

# Connection between $\zeta$ and cutoff regularizations of Casimir energies

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## Abstract

We study the connection between  $\zeta$ - and cutoff-regularized Casimir energies for scalar fields. We show that, in general, both regularization schemes lead to divergent contributions, and to finite parts which do not coincide. We determine the relationships among the various coefficients appearing in one approach and the other. As an application, we discuss the case of scalar fields in  $d$ -dimensional boxes under periodic boundary conditions.

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## 1 Introduction

The task of extracting physically meaningful results from ill defined quantities is a fundamental aspect of quantum field theory. Perhaps the simplest example of this situation is the infinite zero point energy of quantum fields. In 1948, Casimir showed [1] that neutral perfectly conducting parallel plates placed in the vacuum attract each other. The basic idea behind the concept of Casimir vacuum energy is that quantum fields always exist in the presence of external constraints and their zero point field energy is thus modified. Such constraints are idealized as conditions to be satisfied by modes of the field at the boundary of a given manifold.

One of the procedures [2] used for computing Casimir energies is the direct evaluation of infinite sums over zero modes. These sums, which are formally divergent, may be regularized through various techniques [2],[3].

Recently, some work aiming at understanding the relationship between the  $\zeta$  function and exponential cutoff regularizations of Casimir energies for scalar

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fields was performed by the authors of references [4],[5]. In particular, in [5], the connection between both regularizations was established, with the ad-hoc condition that the cutoff-regularized energy presents only polar divergencies. The  $\zeta$ -regularized energy was then shown to be finite and its outcome was proved to be identical to the energy regularized via cutoff, with polar terms subtracted.

In this paper, we extend the study of the above-mentioned connection to the most general case of physical interest : we only restrict the associated boundary problem to guarantee that the energy eigenvalues of the field be real. Then, we make use of some well known results concerning traces of heat kernels and complex powers of differential operators [6], [7], [8] for such boundary problems. We apply Mellin transform techniques to show that both regularizations lead, in general, to divergences and that their finite parts do not coincide.

The paper is organized as follows :

Section 2 contains a presentation of the problem to be studied. There, the formal expression for the Casimir energy of a scalar field through mode summation is given. The associated boundary problem is defined and  $\zeta$  and cutoff regularizations are introduced.

In Section 3 , we present our main result : we show that under fairly general conditions on the associated boundary problem [6],[7] (which guarantee the real nature of the energy eigenvalues), the regularization via  $\zeta$  function does not, in general, lead to a finite result and -consequently- the exponential regularization shows not only poles but also logarithmic singularities. The relationship among the various coefficients in one regularization and the other is established. In particular, finite parts are seen to differ in a well determined fashion.

Section 4 contains a simple example of application : the evaluation of the Casimir energy for a massive scalar field in a  $d$ -dimensional box, subject to periodic boundary conditions. Both regularization schemes are applied to the cases  $d = 1$  and  $d = 2$ , and their outcome is shown to agree with our general result in Section 3. For this particular geometry, and under periodicity conditions, the exponential and  $\zeta$  regularizations are seen to be equivalent only after a physically meaningful prescription is given in order to eliminate divergencies.

Finally, our conclusions are presented in Section 5.

## 2 Casimir energies for scalar fields through mode summation

The evaluation of Casimir energies through the mode summation method involves the direct performance of infinite sums over energy eigenvalues of the zero point field modes [2], [3],

$$E_C = \frac{1}{2} \sum_n \omega_n \tag{1}$$

The energy eigenvalues,  $\omega_n$ , depend on the dimension of the space-time, on the spin of the field under consideration and on the boundary condition imposed on it.

Let us consider the case of a free scalar field in a  $d+1$ - dimensional space-time manifold

$$\mathcal{M}^{(d+1)} = R \times M \quad (2)$$

where  $M$  is a smooth compact  $d$ -dimensional manifold with smooth boundary  $\partial M$ .

