A TOPOLOGICAL MODEL FOR TWO-DIMENSIONAL GRAVITY COUPLED TO MATTER

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June 1991

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Abstract

Starting from a topological gauge theory in two dimensions with symmetry groups $ISO(2,1)$, $SO(2,1)$ and $SO(1,2)$ we construct a model for gravity with non-trivial coupling to matter. We discuss the equations of motion which are connected to those of previous related models but incorporate matter content. We also discuss the resulting quantum theory and finally present explicit formulæ for topological invariants.

1 INTRODUCTION

Lower dimensional theories of gravity have recently attracted much attention [1]-[11]. In particular, considerable progress has been achieved by exploiting the connection between two and three dimensional gravity models and Topological Quantum Field Theories (TQFTs) [3]-[10]. In this way, it has been proven that general relativity in three dimensions is equivalent to the Chern-Simons theory (with gauge groups $ISO(2,1)$, $SO(3,1)$ or $SO(2,2)$ depending on the value of the cosmological constant). Using this connection, it has been shown that the theory is renormalizable and finite, and that it can be solved exactly [3].

Since there is no Chern-Simons like term in two dimensions, other types of TQFTs have to be employed in an attempt to construct two-dimensional gravity models starting from gauge theories [7]-[10]. It is the purpose of this work to present one of such models based on a two-dimensional TQFT which not only includes gauge fields (with symmetry groups $ISO(1,1)$, $SO(2,1)$ or $SO(1,2)$) but also scalar fields, naturally leading to a description of gravity coupled to matter.

One way in which topological theories (of the Witten type [12]) can be obtained is by quantizing a classical action S_{cl} that corresponds to a topological invariant [13]. In a sense, these classical actions are trivial since, being by essence invariant under arbitrary transformations, all fields can be gauged away at the classical level. At the quantum level, this reflects in the reduction of the solution space from an infinite dimensional to a finite dimensional one. The resulting quantum action is intimately related to instanton configurations carrying the topological charge. In order to have instantons in two-dimensional gauge theories, one necessarily has to add Higgs fields. In the Abelian case these instantons are the time-honoured Nielsen-Olesen vortices carrying a topological charge $Q \in Z$ related to the vortex magnetic flux [14]. Non-Abelian extensions can be constructed and the resulting instantons are again vortex-like configurations. The topological charge is again associated to the magnetic flux and takes the form [15]:

$$Q = \frac{1}{2\pi} \int d^2x \sqrt{g} E^{\mu\nu} < \Psi F_{\mu\nu} > \ . \tag{1.1}$$

Here Ψ is one of the scalar fields in the adjoint representation of the gauge group, $F_{\mu\nu}$ is the gauge field curvature and " $< >$ " represents an adequate inner product.

Starting from an action of the form (1.1) and identifying the gauge field with Zweibein and spin connection fields, a highly non-trivial model for two-dimensional gravity has been constructed by Chamseddine and Wyler [7] (see also [8]-[10]). In this model, there is a scalar field which just plays the rôle of a Lagrange multiplier. Any attempt to add to this topological action kinetic energy terms for Ψ either breaks the covariance of the model or implies the appearence of rather complicated self-interactions which obscure the resulting theory.

There is another possibility for constructing TQFTs put forward by Labastida and Pernici [16] (see also [17]). In their approach, instead of starting from a S_{cl} which is a topological charge, one constructs a gaussian action in which instanton defining equations (Bogomol'nyi equations in the two-dimensional case) have a relevant rôle. All fields enter in this action in a self-dual way (in a sense to be precised in next sections) and hence kinetic terms for scalars appear naturally in a way that does not imply a metric dependence at the quantum level. It is this approach the one we follow in the present work. We think it is the most natural one, especially if one takes into account the central rôle that instanton moduli space plays in TQFTs and the fact that in two-dimensional gauge theories instanton equations have non-trivial solutions only when an appropriate number of scalar fields, with their corresponding kinetic terms, are included.

In this way, we arrive at a model for two-dimensional gravity non-trivially coupled to matter. After reviewing the two-dimensional topological gauge theory in Section 2, we establish the connection with the gravity model in Section 3. We there identify Zweibein and spin connection fields and discuss the resulting classical equations of motion. These equations reduce to the Jackiw-Teitelboim equations [1] when scalars are absent and also include as a particular case Chamseddine-Wyler ones [7]. In Section 4 we discuss the quantum action and its symmetries leaving for Section 5 the evaluation of topological invariants. Finally we present a discussion of the model and the conclusions to our work in Section 6.

2 THE GAUGE MODEL

In this Section we briefly review the non-Abelian two-dimensional topological field theory constructed in ref.[18], which is at the basis of the model for two-dimensional gravity to be presented in Section 3.

The model for a non-Abelian two-dimensional gauge field theory that we consider has been constructed, in the manner of Labastida-Pernici [16], starting from a classical action defined on a general two-dimensional manifold M, in which Bogomol'nyi equations play a central rôle,

$$S_{cl}[M] = \int_M d^2x\sqrt{g} < (H_{\mu\nu} - B_{\mu\nu})^2 + |H_\mu - B_\mu|^2 + |\tilde{H}_\mu - \tilde{B}_\mu|^2 > . \quad (2.1)$$

In this expression, $H_{\mu\nu}$, H_μ and $\tilde{H}_\mu$ are auxiliary fields belonging to the algebra of the gauge group $\mathcal{G}$; $H_{\mu\nu} = H^A_{\mu\nu}T_A$, $H_\mu = H^A_\mu T_A$ and $\tilde{H}_\mu = \tilde{H}^A_\mu T_A$, where T_A, $A = 1, ..., dim\mathcal{G}$, are the group generators. They are self-dual fields in the sense that [18]

$$H^{\mu\nu} = E^{\mu\nu} H \quad (2.2)$$

with H the dual of $H^{\mu\nu}$ and ${}^*H^\mu$ the dual of H_μ satisfying

$${}^*H^\mu \equiv iE^{\mu\nu} H_\nu = H^\mu \quad (2.3)$$

($E^{\mu\nu}$ is the contravariant two-dimensional Levi-Civita tensor, $E^{\mu\nu} = \frac{\epsilon^{\mu\nu}}{\sqrt{g}}$, $\epsilon^{01} = -\epsilon^{10} = 1$). $B_{\mu\nu}$, B_μ and $\tilde{B}_\mu$ stand for the following expressions

$$\begin{aligned} B_{\mu\nu} &\equiv F_{\mu\nu} - e_G E_{\mu\nu}\Psi < \Phi^2 - \Phi_0^2 > , && (2.4)\\ B_\mu &\equiv D^+_\mu \Phi + E^+_{\mu\nu}\left[\Psi, D^\nu \Phi\right] , && (2.5)\\ \tilde{B}_\mu &\equiv D^+_\mu \Psi , && (2.6) \end{aligned}$$

so that

$$\begin{aligned} B_{\mu\nu} &= 0 , && (2.7)\\ B_\mu &= 0 , && (2.8)\\ \tilde{B}_\mu &= 0 && (2.9) \end{aligned}$$

represent the Bogomol'nyi equations corresponding to a two-dimensional non-Abelian gauge theory [19]. These equations have to be supplemented with the constraints

$$< \Psi^2 > \;=\; 1\;, \tag{2.10}$$

$$< \Psi\Phi > \;=\; 0\;. \tag{2.11}$$

(see ref.[19] for details). The dynamical fields of the theory defined by S_{cl} are, then, a gauge field A_μ taking values in the algebra of the gauge group $\mathcal{G}$, $A_\mu = A_\mu^A T_A$, and two scalar fields Ψ and Φ in the adjoint representation of the group $\mathcal{G}$, $\Psi = \psi^A T_A$ and $\Phi = \phi^A T_A$. The field strength $F_{\mu\nu}$ is defined as

$$F_{\mu\nu} = F_{\mu\nu}^A T_A \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + e_G[A_\mu, A_\nu]\;, \tag{2.12}$$

accordingly, the covariant derivative D_μ is defined as

$$D_\mu \equiv \partial_\mu + e_G[A_\mu,\ ]\;. \tag{2.13}$$

The "+" symbol appearing in covariant derivatives of expressions (2.5) and (2.6) is defined as follows. Given a vector C_μ,

$$C_\mu^+ \equiv \frac{1}{2}(C_\mu +^* C_\mu) = \frac{1}{2}(C_\mu + iE_{\mu\nu}C^\nu)\;. \tag{2.14}$$

It is easy to prove that C_μ^+ is a self-dual vector (cf. eq.(2.3)); this property implies the self-duality of the expressions (2.5) and (2.6) and, furthermore, as we shall see, *the independence of the quantum action on the metric* $g_{\mu\nu}$. This quality is at the root of the topological character of the quantum theory constructed from S_{cl} [12]; it means that the partition function constructed from S_{cl} does not depend on the choice of any particular metric, it can only depend on the topology of M.

