}(x_3) b_{\mu_4}(x_4) \,, \qquad (28)$$

where both $W^{(2)}_{\mu_1\mu_2}(x_1, x_2)$ and $W^{(4)}_{\mu_1\mu_2\mu_3\mu_4}(x_1, x_2, x_3, x_4)$ are symmetrical functions under a simultaneous permutation of their space arguments and indices. Note that the term linear in b vanishes, as usual, and the possible term with three b's is absent because we are dealing with the Abelian case [2]. In order to find the bosonized action, we have first to solve equation (10), which in terms of expansion (28) may be written in the following form

$$b_\mu(x) = i \int d^3y_1 G_{\mu\nu_1}(x, y_1) f_{\nu_1}(y_1)$$

$$- \frac{1}{3!} \int d^3y_1 d^3y_2 d^3y_3 d^3y_4 \, G_{\mu\nu_1}(x, y_1) W^{(4)}_{\nu_1\nu_2\nu_3\nu_4}(y_1, y_2, y_3, y_4) b_{\nu_2}(y_2) b_{\nu_3}(y_3) b_{\nu_4}(y_4) \qquad (29)$$

where $f_\mu(x) = \epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda$, and $G_{\mu\nu}(x, y) = [W^{(2)}]^{-1}_{\mu\nu}(x, y)$. We have on purpose stopped the expansion up to order three in the field A_μ. The reason is that when this expansion is inserted into the expression for the bosonic action (9), higher order terms would give for the bosonic action terms with more than four A's, which would correspond to correlation

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[2]We should need an $W^{(3)}$ with the properties of being symmetric, transverse, and parity-violating. This cannot be built in three dimensions.

functions of more than four currents. On the other hand, we only know W in the quartic approximation, so the inclusion of those higher order terms would give unreliable results. The bosonized action that follows from expansion (29) is

$$S_{bos}^{cl} = \frac{1}{2} \int d^3x_1 d^3x_2 [W^{(2)}]_{\mu_1\mu_2}^{-1}(x_1, x_2) f_{\mu_1}(x_1) f_{\mu_2}(x_2)$$

$$+ \int d^3x_1 d^3x_2 d^3x_3 d^3x_4 d^3y_1 d^3y_2 d^3y_3 d^3y_4 W_{\mu_1\mu_2\mu_3\mu_4}^{(4)}(x_1, x_2, x_3, x_4)$$

$$[W^{(2)}]_{\mu_1\nu_1}^{-1}(x_1, y_1)[W^{(2)}]_{\mu_2\nu_2}^{-1}(x_2, y_2)[W^{(2)}]_{\mu_3\nu_3}^{-1}(x_3, y_3)[W^{(2)}]_{\mu_4\nu_4}^{-1}(x_4, y_4)$$

$$f_{\nu_1}(y_1) f_{\nu_2}(y_2) f_{\nu_3}(y_3) f_{\nu_4}(y_4) \tag{30}$$

which contains only terms up to order four in A, as promised. It should become evident from the previous equation why the previous known results using the quadratic approximation for W yielded the exact result for the two-current correlation function: In the quadratic approximation no loops are possible, and the result for the bosonized action is just the first term on the rhs of (30). Moreover, we can also affirm that the result (30) yields the exact four-current correlation function.

As a consistency check, we use the classical approximation to evaluate the functional integral over A_μ. Solving equation (13) for S_{bos} found in (30) to determine A_μ as a function of the source s_μ (up to order three in the source) yields

$$f_\mu(x) = -i \int d^3y_1 W_{\mu\nu_1}^{(2)}(x, y_1) s_{\nu_1}(y_1)$$

$$- \frac{i}{3!} \int d^3y_1 d^3y_2 d^3y_3 W_{\mu\nu_1\nu_2\nu_3}^{(4)}(x, y_1, y_2, y_3) s_{\nu_1}(y_1) s_{\nu_2}(y_2) s_{\nu_3}(y_3) . \tag{31}$$

And inserting this in the rhs of (12) produces the result

$$\mathcal{Z}(s) = e^{-W(s)} \tag{32}$$

with

$$W(s) = \frac{1}{2} \int d^3x_1 d^3x_2 W_{\mu_1\mu_2}^{(2)}(x_1, x_2) s_{\mu_1}(x_1) s_{\mu_2}(x_2)$$

$$+ \frac{1}{4!} \int d^3x_1 d^3x_2 d^3x_3 d^3x_4 W_{\mu_1\mu_2\mu_3\mu_4}^{(4)}(x_1, x_2, x_3, x_4) s_{\mu_1}(x_1) s_{\mu_2}(x_2) s_{\mu_3}(x_3) s_{\mu_4}(x_4) , \tag{33}$$

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which is exactly equal (but now as a function of s rather than b) to the original assumption (28) for the fermionic determinant in the presence of the external field. This confirms our statement that the procedure we have followed introduces no errors in the final answer beyond the ones involved in the approximation of the determinant.