After separation of variables, the energy eigenvalues turn out to be

$$\omega_n = \lambda_n^{1/2} \quad (3)$$

where the  $\lambda_n$  satisfy [3] the associated boundary problem

$$D_B \varphi_n = \begin{cases} D\varphi_n = \lambda_n \varphi_n \\ BT\varphi_n = 0 \end{cases} \quad (4)$$

Here,  $D$  is a second order operator on  $M$ ,  $B$  is a tangential operator (which we will take to be differential), defining boundary conditions and  $T$  is the restriction map, which assigns to each section its Cauchy data at  $\partial M$ .

In what follows, we will refer to the boundary problem (4) as  $(D, B)$ . It is clear that, in order for the  $\omega_n$  to make sense as physical energies, the eigenvalues  $\lambda_n$  must be real and positive, which can be achieved by imposing certain well known conditions [6], [7] (to be specified in the next section) on the boundary

problem  $(D, B)$ . As we will see, such conditions also imply that  $\omega_n \rightarrow \infty$  as  $n \rightarrow \infty$

and that they are  $O(n^{1/d})$  for  $n$  large.

Thus, the mode summation (1) is divergent, and a meaning must be given to it through some regularization scheme. In this paper we will be concerned with two of these methods: the exponential cutoff and  $\zeta$  function [9] regularizations.

In the first case, one defines,

$$E_{\text{exp}} \equiv \frac{\mu}{2} \sum_n \frac{\lambda_n^{1/2}}{\mu} e^{-t \frac{\lambda_n^{1/2}}{\mu}} \Big|_{t=0} = -\frac{\mu}{2} \frac{d}{dt} \left( h \left( t, \frac{D_B}{\mu^2} \right) \right) \Big|_{t=0} \quad (5)$$

where

$$h \left( t, \frac{D_B}{\mu^2} \right) = \sum_n e^{-t \frac{\lambda_n^{1/2}}{\mu}} = \text{Tr} \left( e^{-\frac{t}{\mu} D_B^{1/2}} \right) \quad (6)$$

and  $\mu$  is a parameter with dimensions of mass, introduced in order to render  $t$  dimensionless.

As regards  $\zeta$  function regularization [9], the Casimir energy is defined as

$$\begin{aligned}
 E_\zeta &\equiv \frac{\mu}{2} \sum_n \left( \frac{\lambda_n}{\mu^2} \right)^{-\frac{s}{2}} \Big|_{s=-1} = \frac{\mu}{2} \zeta \left( \frac{s}{2}, \frac{D_B}{\mu^2} \right) \Big|_{s=-1} \\
 &= \frac{\mu}{2} Tr \left( \frac{D_B}{\mu^2} \right)^{-\frac{s}{2}} \Big|_{s=-1}
 \end{aligned} \tag{7}$$

From the already mentioned behaviour of  $\lambda_n$  it is easy to see that the sum in (7) is convergent for  $Re(s)$  large enough. So,  $\zeta \left( \frac{s}{2}, \frac{D_B}{\mu^2} \right)$  is holomorphic in the same region. As we will see in the next section, it can be extended to the whole  $s$  plane as a meromorphic function, with only single poles. The  $\zeta$  function regularized energy is then defined as the value of this meromorphic extension at  $s = -1$ . Here, a parameter  $\mu$  has again been introduced, this time in order to render the  $\zeta$  function dimensionless [10].

It will be the subject of the next section to establish the behaviour of Casimir energies regularized in both fashions, and to give the precise relationship between divergent and finite parts appearing in one scheme and the other, thus generalizing the result in reference [5]. We will also show that both regularizations are not, in general, equivalent (i.e., finite parts differ). A particular case will be studied in Section 4, where they will be shown to give the same result after an adequate prescription is imposed to eliminate divergencies.

### 3 Equivalence between regularizations

In this section we present our main result. We study, under fairly general conditions (which are those of physical interest), the connection between exponential and  $\zeta$  function regularizations of Casimir energies for scalar fields. We establish the relationships among coefficients appearing in one case and the other.