The "$< >$" symbol denotes the appropriate invariant, non-degenerate and associative inner product. Its explicit definition depends on the choice of the group $\mathcal{G}$ and will be presented, in the cases of interest, in the next Section.

Bogomol'nyi equations (2.7)-(2.9) are first order differential equations whose solutions solve the (second order) Euler-Lagrange equations for a non-Abelian gauge theory coupled to two Higgs fields defined on the two dimensional manifold M [19]. The number of Higgs fields introduced is such that complete symmetry breaking is achieved so as to ensure non-trivial topology for the gauge field A_μ. In general, Bogomol'nyi equations exist whenever a particular relation between coupling constants hold. For instance, for the model to be considered, two scalar fields are necessary and a potential ensuring complete symmetry breaking is

$$V(\Psi, \Phi) = g_1 < (\Phi^2 - \Phi_0^2)^2 > + g_2 < (\Psi^2 - 1)^2 > + g_3 < (\Psi\Phi)^2 > \quad (2.15)$$

but, due to conditions (2.10) and (2.11), only the first term plays a rôle in the model. In this case, Bogomol'nyi condition relates the gauge coupling constant e_G and the potential strenght g_1,

$$e_G^2 = k g_1 \ , \quad (2.16)$$

the numerical constant k being determined by the inner product definition.

The peculiarity of an action like (2.1) is its invariance under the most general transformations of the dynamical fields (in this case A_μ, Ψ and Φ), provided one adequately choose the transformation laws for the auxiliary fields (namely $H_{\mu\nu}$, H_μ and $\tilde{H}_\mu$). Indeed, if one transforms the gauge and scalar fields in the most general form

$$\delta A_\mu = \epsilon_\mu - D_\mu \epsilon \ , \quad (2.17)$$

$$\delta \Phi = \theta - e_G[\Phi, \epsilon] \ , \quad (2.18)$$

$$\delta\Psi = \tilde{\theta} - e_G[\Psi, \epsilon] \tag{2.19}$$

(we have distinguished usual gauge transformations for later convenience), the classical action (2.1) remains unchanged provided

$$\delta H_{\mu\nu} = \delta B_{\mu\nu} + [H_{\mu\nu} - B_{\mu\nu}, \epsilon] \ , \tag{2.20}$$

$$\delta H_{\mu} = \delta B_{\mu} + [H_{\mu} - B_{\mu}, \epsilon] \ , \tag{2.21}$$

$$\delta \tilde{H}_{\mu} = \delta \tilde{B}_{\mu} + [\tilde{H}_{\mu} - \tilde{B}_{\mu}, \epsilon] \ , \tag{2.22}$$

where variations in the right hand side are to be computed in terms of the variations of the dynamical fields (2.17)-(2.19). It is important to note that not all of the parameters are effective regarding these transformations: if one chooses $\epsilon_\mu = D_\mu \xi$, $\epsilon = \xi$, $\theta = e_G[\Phi, \xi]$ and $\tilde{\theta} = e_G[\Psi, \xi]$ not only the action remains invariant but also all fields do it (on shell).

Transformations (2.17)-(2.19) are enough to select a gauge in which $H_{\mu\nu}$, H_μ and $\tilde{H}_\mu$ vanish. This can be achieved with parameters ϵ_μ, θ and $\tilde{\theta}$ and leaving untouched parameter ϵ [18]. In this gauge, the equations of motion of S_{cl} coincide with the Bogomol'nyi equations (2.7)-(2.9).

This symmetry of the classical action is called a "large" or "topological" symmetry and has to be fixed in the process of quantization. The second generation gauge invariance mentioned in a previous paragraph imposes a refined BRST quantization, for instance by using Batalin-Vilkovisky method [20]. This has been done in detail in ref.[18]. Just for completeness let us indicate the main lines of the quantization procedure. A generic term of the classical action (2.1) takes the form

$$S^1_{cl} = \int_M d^2x \sqrt{g} < |H - B[\Phi]|^2 > \ , \tag{2.23}$$

where Φ is a collection of dynamical fields, $B[\Phi]=0$ is the associated Bogomol'nyi equation and H is the corresponding auxiliary field. Evidently, each one of the terms in (2.1) has this form. This action remains invariant under the large transformations

$$\Phi \rightarrow \Phi+\delta\Phi\ , \tag{2.24}$$

$$H \rightarrow H+\frac{\delta B}{\delta\Phi}\delta\Phi\ . \tag{2.25}$$

Associated with transformations (2.24)-(2.25), we can define BRST commutators

$$\{Q,\Phi\} = \chi\ , \tag{2.26}$$

$$\{Q,H\} = \frac{\delta B}{\delta\Phi}\chi\ . \tag{2.27}$$

The linear transformation $\{Q,\ \}$ is defined by stating that the BRST transformation of a functional F is

$$\delta^{BRST}F=\lambda\{Q,F\}\ , \tag{2.28}$$

with λ a Grassmann odd constant parameter. χ represents the ghost related to the symmetry (2.24)-(2.25). Proceeding à la Batalin-Vilkovisky, the quantum action is constructed from S_{cl} as follows

$$S_q^1=S_{cl}+\{Q,F\} \tag{2.29}$$

where F is a "gauge fermion" [20] introduced to fix the large symmetry. It is evident from (2.25) that H can be gauged away; to this end, one chooses

$$F=\int_M d^2x\sqrt{g}<XH>\ , \tag{2.30}$$

being X an antighost field. We impose the following BRST transformation laws on the antighost field X and Lagrange multiplier D

$$\begin{aligned} \{Q, X\} &= D \,, \\ \{Q, D\} &= 0 \,, \end{aligned} \tag{2.31}$$

with this S_q^1 becomes

$$S_q^1 = S_{cl}^1 + \int_M d^2x\sqrt{g} < X\frac{\delta B}{\delta\Phi}\chi + DH > \ . \tag{2.32}$$

The second term in (2.32) corresponds to the ghost action and the third one to a gauge fixing action. Thus, the partition function $\mathcal{Z}$ for a classical action of the kind (2.23) with a large symmetry (2.24)-(2.25) is

$$\begin{aligned} \mathcal{Z} &= \int \mathcal{D}\Phi\, \mathcal{D}H\, \mathcal{D}\chi\, \mathcal{D}X\, \mathcal{D}D \\ &\qquad \exp[-\int_M d^2x\sqrt{g} < |H - B|^2 + X\frac{\delta B}{\delta\Phi}\chi + DH >] \\ &= \int \mathcal{D}\Phi\, \mathcal{D}\chi\, \mathcal{D}X\ \exp[-\int_M d^2x\sqrt{g} < B^2 + X\frac{\delta B}{\delta\Phi}\chi >] \,. \end{aligned} \tag{2.33}$$

The quantization of the classical action under consideration, eq.(2.1), follows the same steps. However, the actual transformations (2.17)-(2.22) are slightly more complicated and hence our arguments have to be generalized to also include, appart from ghosts of the χ-type associated with the large symmetry, ghosts associated with the usual gauge and second generation invariances. In any case, the final form of S_q can be written in the form (2.29) where F is some functional of the original fields (gauge, scalar and auxiliary fields) and new fields (ghosts and Lagrange multipliers) introduced in the gauge fixing procedure. Moreover, it can be shown that there exists a functional V such that S_q can be written as a BRST commutator

$$S_q = \{Q, V\} \,. \tag{2.34}$$

From this equation it is easy to show that

$$\frac{\delta \mathcal{Z}}{\delta g_{\mu\nu}} = 0 \ , \tag{2.35}$$

the defining equation for TQFTs. Furthermore, the partition function is, for similar reasons, independent of the gauge coupling constant e_G, as long as e_G is different from zero. This can be easily demonstrated by going through new field variables in such a way that $\frac{1}{e_G^2}$ is factorized from the quantum action, which remains gauge coupling independent. This property permits to exactly evaluate $\mathcal{Z}$ by going to the small e_G^2 limit where it is dominated by the classical minima, that is, the solutions to Bogomol'nyi equations. In the next Section we shall give an explicit form for S_q [18].

3 THE GRAVITATIONAL MODEL

Let us now construct a model for two-dimensional gravity based on the topological model presented in Section 2. We consider the symmetry groups $ISO(1,1)$, $SO(2,1)$ and $SO(1,2)$, with generators which will be identified with the T_A's introduced in the previous Section. In order to describe the three cases with a sole algebra, we write $(T_A) = (P_a, J)$, $a = 0, 1$,

$$\begin{aligned} [P_a, P_b] &= \Lambda \epsilon_{ab} J \ , \\ [P_a, J] &= -\epsilon_a^{\ b} P_b \ , \\ [J, J] &= 0 \ . \end{aligned}$$

In this context, P_a and J will play the rôle of the generators of the two translations and Lorentz rotation, respectively, on the two-dimensional manifold M. Latin indices are raised and lowered with an internal metric η_{ab}. We shall see that a choice of signature for the η_{ab} will fix the corresponding signature for the metric in our gravity model. With our conventions ϵ^{ab} is such that $\epsilon^{01} = -\epsilon^{10} = 1$.