We now show the meaning of the cancellation of one-loop diagrams, as an illustration of the general result presented in the previous section. We shall of course take into account that now we are dealing with an approximation to the exact determinant because we have truncated the expansion at the quartic term, so that the cancellation will show up to this order. It is a straightforward exercise to show that, if (21) is expanded to one-loop order, we obtain the relation

$$\det[\frac{\delta^2 \hat{S}(\hat{A}_s)}{\delta A_\mu(x)\delta A_\nu(y)}] \ \det[\frac{\delta^2 W(\hat{b}_{\hat{A}_s})}{\delta b_\mu(x)\delta b_\nu(y)}] \ = \ 1 \ . \tag{34}$$

¿From the previous example, we can of course extract the values of the two functional derivatives. It is simpler for the case of the derivatives with respect to A_μ to consider the derivatives with respect to $f_\mu = \epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda$. At the end, the determinants will only differ in the determinant of a field independent operator.

$$\frac{\delta^2 \hat{S}(\hat{A}_s)}{\delta f_{\mu_1}(x_1)\delta f_{\mu_2}(x_2)} \ = \ [W^{(2)}]^{-1}_{\mu_1\mu_2}(x_1, x_2) - \frac{1}{2}\int d^3z_1 d^3z_2 d^3y_1 d^3y_2 [W^{(2)}]^{-1}_{\mu_1\rho_1}(x_1, z_1)[W^{(2)}]^{-1}_{\mu_2\rho_2}(x_2, z_2)$$

$$W^{(4)}_{\rho_1\rho_2\nu_1\nu_2}(z_1, z_2, y_1, y_2)s_{\nu_1}(y_1)s_{\nu_2}(y_2)$$

$$\frac{\delta^2 W(\hat{b}_{\hat{A}_s})}{\delta b_{\mu_1}(x_1)\delta b_{\mu_2}(x_2)} \ = \ W^{(2)}_{\mu\nu}(x_1, x_2) \ + \ \frac{1}{2}\int d^3y_1 d^3y_2 W^{(4)}_{\mu_1\mu_2\nu_1\nu_2}(x_1, x_2, y_1, y_2)s_{\nu_1}(y_1)s_{\nu_2}(y_2) \ .$$

$$\tag{35}$$

The evaluation of these determinant up to the order we are dealing with yields

$$\log\det[\frac{\delta^2 \hat{S}(\hat{A}_s)}{\delta A_{\mu_1}(x_1)\delta A_{\mu_2}(x_2)}] \ =$$

$$-\frac{1}{2}\text{Tr}\left\{\int d^3z_1 d^3y_1 d^3y_2 [W^{(2)}]^{-1}_{\mu_1\rho_1}(x_1, z_1)W^{(4)}_{\rho_1\mu_2\nu_1\nu_2}(z_1, x_2, y_1, y_2)s_{\nu_1}(y_1)s_{\nu_2}(y_2)\right\}$$

$$\log\det[\frac{\delta^2 W(\hat{b}_{\hat{A}_s})}{\delta b_{\mu_1}(x_1)\delta b_{\mu_2}(x_2)}] \ =$$

$$+ \frac{1}{2}\mathrm{Tr}\left\{\int d^3z_1 d^3y_1 d^3y_2 [W^{(2)}]^{-1}_{\mu_1\rho_1}(x_1,z_1) W^{(4)}_{\rho_1\mu_2\nu_1\nu_2}(z_1,x_2,y_1,y_2) s_{\nu_1}(y_1) s_{\nu_2}(y_2)\right\} \qquad (36)$$

where the cancellation becomes evident.

IV. A 'CLASSICAL' APPROACH TO BOSONIZATION

A. Abelian Case

In section II it was shown that the relation between the bosonic Action for A_μ, and the corresponding one for the b_μ field is given by

$$S_{bos}[A] = W[b] - i\int d^3x \epsilon_{\mu\nu\alpha} b_\mu \partial_\nu A_\alpha$$

where the b_μ field has to be evaluated over the solution to the equation

$$\frac{\delta W[b]}{\delta b_\mu} - i\epsilon_{\mu\nu\alpha}\partial_\nu A_\alpha = 0 . \qquad (37)$$

One possible way to mimic this construction would be to start from the generating functional of connected current correlation functions, $W[s]$, which is, of course, independent of the 'dynamical' field b_μ. We see that due to the gauge invariance of $W[s]$ under transformations of s, we can write

$$W[s_\mu + b_\mu]|_{F_{\mu\nu}[b]=0} = W[s_\mu] \qquad (38)$$

We can represent the zero curvature condition using a Lagrange multiplier field A_μ and rewrite eq.(38),

$$W[s] = \mathcal{W}[s,A,b]|_{A,b} \qquad (39)$$

where we have defined,

$$\mathcal{W}[s,A,b] = W[s+b] - i\int d^3x \epsilon_{\mu\nu\alpha} A_\mu \partial_\nu b_\alpha. \qquad (40)$$