Before going to our main result, we reproduce, without proof, some well known facts concerning elliptic boundary problems [6],[7], as applied to the case of interest :

#### Lemma 1

Let  $M$  be a smooth compact  $d$ -dimensional manifold, with smooth boundary  $\partial M$ . Let  $D$  be an elliptic second order partial differential operator, and let  $B$  be a tangential differential operator over  $\partial M$ .

If the boundary problem  $(D, B)$  is self-adjoint and elliptic with respect to  $C - R_+$  (i.e., it has an Agmon's cone [11] including the negative axis), then :

- a) We can find a complete orthonormal system  $\{\phi_n\}_{n=1}^\infty$  with  $D\phi_n = \lambda_n \phi_n$ .
- b)  $\phi_n$  satisfy the boundary condition  $BT\phi_n = 0$  (here,  $T$  is the restriction map, which assigns to any smooth section its Cauchy data).

c)  $\lambda_n \in R$  and  $\lim_{n \rightarrow \infty} |\lambda_n| = \infty$ . If we order the  $\lambda_n$  such that  $|\lambda_1| \leq |\lambda_2| \leq \dots$ , then there exists  $n_0$  so that  $|\lambda_n| > n^{\frac{2}{d}}$  for  $n > n_0$ .

d) The  $\lambda_n$  are bounded from below and  $\text{spec}(D_B)$  is contained in  $[-C, \infty]$  for some constant  $C$ .

(In what follows we will assume, without losing generality that  $\text{spec}(D_B)$  is positive).

### Lemma 2

Under the conditions of the previous lemma [7] :

a)  $Y\left(t, \frac{D_B}{\mu^2}\right) = \text{Tr}\left(e^{-t \frac{D_B}{\mu^2}}\right)$  is holomorphic in a sector  $V_{\theta_0}$  (for some  $\theta_0 \in (0, \frac{\pi}{2})$ ,  $V_{\theta_0} = \{t = re^{i\theta}/r > 0, |\theta| < \theta_0\}$ ).

b)  $Y\left(t, \frac{D_B}{\mu^2}\right)$  has the asymptotic expansion :  $Y(t) \sim \sum_{j=0}^{\infty} a_j t^{\frac{j-d}{2}}$  , for  $t \rightarrow 0$  uniformly for  $t \in V_{\delta}$ , for each  $\delta < \theta_0$ .

Here, the  $a_j$  can be evaluated from Seeley's coefficients [7], including volume as well as boundary contributions.

c)  $Y\left(t, \frac{D_B}{\mu^2}\right)$  decreases exponentially for  $|t| \rightarrow \infty$  in  $V_{\delta}$ .

### Lemma 3

Under the same conditions [7],[12] :

a)

$$\Gamma\left(\frac{s}{2}\right) \zeta\left(\frac{s}{2}, \frac{D_B}{\mu^2}\right) = \Gamma\left(\frac{s}{2}\right) \text{Tr}\left(\left(\frac{D_B}{\mu^2}\right)^{-\frac{s}{2}}\right) = \int_0^{\infty} t^{\frac{s}{2}-1} Y\left(t, \frac{D_B}{\mu^2}\right) dt \quad (8)$$

is the Mellin transform of  $Y(t, \frac{D_B}{\mu^2})$ . It is holomorphic for  $\text{Re}(s) > d$  and extends to a meromorphic function, with isolated simple poles :

$$\Gamma\left(\frac{s}{2}\right) \zeta\left(\frac{s}{2}, \frac{D_B}{\mu^2}\right) = \sum_{j=0}^{\infty} \frac{2a_j}{s+j-d} + r\left(\frac{s}{2}\right) \quad (9)$$

where  $r\left(\frac{s}{2}\right)$  is an entire function.