The constant Λ behaves as a dimensionless cosmological constant[1]; the values $\Lambda = 0$, $\Lambda > 0$ and $\Lambda < 0$ give rise to the $ISO(1,1)$ group (the isometry group of two-dimensional flat Minkowski space-time), the $SO(2,1)$ group (the isometry group of two-dimensional de Sitter space-times) and the $SO(1,2)$ group (the isometry group of two-dimensional anti-de Sitter space-times), respectively.

For the $SO(2,1)$ and $SO(1,2)$ groups we define the inner product by using the Killing metric arising from their algebras; thus,

$$
\begin{aligned}
< P_a, P_b > &= \Lambda \eta_{ab} \ , \\
< P_a, J > &= 0 \ , \\
< J, J > &= 1 \ .
\end{aligned}
\tag{3.1}
$$

We cannot proceed in an analogous way in the case of the $ISO(1,1)$ group because of the degeneracy of its Killing metric. We can, however, overcome this difficulty by defining the following inner product [8]

$$< P_a, P_b > = \eta_{ab} \ , \tag{3.2}$$

$$< P_a, J > = 0 \ , \tag{3.3}$$

$$< J, J > = 1 \ . \tag{3.4}$$

It is not possible, though, to avoid the degeneracy of the Casimir operator which is still taken in the form $W = P^a P_a$.

[1] Dimensionfull magnitudes should be constructed by using the gauge coupling constant e_G which has dimensions of mass.

Since our aim is to make contact with two-dimensional gravity, we introduce the following notation

$$
\begin{aligned}
A_\mu^a &= e_\mu^a \,, && (3.5)\\
A_\mu^3 &= f_\mu \,, && (3.6)
\end{aligned}
$$

attempting to identify two of the vector potential components (those along the "translation directions" P_a) as a Zweibein and to relate the remaining vector potential component (the one along the "Lorentz rotation" direction J) to the spin connection. Then, the covariant derivative D_μ (eq.(2.13)) acting on an algebra valued field $C = (c^a, c)$ becomes

$$
D_\mu C = D_\mu[e,f]C = \mathcal{D}_\mu^{ab}[f] c_b P_a + (\partial_\mu c + e_G \Lambda \, \epsilon_{ab} e_\mu^a c^b) J \,, \tag{3.7}
$$

where $\mathcal{D}_\mu^{ab}[f]$ is given by

$$
\mathcal{D}_\mu^{ab}[f] \equiv \delta^{ab}\partial_\mu - e_G \epsilon^{ab} f_\mu \,. \tag{3.8}
$$

Concerning the Higgs fields, we denote them

$$
\begin{aligned}
\Psi &= (\psi^a, \psi) \,, && (3.9)\\
\Phi &= (\phi^a, \phi) \,, && (3.10)
\end{aligned}
$$

while for auxiliary fields $H_{\mu\nu}$, H_μ and $\tilde{H}_\mu$ and expressions $B_{\mu\nu}$, B_μ and $\tilde{B}_\mu$ we write

$$
\begin{aligned}
H_{\mu\nu} &= (h_{\mu\nu}^a, h_{\mu\nu}) & B_{\mu\nu} &= (b_{\mu\nu}^a, b_{\mu\nu}) \,, \\
H_\mu &= (h_\mu^a, h_\mu) & B_\mu &= (b_\mu^a, b_\mu) \,, \\
\tilde{H}_\mu &= (\tilde{h}_\mu^a, \tilde{h}_\mu) & \tilde{B}_\mu &= (\tilde{b}_\mu^a, \tilde{b}_\mu) \,.
\end{aligned} \tag{3.11}
$$

Then,

$$
\begin{aligned}
b^a_{\mu\nu} &= \mathcal{D}^{ab}_\mu[f]e_{\nu b} - \mathcal{D}^{ab}_\nu[f]e_{\mu b} - e_G E_{\mu\nu}\,\psi^a\,v(\Phi)\ , && (3.12)\\
b_{\mu\nu} &= \partial_\mu f_\nu - \partial_\nu f_\mu + e_G\Lambda E_{\mu\nu} - e_G E_{\mu\nu}\,\psi\,v(\Phi)\ , && (3.13)\\
b^a_\mu &= \Delta^{+a}_\mu(f,\phi_c,\phi) - E^+_{\mu\nu}\epsilon^{ab}\psi\Delta^\nu_b(f,\phi_c,\phi) + \\
&\quad E^+_{\mu\nu}\epsilon^{ab}\,\psi_b\,\partial^\nu\phi\ , && (3.14)\\
b_\mu &= \partial^+_\mu\phi + e_G\Lambda\,\epsilon_{ab}\,e^{+a}_\mu\psi^b + \Lambda\,E^+_{\mu\nu}\epsilon^{ab}\,\psi_a\,\Delta^\nu_b(f,\phi_c,\phi)\ , && (3.15)\\
\tilde{b}^a_\mu &= \Delta^{+a}_\mu(f,\psi_b,\psi)\ , && (3.16)\\
\tilde{b}_\mu &= \partial^+_\mu\psi + e_G\Lambda\,\epsilon_{ab}\,e^{+a}_\mu\,\psi^b\ . && (3.17)
\end{aligned}
$$

where

$$v(\Phi) \equiv < \Phi^2 - \Phi_0^2 >\ , \qquad (3.18)$$

and $\Delta^a_\mu(f,\phi_b,\phi)$ stands for

$$\Delta^a_\mu(f,\phi_b,\phi) \equiv \mathcal{D}^{ab}_\mu[f]\phi_b + e_G\epsilon^{ab}\,e_{\mu b}\,\phi\ . \qquad (3.19)$$

With this notation, the equations of motion of the theory become

$$
\begin{aligned}
h^a_{\mu\nu} &= b^a_{\mu\nu}\ , && (3.20)\\
h_{\mu\nu} &= b_{\mu\nu}\ , && (3.21)\\
h^a_\mu &= b^a_\mu\ , && (3.22)\\
h_\mu &= b_\mu\ , && (3.23)\\
\tilde{h}^a_\mu &= \tilde{b}^a_\mu && (3.24)\\
\tilde{h}_\mu &= \tilde{b}_\mu\ . && (3.25)
\end{aligned}
$$

Similarly, the constraints (2.10) and (2.11) are

$$
\begin{aligned}
\Lambda\psi^a\psi_a + \psi^2 &= 1\ , && (3.26)\\
\Lambda\psi^a\phi_a + \psi\phi &= 0\ , && (3.27)
\end{aligned}
$$

in the cases of the $SO(2,1)$ and $SO(1,2)$ groups and

$$\psi^a\psi_a + \psi^2 = 1 \,, \tag{3.28}$$

$$\psi^a\phi_a + \psi\phi = 0 \,, \tag{3.29}$$

in the case of the $ISO(1,1)$ group.

In order to confirm the identification between the gauge field components A^a_μ and the Zweibein e^a_μ (so as to interpret the topological model presented in Section 2 as a model for two-dimensional gravity) it is convenient at this point to analyse Bogomol'nyi equations which are, in fact, the equations of motion for the topological model with quantum action S_q (2.34) in the small e_G limit. As we stated above, the gauge freedom (see eqs.(2.17)-(2.22)) allows us to gauge away auxiliary fields $H_{\mu\nu}$, H_μ and $\tilde{H}_\mu$ so that the equations of motion (3.20)-(3.25) become the Bogomol'nyi equations. Furthermore, as we explained at the end of the previous Section, $\mathcal{Z}$ is independent of e_G and can be evaluated by taking the limit for which the path integral is dominated by Bogomol'nyi equations solutions.

As we shall show below, the first two equations of our gravity model become equations for torsion and curvature similar to those presented in ref.[1] but with extra terms added to the cosmological constant. The rest of the equations are directly related to the matter content of the system. To see this, let us obtain from eq.(3.20) an explicit expression for f_μ in terms of e^a_μ, ψ^a and Φ under the assumption that e^a_μ is invertible (i.e. there exists e^μ_b such that, $e^a_\mu e^\mu_b = \delta^a_b$ and $e^a_\mu e^\nu_a = \delta^\nu_\mu$). The expression is

$$f_\mu = \frac{1}{e_G} E^{\alpha\beta}(\partial_\alpha e^a_\beta) e_{\mu a} - e^a_\mu \psi_a v(\Phi) \,. \tag{3.30}$$

Using it, eq.(3.21) transforms into

$$\frac{2}{e_G} E^{\mu\nu} \partial_\mu w_\nu - 2E^{\mu\nu} \partial_\mu (e^a_\nu \psi_a v(\Phi)) - 2e_G \, \psi v(\Phi) + 2e_G \Lambda = 0 \tag{3.31}$$

where w_μ is defined as follows

$$w_\mu \equiv E^{\alpha\beta} (\partial_\alpha e^a_\beta) e_{\mu a} \ . \tag{3.32}$$