Eq.(39) means that $W = \mathcal{W}$ when the fields A, b are eliminated by using their equation of motion. Indeed, if the equation for A is used first,

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$$\frac{\delta W[s]}{\delta A_\mu} = 0 \rightarrow \epsilon_{\mu\nu\alpha}\partial_\nu b_\alpha = 0 \qquad (41)$$

we see that b can only be a pure gradient, gauge invariance of W is used to prove (39). If, in turn, the equation of motion of b were used first, one would recover the minimal bosonized action of the previous section.

B. Non-Abelian Case

We will proceed along similar lines to those of the previous (Abelian) case. We start by writing the functional $W[s]$ as,

$$W[s] = W[b]\big|_{F_{\mu\nu}[s]=F_{\mu\nu}[b]} \qquad (42)$$

The condition $F_{\mu\nu}[s] = F_{\mu\nu}[b]$ implies, on a particular section, $b_\mu = s_\mu$, as can be seen in [11] It will be useful to rewrite eq (42) defining as in the previous subsection,

$$\mathcal{W}[s, A, b]\big|_{A,b} = W[s]$$

and

$$\mathcal{W}[s, A, b] = W[b] + \frac{i}{2} \int d^3x \epsilon_{\mu\nu\alpha}((A-b)_\mu F_{\nu\alpha}[b] + (s-A)_\mu F_{\nu\alpha}[s] - 2\bar{c}_\mu D_\nu[b]c_\alpha) . \qquad (43)$$

Here we have introduced the fields A_μ transforming as a vector under gauge transformations, $\bar{c}_\mu$ and c_μ a pair of anticommuting ghost, transforming covariantly under gauge transformations. We have introduced these fields in a way reminiscent to that of ref. [11]. The purpose of writing eq.(43) is to have an analogous identity as the one derived in the Abelian case, in a way such that, after equations of motion for the 'dynamical' fields are used, eq.(43) holds.

The ghost term in the previous equation may be thought of as a coming from a (partial) gauge fixing for the topological gauge invariance $b_\mu \rightarrow b_\mu + \epsilon_\mu$ present in $W[s]$, since it is in fact *independent* of b_μ. It is partial because there remains a non-Abelian gauge invariance (of the usual kind), since b_μ is only fixed up to gauge transformations. The Fadeev-Popov like term is the one just needed for the measure of the functional integral to be well-defined,

we add this term in the classical action (43) in a way reminiscent to that in the quantum theory (see ref. [11]). It has to be included if the quantum theories following from different gauge fixings are to be equivalent.

Differentiation of both sides of (43) gives the eqs. of motion for the 'dynamical' fields. Indeed, differentiation with respect to b_μ gives,

$$\frac{\delta W[b]}{\delta b_\mu} - 2i\epsilon_{\mu\nu\alpha}(\frac{1}{2}F_{\nu\alpha}[b] + D_\nu[b](A - b)_\alpha + [\bar{c}_\alpha, c_\nu]) = 0, \tag{44}$$

with respect to A_μ

$$\epsilon_{\mu\nu\alpha}F_{\nu\alpha}[b] = \epsilon_{\mu\nu\alpha}F_{\nu\alpha}[s] \tag{45}$$

and with respect to the ghost fields,

$$\epsilon_{\mu\nu\alpha}D_\nu[b]\bar{c}_\alpha = 0$$

$$\epsilon_{\mu\nu\alpha}D_\nu[b]c_\alpha = 0 \tag{46}$$

Inserting the solution of eqs. (45-44) $b_\mu = s_\mu$, $\bar{c}_\mu = c_\mu = 0$ in eq.(44) we obtain,

$$\frac{\delta W[s]}{\delta s_\mu} - i\epsilon_{\mu\nu\alpha}(1/2F_{\nu\alpha}[s] + D_\nu[s](A - s)_\alpha) = 0, \tag{47}$$

This is the analogous of the equations previously obtained, to which it reduces in the Abelian case.

The outcome of this discussion is that, by construction, one can indeed start from the functional $W[b]$, calculated in some approximation, and define *classically* a bosonized form. The procedure is just to evaluate $\mathcal{W}$ on the equations of motion for b_μ, what allows one to get an expression that depends on A, the ghosts, and the source. Taking derivatives with respect to the source the bosonization rules are derived. As the whole procedure stems from an exact classical relation, it is evident that no quantum corrections are required. However, the derivation of the exact classical bosonized action shall require classical perturbation theory, since the equations that determine this functional are in general non-linear (except when the quadratic approximation for $W[s]$ is used).

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