b) For each real  $c_1, c_2$  and each  $\delta < \theta_0$ ,

$$\left| \Gamma\left(\frac{s}{2}\right) \zeta\left(\frac{s}{2}, \frac{D_B}{\mu^2}\right) \right| \leq C(c_1, c_2, \delta) e^{-\delta |Im \frac{s}{2}|} \quad , \quad \left| Im \frac{s}{2} \right| \geq 1, c_1 \leq Re \frac{s}{2} \leq c_2 \quad (10)$$

With these elements at hand, we are now able to prove the following Lemma, which is the basis of our main result :

Lemma 4

Under the same assumptions as before :

a)  $h\left(t, \frac{D_B}{\mu^2}\right) = Tr\left(e^{-\frac{t}{\mu} D_B^{1/2}}\right) = \sum_n e^{-t \frac{\lambda_n^{1/2}}{\mu}}$  has the asymptotic expansion

$$\begin{aligned} h\left(t, \frac{D_B}{\mu^2}\right) &\sim \sum_{k=0}^d \frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} a_{d-k} \left(\frac{t}{2}\right)^{-k} + \sum_{k=1}^{\infty} \frac{\Gamma\left(-k+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} a_{d+2k} \left(\frac{t}{2}\right)^{2k} \\ &+ \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{\Gamma\left(\frac{1}{2}\right)\Gamma(k)} \left(\frac{t}{2}\right)^{2k-1} \left[ r\left(-k+\frac{1}{2}\right) + a_{d+2k-1} \left( \Psi(1) - \sum_{l=1}^{k-2} \frac{1}{l-k+1} \right) \right. \\ &\quad \left. + \sum_{j \neq d+2k-1} \frac{2a_j}{j-d-2k+1} - 2a_{d+2k-1} \ln\left(\frac{t}{2}\right) \right] \end{aligned} \quad (11)$$

for  $t \rightarrow 0$ , uniformly for  $t \in V_\delta$ , for each  $\delta < \theta_0$ .

Proof

Notice, in the first place, that

$$\Gamma(s) \zeta\left(\frac{s}{2}, \frac{D_B}{\mu^2}\right) = \int_0^\infty t^{s-1} h\left(t, \frac{D_B}{\mu^2}\right) dt \quad (12)$$

is the Mellin transform of  $h\left(t, \frac{D_B}{\mu^2}\right)$ . Now,

$$\begin{aligned} \Gamma(s) \zeta\left(\frac{s}{2}, \frac{D_B}{\mu^2}\right) &= \frac{\Gamma(s)}{\Gamma\left(\frac{s}{2}\right)} \left[ \Gamma\left(\frac{s}{2}\right) \zeta\left(\frac{s}{2}, \frac{D_B}{\mu^2}\right) \right] = \\ &= \frac{2^{s-1}}{\sqrt{\pi}} \Gamma\left(\frac{s+1}{2}\right) \left[ \Gamma\left(\frac{s}{2}\right) \zeta\left(\frac{s}{2}, \frac{D_B}{\mu^2}\right) \right] \end{aligned} \quad (13)$$

From Lemma 3 a), and the well known singularity structure of  $\Gamma\left(\frac{s+1}{2}\right)$ , it turns out that (13) is holomorphic for  $Res > d$ , and

$$h\left(t, \frac{D_B}{\mu^2}\right) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \quad t^{-s} \frac{2^{s-1}}{\sqrt{\pi}} \Gamma\left(\frac{s+1}{2}\right) \left[ \Gamma\left(\frac{s}{2}\right) \zeta\left(\frac{s}{2}, \frac{D_B}{\mu^2}\right) \right] \quad , \quad c > d \quad (14)$$

Moreover, from Lemma 3 b), together with the fact that  $\Gamma\left(\frac{s+1}{2}\right)$  is  $O\left(e^{(-\frac{\pi}{2}+\epsilon)|Im\frac{s}{2}|}\right)$ , for any  $\epsilon > 0$ , an asymptotic expansion for  $h\left(t, \frac{D_B}{\mu^2}\right)$  can be obtained by shifting the contour of integration in (14) past the poles of  $\Gamma\left(\frac{s+1}{2}\right) \left[ \Gamma\left(\frac{s}{2}\right) \zeta\left(\frac{s}{2}, \frac{D_B}{\mu^2}\right) \right]$ . These poles are located at  $s = d - j$ .