From eqs.(3.31) and (3.32), one can see that it is consistent to identify e^a_μ with a Zweibein so that the two-dimensional metric $g_{\mu\nu}$ be given by

$$g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab} \ . \tag{3.33}$$

Indeed, in two dimensions the affine spin connection (under the condition of metricity) can be written in the form

$$w^{ab}_\mu = \epsilon^{ab} \Omega_\mu \ . \tag{3.34}$$

If we identify the connection Ω_μ with w_μ as given by eq.(3.32),

$$w^{ab}_\mu = \epsilon^{ab} w_\mu \ , \tag{3.35}$$

the first term in the left hand side of eq.(3.31) becomes proportional to the curvature scalar R,

$$R = 2E^{\mu\nu} \partial_\mu w_\nu \tag{3.36}$$

and the complete equation of motion (3.31) takes the form

$$R + 2e_G^2 \Lambda = 2e_G \, E^{\mu\nu} \partial_\mu (e^a_\nu \psi_a v(\Phi)) - 2e_G^2 \, \psi v(\Phi) \equiv e_G \tau \ . \tag{3.37}$$

Were the scalar field Φ absent, this equation would reduce to

$$R + 2e_G^2 \Lambda = 0 \ . \tag{3.38}$$

This is precisely one of the equations of motion for the Jackiw-Teitelboim model [1] for two-dimensional gravity (also discussed in refs.[7]-[9]). Moreover, the second equation of motion for the Jackiw-Teitelboim model, which gives the vanishing torsion condition, follows immediately from eqs. (3.20) and (3.32):

$$E^{\mu\nu} T^a_{\mu\nu} \equiv E^{\mu\nu} \mathcal{D}_\mu [\frac{1}{e_G} w] e^a_\nu = 0 \ . \tag{3.39}$$

Concerning the new terms induced by scalar fields, they act as an effective energy momentum tensor trace τ. Hence, our topological model can be interpreted as a theory for two-dimensional gravity non-trivially coupled to matter. This has been achieved by using a two-dimensional topological model defined through a classical action given by eq.(2.1). The fact that S_{cl} is constructed from self-dual auxiliary fields allows terms such as $< E^{\mu\nu} B_{\mu\nu} B >$ or $< E^{\mu\nu} H_\nu B_\mu >$ to be present; they induce matter interactions in the sense that there is no dependence on the metric at the quantum level, as it will be demonstrated in the next Section. Had we started from a topological action à la Baulieu-Singer [13] (as in refs.[7]-[9]), we would have faced the problem mentioned by Chamseddine and Wyler [7]: matter interactions would require the introduction of a metric in a non-trivial way (thus imposing non-covariant couplings of the gauge field once it has been identified with the Zweibein) or rather complicated terms.

Let us now study the equations for matter, namely eqs.(3.22)-(3.25). For the sake of clarity we shall distinguish between the $\Lambda = 0$ and the $\Lambda \neq 0$ cases.

A) $\Lambda = 0$.

In this case, from eq.(3.25) we have that ψ is constant

$$\psi = \eta \tag{3.40}$$

and, from eq.(3.24) we can in principle determine the other components of the Ψ field in terms of the Zweibein and the other scalar field Φ:

$$\Delta^a_\mu(w, \psi_b, \eta) - e_G \, \epsilon^{ab} \, e^c_\mu \, \psi_c \psi_b \, v(\Phi) = 0 \, . \tag{3.41}$$

Concerning the Φ field, eq.(3.23) implies that also ϕ is constant

$$\phi = \lambda \, . \tag{3.42}$$

After some calculations, it can be shown that eq.(3.22) reduces to the following pair of equations

$$(\eta^2 - 1) \, [\epsilon^{ab} \psi_a \Delta^\mu_b(f, \phi_b, \lambda)] \;\; = \;\; 0 \, , \tag{3.43}$$

$$(\eta^2 - 1) \, [\psi^a \Delta^\mu_a(f, \phi_b, \lambda)] \;\; = \;\; 0 \, . \tag{3.44}$$

These equations have as one obvious possible solution $\eta = \pm 1$. If this were the case, the constraints reduce to $\psi^a \psi_a = 0$ and $\phi^a \psi_a \pm \lambda = 0$. If the flat metric η^{ab} is Euclidean, the unique solution to the former constraint is $\psi^a \equiv 0$, but this implies, through eq.(3.41), the vanishing of the Zweibein. Then, $\eta \pm 1$ is not a sensible solution in Euclidean space-time. If, on the other hand, the flat metric η^{ab} is Minkowskian, the first constraint has solutions different from zero and further analysis of the complete system is required to find explicit solutions.

For $\eta \neq \pm 1$, the matter system reduces to eq.(3.41) and the two equations stemming from eqs.(3.43) and (3.44), supplemented with the constraints $\psi^a \psi_a + \eta^2 = 1$ and $\psi^a \phi_a + \eta \lambda = 0$, coupling ψ^a, ϕ^a and e^a_μ. This is a coupled non linear system which has to be studied together with the equation (3.37) for the curvature scalar.

B) $\Lambda \neq 0$.

In this case, we can solve ψ^a in terms of ψ and the Zweibein from eq.(3.25)

$$\psi^a = \frac{1}{2e_G\Lambda}\epsilon^{ab}e^\mu_b\,\partial_\mu\psi\ . \tag{3.45}$$

Using this result and the constraint (3.26) we obtain the following equation for ψ

$$\Box\psi + 4e^2_G\,(1-\psi^2)\,v(\Phi) + 4e^2_G\Lambda\,\psi = 0\ . \tag{3.46}$$

Once again, were the scalar field Φ absent we would recover the Klein-Gordon equation in de Sitter space for the model of ref.[7]. The additional term we have corresponds to a self-interaction, highly non-linear and typical of theories with a Higgs potential.

It is still pending the study of the equations (3.22) and (3.23). The analysis of the former is similar to the case $\Lambda = 0$; it just appears one extra term in each of the equations (3.43) and (3.44)

$$(\psi^2-1)\,[\epsilon^{ab}\psi_a\Delta^\mu_b(f,\phi_b,\phi)] - E^{\mu\nu}\partial_\nu\phi] \;=\; 0\ , \tag{3.47}$$

$$(\psi^2-1)\,[\psi^a\Delta^\mu_a(f,\phi_b,\phi) + \psi\partial^\mu\phi] \;=\; 0\ . \tag{3.48}$$

Though $\psi^2 = 1$ is a solution to these equations, recalling eq.(3.46) we see that it solves the complete system only if Λ equals zero. Hence, we have to leave aside $\psi = \pm 1$ and study the vanishing of the brackets in eqs.(3.47) and (3.48), together with eq.(3.46). We arrive at the following equation

$$\partial_\mu\phi(1-\Lambda) - \frac{1}{2}\partial_\mu\psi = 0 \tag{3.49}$$

which distinguishes between Λ equal or different from one. In the former case, ψ must be a constant which implies (through eq.(3.45)) that ψ^a vanishes and then that η must be equal to ± 1, leaving no solutions to the system. In the latter case, one has to select a given manifold M in order to go further. For example, if we take M to be a manifold with boundary, the solution to (3.49) can be written in the form

$$\phi = \frac{1}{2(1-\Lambda)}\psi + (\phi_0 - \frac{\psi_0}{2(1-\Lambda)}) \tag{3.50}$$

where we have imposed $\psi \to \psi_0$ and $\phi \to \phi_0$ at the boundary.

In general, the complete resolution of the full system (3.20)-(3.25) both in the $\Lambda = 0$ and $\Lambda \neq 0$ cases, depends on the topological structure of the two-dimensional manifold M. Two different situation can be envisaged:

1. M is such that there exists a finite number of isolated classical solutions, that is, the "moduli space" $\mathcal{M}$ contains a finite number of points. The dimension of $\mathcal{M}$ is then $d(\mathcal{M}) = 0$.

2. M is such that the moduli space has dimension different from zero, $d(\mathcal{M}) \neq 0$.

We shall come back to this point, in connection with the evaluation of topological invariants, in the next Section.

We summarize in table 1 what we have learnt about the equations of motion and their solutions.

4 SYMMETRIES AND QUANTUM ACTION

4.1 Symmetries of the gravitational model

It is interesting to recover from topological transformation laws (2.17)-(2.19) the usual transformation laws of two-dimensional gravity, viz. diffeomorphism and Lorentz transformations.

Let us start by writing the parameters ϵ_μ, ϵ, θ and $\tilde{\theta}$ appearing in (2.17)-(2.19) in the form

$$\begin{aligned} \epsilon_\mu &= \varepsilon_\mu^a P_a + \varepsilon J \ , \\ \epsilon &= \varepsilon^a P_a + \varepsilon J \ , \\ \theta &= \vartheta^a P_a + \vartheta J \ , \\ \tilde{\theta} &= \tilde{\vartheta}^a P_a + \tilde{\vartheta} J \ . \end{aligned} \tag{4.1}$$

With this, the transformation laws for e_μ^a, f_μ, ϕ^a, ϕ, ψ^a and ψ can be readily recognized to be

$$\delta e_\mu^a = \varepsilon_\mu^a - \partial_\mu \varepsilon^a - e_G\, \epsilon^{ab}(\varepsilon e_{\mu b} - f_\mu \varepsilon_b) \ , \tag{4.2}$$

$$\delta f_\mu = \varepsilon_\mu - \partial_\mu \varepsilon + e_G\, \Lambda \epsilon_{ab}\, \varepsilon^a e_\mu^b \ , \tag{4.3}$$

$$\delta \phi^a = \vartheta^a + e_G\, \epsilon^{ab}\, (\phi \varepsilon_b - \varepsilon \phi_b) \ , \tag{4.4}$$

$$\delta \phi = \vartheta - e_G\, \Lambda \epsilon_{ab} \phi^a \varepsilon^b \ , \tag{4.5}$$

$$\delta \psi^a = \tilde{\vartheta}^a + e_G\, \epsilon^{ab}(\psi \varepsilon_b - \varepsilon \psi_b) \ , \tag{4.6}$$

$$\delta \psi = \tilde{\vartheta} - e_G \Lambda \epsilon_{ab}\, \psi^a \varepsilon^b \ . \tag{4.7}$$

Our first purpose is to compare these transformations with diffeomorphism transformations δ_D

$$\delta_D e_\mu^a = v^\alpha(\partial_\alpha e_\mu^a - \partial_\mu e_\alpha^a) + \partial_\mu(v^\alpha e_\alpha^a) \ , \tag{4.8}$$

$$\delta_D f_\mu = v^\alpha(\partial_\alpha f_\mu - \partial_\mu f_\alpha) + \partial_\mu(v^\alpha f_\alpha) \ , \tag{4.9}$$

$$\delta_D \Phi = v^\alpha \partial_\alpha \Phi \ , \tag{4.10}$$

$$\delta_D\Psi = v^\alpha\partial_\alpha\Psi \; , \tag{4.11}$$

where v^α is the local parameter transforming x^α, $\delta_D x^\alpha = v^\alpha$. In order to find a connection between topological and diffeomorphism transformations, let us consider the following subset of parameters

$$\begin{aligned}
\varepsilon^a &= -\tfrac{1}{e_G} v^\alpha e^a_\alpha \; , \\
\varepsilon &= -\tfrac{1}{e_G} v^\alpha f_\alpha \; , \\
\varepsilon^a_\mu &= v^\alpha E_{\alpha\mu}\psi^a v(\Phi) \; , \\
\varepsilon_\mu &= v^\alpha E_{\alpha\mu}\psi v(\Phi) \; .
\end{aligned} \tag{4.12}$$

With this we find, from (4.2) and (4.8),

$$\delta e^a_\mu - \delta_D e^a_\mu = \frac{1}{e_G} v^\alpha [\mathcal{D}^{ab}_\mu[f] e_{\alpha b} - \mathcal{D}^{ab}_\alpha[f] e_{\mu b} - e_G E_{\mu\alpha}\psi^a v(\Phi)] \tag{4.13}$$

or, using the equation of motion (3.20) [2]

$$\delta e^a_\mu - \delta_D e^a_\mu|_{o.s.} = -\frac{1}{e_G} v^\alpha h^a_{\alpha\mu} \; . \tag{4.14}$$

Concerning f_μ, a similar procedure shows that the difference between large and diffeomorphism transformations is, using equation (3.21),

$$\delta f_\mu - \delta_D f_\mu|_{o.s.} = -\frac{1}{e_G} v^\alpha h_{\alpha\mu} \; . \tag{4.15}$$

With respect to the scalar field Ψ, once the parameters ϵ and ϵ_μ have been fixed, it is simple to prove from eqs.(4.6) and (4.11) the following identity

$$\delta\psi^a - \delta_D\psi^a = \tilde{\vartheta}^a - \frac{1}{e_G} v^\alpha \Delta^a_\alpha(f, \psi_b, \psi) \tag{4.16}$$

then, choosing $\tilde{\vartheta}^a = 0$ and using the equation of motion (3.24) we have

$$\delta\psi^a - \delta_D\psi^a|_{o.s.} = -\frac{1}{e_G} v^\alpha \tilde{h}^a_\alpha \; . \tag{4.17}$$

[2] We represent the use of the equations of motion by $|_{o.s}$.

The analysis for ψ is analogous: choosing $\tilde{\vartheta}=0$ and using eq.(3.25)

$$\delta\psi - \delta_D\psi|_{o.s.} = -\frac{1}{e_G}v^\alpha \tilde{h}_\alpha \ . \tag{4.18}$$

Finally, the difference between variations of the components of the field Φ are

$$\begin{aligned} \delta\phi^a - \delta_D\phi^a &= \vartheta^a - \frac{1}{e_G}v^\alpha \Delta^a_\alpha(f,\phi_b,\phi) \ , & (4.19)\\ \delta\phi - \delta_D\phi &= \vartheta - \frac{1}{e_G}v^\alpha(\partial_\alpha\phi + e_G\Lambda\,\epsilon_{ab}\phi^a e^b_\alpha) \ . & (4.20) \end{aligned}$$

Now, we can select ϑ^a and ϑ in the following way

$$\begin{aligned} \vartheta^a &= \frac{1}{e_G}v^\alpha \Delta^a_\alpha(f,\phi_b,\phi) \ , & (4.21)\\ \vartheta &= \frac{1}{e_G}v^\alpha(\partial_\alpha\phi + e_G\Lambda\,\epsilon_{ab}\phi^a e^b_\alpha) \ , & (4.22) \end{aligned}$$

which implies

$$\begin{aligned} \delta\phi^a - \delta_D\phi^a &= 0 \ , & (4.23)\\ \delta\phi - \delta_D\phi &= 0 \ . & (4.24) \end{aligned}$$

Similarly, we can show that the difference between a topological transformation ((2.20)-(2.22)) and a diffeomorphism transformation for each auxiliary field is proportional to the corresponding auxiliary field.

In summary, working in the gauge in which all auxiliary fields vanish, $\delta H_{\mu\nu} = \delta_D H_{\mu\nu} = 0$, $\delta H_\mu = \delta_D H_\mu = 0$, $\delta\tilde{H}_\mu = \delta_D\tilde{H}_\mu = 0$, we have

$$\begin{aligned} \delta e^a_\mu - \delta_D e^a_\mu|_{o.s} &= 0 \ ,\\ \delta f_\mu - \delta_D f_\mu|_{o.s} &= 0 \ ,\\ \delta\Phi - \delta_D\Phi|_{o.s} &= 0 \ ,\\ \delta\Psi - \delta_D\Psi|_{o.s} &= 0 \ . \end{aligned} \tag{4.25}$$

Concerning Lorentz transformations δ_L, again an appropriate choice of parameters allows their identification with transformations (4.2)-(4.7). Indeed, if we choose

$$\varepsilon^a_\mu = \varepsilon_\mu = \varepsilon^a = \vartheta^a = \vartheta = \tilde{\vartheta}^a = \tilde{\vartheta} = 0 \tag{4.26}$$

and

$$\varepsilon = -\frac{\kappa}{e_G} \tag{4.27}$$

we have

$$\begin{aligned} \delta e^a_\mu &= \kappa \epsilon^{ab} e_{\mu b} \ , \\ \delta f_\mu &= \partial_\mu \kappa \ , \\ \delta \phi^a &= \kappa \epsilon^{ab} \phi_b \ , \\ \delta \psi^a &= \kappa \epsilon^{ab} \psi_b \ , \\ \delta \phi &= 0 \ , \\ \delta \psi &= 0 \ . \end{aligned} \tag{4.28}$$

The right hand side of eqs.(4.28) precisely corresponds to Lorentz transformations δ_L with parameter κ and then,

$$\begin{aligned} \delta e^a_\mu &= \delta_L e^a_\mu \ , \\ \delta f_\mu &= \delta_L f_\mu \ , \\ \delta \Phi &= \delta_L \Phi \ , \\ \delta \Psi &= \delta_L \Psi \end{aligned} \tag{4.29}$$

and $H_{\mu\nu} = \delta_L H_{\mu\nu} = 0$, $\delta H_\mu = \delta_L H_\mu = 0$, $\delta \tilde{H}_\mu = \delta_L \tilde{H}_\mu = 0$ in the gauge in which all auxiliary fields vanish.

We then see from eqs.(4.25) and (4.29) that, as expected, the topological model defined from the classical action (2.1) can be used as a model for two-dimensional gravity with its topological transformations interpreted as diffeomorphism and Lorentz transformations. In order to make such an identification we have restricted the parameter space to a subspace satisfying (4.12), (4.26) and (4.27) relations. In this sense, the whole topological invariance is larger than the usual invariances for gravity.

4.2 The Quantum Action

As explained in Section 2, because of the large topological symmetry (eqs. (2.17)-(2.22)) of the classical action (2.1), one has to proceed to a careful BRST quantization in which ghosts and ghosts for ghosts appear through the process of gauge fixing. We shall skip the details (given in ref.[18] for the gauge theory and sketched in Section 2) and just quote the result for the quantum action

$$\begin{aligned} S_q[M] &= \int_M d^2x\sqrt{g} < B_{\mu\nu}D^{\mu\nu} - \frac{1}{4}D_{\mu\nu}D^{\mu\nu} + B_\mu D^\mu + \\ &\qquad \tilde{B}_\mu \tilde{D}^\mu > + S_{gf}[M] + S_{gh}[M] \ . \end{aligned} \tag{4.30}$$

The explicit expression for the classical part of S_q in gravity language is straightforwardly obtained calculating the adequate inner product and expressing $B_{\mu\nu}$, B_μ and $\tilde{B}_\mu$ components as in eqs.(3.12)-(3.17). In (4.30) auxiliary fields $H_{\mu\nu}$, H_μ and $\tilde{H}_\mu$ have been traded for Lagrange multipliers $D_{\mu\nu} = (d^a_{\mu\nu}, d_{\mu\nu})$, $D_\mu = (d^a_\mu, d_\mu)$ and $\tilde{D}_\mu = (\tilde{d}^a_\mu, \tilde{d}_\mu)$. Of course, the equations of motion arising from this classical part coincide with those gotten from eq.(2.1) in the $H_{\mu\nu} = 0$, $H_\mu = 0$ and $\tilde{H}_\mu = 0$ gauge, and also the metric and coupling constant independence is maintained. From the explicit expression of S_q one also sees that e_G^2 can be identified with Newton's gravitational constant.

Concerning the gauge fixing action S_{gf}, it cannot be expressed in a covariant way and the introduction of a metric is unavoidable. The metric $g_{\alpha\beta}$ on M selected to incorporate matter couplings is here again used; evidently, physical results should be independent of this choice. A particularly advantageous gauge is the Landau gauge. In order to appropriately introduce it we define a covariant derivative $\mathcal{D}_\alpha[e_{cl}, w_{cl}]$ which acts on a vector C_β taking values in the algebra of the gauge group in the following way

$$\mathcal{D}_\alpha[e_{cl}, w_{cl}]C_\beta = (\partial_\alpha c^a_\beta - \Gamma^\sigma_{\alpha\beta}[g]\, c^a_\sigma - e_G\, \epsilon^{ab} w_{cl\,\alpha} c_{\beta b})\, P_a + \quad (4.31)$$

$$(\partial_\alpha c_\beta - \Gamma^\sigma_{\alpha\beta}[g]\, c_\sigma + e_G \Lambda\, \epsilon_{bc} e_{cl}{}^b_\alpha c^c_\beta)\, J\ . \quad (4.32)$$

Here, we have used the gravitational covariant derivative plus a term containing background Zweibein and spin connection fields e_{cl} and w_{cl} which are solutions to the equations of motion (these last have been introduced to handle with zero mode problems). With this notation the Landau gauge condition reads

$$\mathcal{D}^\alpha[e_{cl}, w_{cl}](e_\beta, f_\beta) = 0 \quad (4.33)$$

$$\mathcal{D}^\alpha[e_{cl}, w_{cl}]\chi_\beta = 0 \quad (4.34)$$

and S_{gf} is

$$S_{gf}[M] = \int_M d^2x \sqrt{g} < Y \mathcal{D}_\mu[e_{cl}, w_{cl}](e^\mu, f^\mu) + \tilde{Y} \mathcal{D}_\mu[e_{cl}, w_{cl}]\chi^\mu > \quad (4.35)$$

where $Y = (y^a, y)$ and $\tilde{Y} = (\tilde{y}^a, \tilde{y})$ are Lagrange multipliers enforcing the gauge conditions. The corresponding ghost action takes the form

$$S_{gh}[M] = \int_M d^2x \sqrt{g} < X(E^{\mu\nu} D_\mu[e,f]\chi_\nu - 2e_G\, \Psi < \Phi\rho > - e_G < (\Phi^2 - \Phi_0^2)\tilde{\rho} >) + \frac{1}{4} e_G\, [X, X]\sigma + X^+_\mu(-e_G[\Phi, \chi^\mu] +$$

$$
\begin{aligned}
&D^\mu[e,f]\rho - E^{\mu\nu}[\Psi,[\Phi,\chi_\nu]] + E^{\mu\nu}[\Psi, D_\nu[e,f]\rho] - \\
&e_G\, E^{\mu\nu}[D_\nu[e,f]\Phi,\tilde\rho]) + \frac{1}{4}e_G\,[X_\mu, X^\mu]\sigma + \\
&\tilde X^+_\mu(-e_G[\Psi,\chi^\mu] + D^\mu[e,f]\tilde\rho) + \frac{1}{4}e_G\,[\tilde X_\mu, \tilde X^\mu]\sigma + \\
&(-\mathcal{D}_\mu[e_{cl}, w_{cl}]\overline{C} + e_G[\chi_\mu,\overline{\sigma}])(\chi^\mu - D^\mu[e,f]C) + \\
&\overline{\sigma}\mathcal{D}_\mu[e_{cl}, w_{cl}](D^\mu[e,f]\sigma + e_G[C,\chi^\mu]) \;>\; . \qquad (4.36)
\end{aligned}
$$

Fields $\mathcal{C} = (\chi_\mu, C, \rho, \tilde\rho, \sigma)$ with ghost numbers $(1,1,1,1,2)$ are the ghosts associated with each of the symmetries of the classical action. To be more precise, they are related as follows

$$\epsilon_\mu \;\rightarrow\; \chi_\mu \;, \qquad (4.37)$$

$$\epsilon \;\rightarrow\; C \qquad (4.38)$$

$$\theta \;\rightarrow\; \rho \qquad (4.39)$$

$$\tilde\theta \;\rightarrow\; \tilde\rho \qquad (4.40)$$

$$\xi \;\rightarrow\; \sigma \;. \qquad (4.41)$$

The corresponding antighosts are written as $\overline{\mathcal{C}} = (\overline{\sigma}, \overline{C}, X_\mu, \tilde X_\mu, X)$ with ghost numbers $(-2,-1,-1,-1,-1)$. X_μ, $\tilde X_\mu$ and X are self-dual fields in the sense of eqs.(2.3) and (2.2), respectively. The covariant derivative $D_\mu[e,f]$ has been introduced in eq.(3.7).

The partition function for our model, when written in gravity language is, then,

$$\mathcal{Z}[M] = \int \mathcal{D}\text{fields}\;\; e^{-S_q[M]} \;. \qquad (4.42)$$

The fields of the theory and their corresponding ghost numbers are summarized in table 2.

Given the topological invariance of the action S_{cl} (eqs.(4.2)-(4.7)), it is easy to find the associated BRST commutators (2.28) for gravity and matter fields

$$\begin{array}{rcl}\{Q, e^a_\mu\} & = & \chi^a_\mu - \partial_\mu c^a - e_G\,\epsilon^{ab}(c e_{\mu b} - f_\mu c_b)\ ,\\ \{Q, f_\mu\} & = & \chi_\mu - \partial_\mu c + e_G \Lambda\,\epsilon_{ab}\,c^a e^b_\mu\ ,\\ \{Q, \Phi\} & = & \rho - e_G[\Phi, C]\ ,\\ \{Q, \Psi\} & = & \tilde{\rho} - e_G[\Psi, C]\ ,\end{array} \tag{4.43}$$

for ghosts and antighosts

$$\begin{array}{rclrcl}\{Q, \chi_\mu\} & = & -D_\mu[e,f]\sigma + e_G[C, \chi_\mu]\ , & \{Q, C\} & = & -(\sigma + \frac{1}{2}e_G[C, C])\ ,\\ \{Q, \rho\} & = & -e_G\left([\Phi, \sigma] + [C, \rho]\right)\ , & \{Q, \sigma\} & = & e_G\,[\sigma, C]\ ,\\ \{Q, \tilde{\rho}\} & = & -e_G\left([\Psi, \sigma] + [C, \tilde{\rho}]\right)\ , & \{Q, \overline{C}\} & = & Y\ ,\\ \{Q, \overline{\sigma}\} & = & \tilde{Y}\ , & & & \end{array}$$

and for Lagrange multipliers

$$\begin{array}{rcl}\{Q, Y\} & = & 0\ ,\\ \{Q, \tilde{Y}\} & = & 0\ ,\\ \{Q, X\} & = & \frac{1}{2}E^{\mu\nu}D_{\mu\nu} - e_G[X, C]\ ,\\ \{Q, X_\mu\} & = & D_\mu - e_G[X_\mu, C]\ ,\\ \{Q, \tilde{X}_\mu\} & = & \tilde{D}_\mu - e_G[\tilde{X}, C]\ ,\\ \{Q, D_{\mu\nu}\} & = & e_G\left([D_{\mu\nu}, C] + E_{\mu\nu}[X, \sigma]\right)\ ,\\ \{Q, D_\mu\} & = & e_G\left([D_\mu, C] + [X_\mu, \sigma]\right)\ ,\\ \{Q, \tilde{D}_\mu\} & = & e_G\left([\tilde{D}_\mu, C] + [\tilde{X}_\mu, \sigma]\right)\ .\end{array}$$

It is straightforward but tedious to corroborate the BRST invariance of S_q. Moreover, it can be also proved that, as announced in the previous Section,

$$S_q = \{Q, V\} \tag{4.44}$$

where the functional V is

$$V \;=\; \int_M d^2x\sqrt{g} < \frac{1}{4}XE^{\mu\nu}D_{\mu\nu} - XE^{\mu\nu}B_{\mu\nu} + \frac{1}{4}X^\mu D_\mu -$$

$$X^\mu B_\mu + \frac{1}{4}\tilde{X}^\mu \tilde{D}_\mu - \tilde{X}^\mu \tilde{B}_\mu -$$
$$\overline{C}\mathcal{D}_\mu[e_{cl}, w_{cl}](e^\mu, f^\mu) - \overline{\sigma}\mathcal{D}_\mu[e_{cl}, w_{cl}]\chi^\mu > \ . \quad (4.45)$$

This property guarantees that $\mathcal{Z}[M]$ only depends on the topology of M and not on the choice of the selected metric. In fact,

$$\frac{\delta S_q}{\delta g^{\mu\nu}} = \{Q, \frac{\delta V}{\delta g^{\mu\nu}}\} \ , \quad (4.46)$$

which ensures that the metric dependence of the quantum action is trivial in the sense that its variation with respect to the metric gives a BRST commutator which has no effect at the physical level. More precisely, a possible dependence of the partition function measure on the metric must be taken into account to finally establish the independence of $\mathcal{Z}$ on the metric. This has been done in ref.[21] for Witten type TQFTs and it has been there confirmed that, or this kind of theories, $\mathcal{Z}$ is indeed metric independent. Furthermore, (4.44) implies the independence of $\mathcal{Z}[M]$ on the gauge coupling constant e_G.

5 TOPOLOGICAL INVARIANTS

In view of the independence of the partition function on the metric signaled above, the simplest topological invariant to be considered is, precisely, the partition function $\mathcal{Z}[M]$.

In order to clarify our derivation of topological invariants, we shall again first consider the simplified action S^1_{cl}, eq.(2.23). It can be easily shown that the zero mode equation associated with the ghost field χ appearing in S^1_q coincides with the equation describing the moduli space for Bogomol'nyi solutions. Indeed, given a solution Φ_{cl} to Bogomol'nyi equations,

$$B[\Phi_{cl}] = 0 \ , \tag{5.1}$$

a nearby configuration $\Phi_{cl} + \delta\Phi_{cl}$ will also be a solution provided

$$\frac{\delta B}{\delta \Phi}|_{\Phi_{cl}} \delta\Phi_{cl} = 0 \ . \tag{5.2}$$

Since the ghost action in S^1_q is

$$S^1_{gh} = \int_M d^2x \sqrt{g} < X \frac{\delta B}{\delta \Phi} \chi > \ , \tag{5.3}$$

the equation of motion for X, giving the zero mode equation for χ, coincides with eq.(5.2) for $\delta\Phi_{cl}$ when $\Phi = \Phi_{cl}$. (For simplicity we shall suppose that X has no zero modes.)

As for solutions to eq.(5.2), there are two possibilities; either no non-trivial solution exists or there are solutions which span the moduli space; $d(\mathcal{M})$ is equal or different from zero, respectively.

Concerning the case $d(\mathcal{M}) = 0$, $\mathcal{Z}[M]$ can be exactly evaluated, a basic property of topological models, related to the Q-symmetry of S_q. Indeed, $\mathcal{Z}[M]$ is independent of e_G and then it can be computed in the e_G going to zero limit where the path integral is dominated by configurations $(\Phi, \chi, X) = (\Phi^i_{cl}, 0, 0)$, with $i = 1, 2, ...n$ labelling isolated Bogomol'nyi solutions. Calling φ the fluctuations around $\Phi = \Phi^i_{cl}$ we have

$$\begin{aligned} \mathcal{Z}[M] \ = \ & \sum_{i=1}^{n} \int \mathcal{D}\varphi \, \mathcal{D}\chi \, \mathcal{D}X \ \exp[- \int_M d^2x \sqrt{g} < \varphi \frac{\delta B}{\delta \Phi}|_{\Phi^i_{cl}} \frac{\delta B}{\delta \Phi}|_{\Phi^i_{cl}} \varphi + \\ & X \frac{\delta B}{\delta \Phi}|_{\Phi^i_{cl}} \chi >] \end{aligned} \tag{5.4}$$

or

$$\mathcal{Z}[M] \ = \ \sum_{i=1}^{n} \frac{\mathrm{Pfaff}(\frac{\delta B}{\delta \Phi}|_{\Phi^i_{cl}})}{\sqrt{\det(\frac{\delta B}{\delta \Phi}|_{\Phi^i_{cl}} \frac{\delta B}{\delta \Phi}|_{\Phi^i_{cl}})}} \ ,$$

$$\mathcal{Z}[M] \quad = \quad \sum_{i=1}^{n}(-1)^{n_i} \tag{5.5}$$

where $n_i = 0, 1$ according to the way one determines the sign of the Pfaffian (see ref.[12]). Since in topological theories $\mathcal{Z}[M]$ is metric independent, the right hand side of eq.(5.5) gives the explicit way of computing a topological invariant.

The derivation we have presented for this simple example can be straightforwardly extended to the model of interest with classical action (2.1). Simply, in view of the symmetry (2.17)-(2.22), the gauge fermion F has been taken as

$$\begin{aligned} F \quad = \quad & \int_M d^2x\sqrt{g} < XH + X_\mu H^\mu + \tilde{X}_\mu \tilde{H}^\mu + \overline{C}\mathcal{D}_\mu[e_{cl}, w_{cl}](e^\mu, f^\mu) + \\ & \overline{\sigma}\mathcal{D}_\mu[\tilde{e}, w_{cl}]\chi^\mu > , \end{aligned} \tag{5.6}$$

so that the quantum action, when written in terms of gravitational fields, is given by eq.(4.30). Again, the bosonic and fermionic contributions to $\mathcal{Z}[M]$ cancel up to a sign around each classical solution. These signs have to be computed from the quantum action for our gravity model, eq.(4.30). In order to do so, one first performs an expansion around the classical solutions discussed in Section 3 up to quadratic terms and then computes bosonic an fermionic determinants once an assignment for the Pfaffian sign is adopted. Each n_i can then be determined and one can again conclude that $\mathcal{Z}[M]$ takes the form

$$\mathcal{Z}[M] = \sum_{i=1}^{n}(-1)^{n_i} \ , \tag{5.7}$$

and is a topological invariant in the $d(\mathcal{M}) = 0$ case.

Let us now discuss the evaluation of topological invariants in the $d(\mathcal{M}) \neq 0$ case. In this case, the Pfaffian vanishes and, as explained in ref.[12], topological invariants have to be computed from vacuum expectation values of BRST invariant and metric independent functionals containing a product of an appropriate number of fields so as to absorb zero modes. In ref.[18] the construction of such invariants was discussed for the gauge theory defined by action (2.1). One starts by constructing functionals W_k satisfying

$$\begin{aligned} 0 &= \{Q, W_0\} \ , \\ dW_0 &= \{Q, W_1\} \ , \\ dW_1 &= \{Q, W_2\} \ , \\ dW_2 &= 0 \ . \end{aligned} \tag{5.8}$$

and using the notation of Section 2 one easily finds

$$\begin{aligned} W_0 &= \tfrac{1}{2} < \sigma^2 > \ , \\ W_1 &= < \sigma \chi_\mu > dx^\mu \ , \\ W_2 &= < \sigma F_{\mu\nu} > dx^\mu \wedge dx^\nu \ . \end{aligned} \tag{5.9}$$

These functionals have ghost number $4 - k$. Given a moduli dimension $d(\mathcal{M}) \neq 0$, a non-trivial topological invariant takes the form

$$\mathcal{Z}(\gamma_1, ..., \gamma_r) = \int \mathcal{D}\text{fields} \quad \prod_{i=1}^{r} I^{(\gamma_i)} \quad e^{-S_q[M]} \ , \tag{5.10}$$

with $\gamma_1, ..., \gamma_r$ homology cycles of dimension $k_1, ..., k_r$ such that

$$\sum_{i=1}^{r} (4 - k_i) = d(\mathcal{M}) \tag{5.11}$$

and $I^{(\gamma_i)}$ defined as

$$I^{(\gamma_i)} = \int_{\gamma_{k_i}} W_{k_i} \ . \tag{5.12}$$

In order to obtain explicit formulæ for topological invariants, computed as vacuum expectation values (*vev*'s) in the form (5.10), one proceeds as follows. As in the partition function case, the lowest order in the e_G^2 expansion gives the exact result for the path integral defining the *vev*, then, the dynamical fields can be replaced by their classical configurations solving the Bogomol'nyi equations. In the present case, the only dynamical field appearing in W_k's is the gauge field A_μ which is replaced by A_μ^{cl}. The ghost χ_μ appearing in W_1, whose zero modes probe the moduli space (together with ρ and $\tilde{\rho}$ zero modes), have to be replaced by its zero mode configuration χ_μ^0. Concerning the ghost for ghost σ, one has to perform the corresponding integration. For example, the *vev* of σ^A ($\sigma = \sigma^A T_A$) is computed as follows:

$$\begin{aligned} < \sigma^A >_{vev} &= \int \mathcal{D}\sigma\, \mathcal{D}\overline{\sigma}\, \sigma^A(x) \exp[-\int_M d^2y\sqrt{g} < \overline{\sigma} D_\mu D^\mu \sigma + \\ & [\chi_\mu^0, \chi^{\mu 0}]\overline{\sigma} + ... >] \ . \end{aligned} \tag{5.13}$$

The dots in the exponential represent irrelevant terms to lowest order in e_G^2. Expanding the second term and performing the integration over σ and $\overline{\sigma}$, one has

$$< \sigma^A >_{vev} = \int_M d^2y\sqrt{g} < [\chi_\mu^0(y), \chi^{\mu 0}(y)] T_B > \Delta^{AB}(y-x) \ , \tag{5.14}$$

where

$$(D_\mu D^\mu \Delta)^{AB}(z) = \delta^{AB}\delta(z) \ . \tag{5.15}$$

Replacing σ by $< \sigma >_{vev}$ whenever it appears in $I^{(\gamma_i)}$, one obtains the following expressions for $I^{(\gamma_i)}$'s

$$I^{(\gamma_0)} = \int_{\gamma_0} << \sigma >_{vev}^2 > \ , \tag{5.16}$$

$$I^{(\gamma_1)} = \int_{\gamma_1} << \sigma >_{vev} \chi_\mu^0 > dx^\mu \ , \tag{5.17}$$

$$I^{(\gamma_2)} = \int_{\gamma_2} << \sigma >_{vev} F_{\mu\nu}^{cl} > dx^\mu \wedge dx^\nu \ . \tag{5.18}$$

Of course, to go further into the evaluation of topological invariants one has to know the structure of the moduli space, the explicit form of A^{cl}_μ, χ^0_μ, etc.

We just conclude by writing the results presented above in terms of the fields appearing in our gravity model. The *vev* of σ is still given by eq.(5.14) with $\Delta^{AB}(z)$ satisfying

$$(\mathcal{D}_\mu[e_{cl}, w_{cl}]\mathcal{D}^\mu[e,f]\Delta)^{AB}(z) = \delta^{ab}\delta(z) \ . \tag{5.19}$$

Then, $I^{(\gamma_0)}$ and $I^{(\gamma_1)}$ are computed from eqs.(5.16), (5.17) and (5.19) with χ^0_μ the zero modes of the fermionic operator in (4.30). Concerning $I^{(\gamma_2)}$ note that

$$F^{cl}_{\mu\nu} dx^\mu \wedge dx^\nu = e_G \Psi^{cl} \ v(\Phi^{cl}) \ d^2x \ , \tag{5.20}$$

through the use of Bogomol'nyi equation (2.7) and then,

$$I^{(\gamma_2)} = e_G \int_{\gamma_2} d^2x << \sigma >_{vev} \Psi^{cl} > v(\Phi^{cl}) \ . \tag{5.21}$$

6 SUMMARY AND DISCUSSION

In this work, we have succeeded in constructing a two-dimensional model for the gravitational field with a non-trivial coupling to matter. This has been achieved starting from the topological gauge model presented in ref.[18] and interpreting the gauge fields as a Zweibein and (effective) connection fields. In this way, the original TQFT has been expressed in geometrical terms so that its classical equations of motion become gravitational field equations coupled to matter (see table 1). The basic property of (Witten type) TQFTs, i.e. the fact that $S_q = \{Q, V\}$ has been fundamental to get a gravitational model with matter coupling. Indeed, since $\delta S_q/\delta g^{\mu\nu} = \{Q, \lambda_{\mu\nu}\}$, the quantum theory does not depend on the background metric used to introduce matter couplings and to fix the gauge. The same property ensures the model independence on all of the parameters, in particular on ϕ_0, the minimum of the Higgs potential. Thus the small e_G expansion performed to calculate expectation values of interest is, in this case exact and, furthermore, the model is scale invariant.

It is interesting to point that, if all scalar fields are put to zero (i.e. matter is absent) our equations of motion become those of the Jackiw-Teitelboim model for two-dimensional gravity [1]. If only one scalar field (that appearing with a symmetry breaking potential) is set to zero, then the model becomes that constructed by Chamseddine and Wyler [7]. To be more precise, the classical equations of our model coincide with those of ref.[7] when Φ is absent. At the quantum level, Chamseddine and Wyler quantized a topological theory à la Baulieu-Singer [13], starting from a classical action which is a topological invariant while we proceeded to quantization à la Labastida-Pernici [16] starting from a quantum action where Bogomol'nyi equations play a central rôle.

We have explicitely shown how the large symmetry, characteristic of topological theories, corresponds to diffeomorphism and local Lorentz symmetries in a certain subspace of transformation parameter space. Thus, as expected, the basic gravitational symmetries are incorporated in our model.

As stated above, the exact quantum description of our model can be made in the limit of small gauge coupling constant (which can be here interpreted as Newton's gravitational constant). In particular, the partition function can be computed exactly by performing a semiclassical expansion, this leading to an explicit expression for a topological invariant (when the moduli space dimension is zero). Other topological invariants have been discussed by exploiting the BRST invariance of the gauge theory.

Our results extend those of refs.[7]-[9], in which topological theories of pure gravitational fields in two dimensions have been constructed, to a gravity-matter theory. In all of these models, the large topological symmetry of the action reduces the space of states to a finite dimensional one. It would be worthwhile to investigate this issue following, for example, Horowitz approach to the computation of state functions for TQFTs [6], to probe whether there exists a unique solution as it is the case in several cases. Finally, it should be stressed that if one takes our model as a toy model for gravity, the large topological symmetry should be broken. These and related problems should be studied more thoroughtfully.

Acknowledgements:

One of the authors (L.F.C.) would like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste.

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	$\Lambda = 0$	$\Lambda \neq 0$
Scalar curvature equation	$R = e_G \tau$	$R + 2e_G^2 \Lambda = e_G \tau$
Vanishing torsion equation	$D_\mu^{ab}[\frac{w}{e_G}] e_\nu^b = 0$	$D_\mu^{ab}[\frac{w}{e_G}] e_\nu^b = 0$
Ψ field	$\psi = \eta$	$\Box \psi + 4e_G^2 \, (1 - \psi^2) \, v(\Phi) + 4e_G^2 \Lambda \, \psi = 0$
	$\Delta_\mu^a(f, \psi_b, \eta) = 0$	$\psi^a = \frac{1}{2e_G \Lambda} \epsilon^{ab} e_b^\mu \partial_\mu \psi$
Φ field	$\phi = \lambda$	$\Lambda \neq 1$: $\phi = \frac{1}{2(1-\Lambda)} \psi + (\phi_0 - \frac{\psi_0}{2(1-\Lambda)})$ $\Lambda = 1$: no solutions
	$\eta = \pm 1$: ϕ^a to be determined from the constraint $\eta \neq \pm 1$: $\epsilon^{ab} \psi_a \Delta_b^\mu(f, \phi^a, \lambda) = 0$ $\psi^a \Delta_a^\mu(f, \phi^a, \lambda) = 0$	$\epsilon^{ab} \psi_a \Delta_b^\mu(f, \phi_b, \phi) - E^{\mu\nu} \partial_\nu \phi = 0$ $\psi^a \Delta_a^\mu(f, \phi_b, \phi) + \psi \partial^\mu \phi = 0$
Constraints	$\psi^a \psi_a + \eta^2 = 1$ $\psi^a \phi_a + \eta \lambda = 0$	$\Lambda \psi^a \psi_a + \psi^2 = 1$ $\Lambda \psi^a \phi_a + \psi \phi = 0$

Table 1: Equations of motion and their solutions.

Field		Ghost number
Zweibein	e_μ^a	0
(related to the spin connection)	f_μ	0
Scalar field	Φ	0
Scalar field	Ψ	0
	$D_{\mu\nu}$	0
	D_μ	0
Lagrange multipliers	$\tilde{D}_\mu$	0
	Y	0
	$\tilde{Y}$	0
	χ_μ	1
	C	1
Ghost fields	ρ	1
	$\tilde{\rho}$	1
	σ	2
	$\overline{\sigma}$	-2
	$\overline{C}$	-1
Antighost fields	X_μ	-1
	$\tilde{X}_\mu$	-1
	X	-1

Table 2: Fields of the theory, ghost numbers and Grassmann parities.