For  $s = d - j = k \geq 0$  ( $j \leq d$ ) they are simple poles, and they contribute to the Cauchy integral with

$$\frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} a_{d-k} \left(\frac{t}{2}\right)^{-k}, \quad k = 0, 1, \dots, d \quad (15)$$

For  $s = d - j = -2k$  ( $k = 1, 2, \dots$ ) they are also simple, and their contribution is

$$\frac{\Gamma\left(-k + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} a_{d+2k} \left(\frac{t}{2}\right)^{2k}, \quad k = 1, 2, \dots \quad (16)$$

For  $s = d - j = -(2k - 1)$  ( $k = 1, 2, \dots$ ) they are simple and double poles, which contribute :

$$\frac{(-1)^{k-1}}{\Gamma\left(\frac{1}{2}\right)\Gamma(k)} \left(\frac{t}{2}\right)^{2k-1} \left[ r\left(-k + \frac{1}{2}\right) + \sum_{j \neq d+2k-1} \frac{2a_j}{j-d-2k+1} \right] \quad (17)$$

$$\frac{(-1)^k}{\Gamma\left(\frac{1}{2}\right)\Gamma(k)} \left(\frac{t}{2}\right)^{2k-1} a_{d+2k-1} \left[ 2 \ln\left(\frac{t}{2}\right) - \left( \Psi(1) - \sum_{l=1}^{k-2} \frac{1}{l-k+1} \right) \right] \quad (18)$$

(Notice that the last sum in (18) is to be included whenever it makes sense).

So, shifting the path of integration in (14) up to, and including, the singularity at  $s = -2K$  we have,

$$\begin{aligned} h\left(t, \frac{D_E}{\mu^2}\right) &= \sum_{k=0}^d \frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} a_{d-k} \left(\frac{t}{2}\right)^{-k} + \sum_{k=1}^K \frac{\Gamma\left(-k + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} a_{d+2k} \left(\frac{t}{2}\right)^{2k} \\ &+ \sum_{k=1}^K \frac{(-1)^{k-1}}{\Gamma\left(\frac{1}{2}\right)\Gamma(k)} \left(\frac{t}{2}\right)^{2k-1} \left[ r\left(-k + \frac{1}{2}\right) + a_{d+2k-1} \left( \Psi(1) - \sum_{l=1}^{k-2} \frac{1}{l-k+1} \right) \right] \\ &+ \sum_{j \neq d+2k-1} \frac{2a_j}{j-d-2k+1} - 2a_{d+2k-1} \ln\left(\frac{t}{2}\right) \Big] + \rho_K(t) \end{aligned} \quad (19)$$

The remainder  $\rho_K(t)$  is given by an integral like (14), but with  $c < -2K$  which, as a result of Lemma 3 b) and the estimate for  $|\Gamma\left(\frac{s+1}{2}\right)|$  already discussed, is  $O\left(\left|\frac{t}{2}\right|^{2K+1-\epsilon}\right)$ , which completes the proof.

This asymptotic development can be differentiated term by term, to obtain an asymptotic development for  $\frac{dh}{dt}$ . When evaluated at  $t = 0$ , it gives for the Casimir energy,

$$\begin{aligned} E_{\text{exp}} &= -\frac{\mu}{2} \left. \frac{dh\left(t, \frac{D_E}{\mu^2}\right)}{dt} \right|_{t=0} = -\frac{\mu}{2} \sum_{k=1}^d (-k) \frac{\Gamma\left(\frac{k+1}{2}\right)}{2^{-k}\Gamma\left(\frac{1}{2}\right)} a_{d-k} (t)^{-k-1} \Big|_{t=0} \\ &- \frac{\mu}{4\Gamma\left(\frac{1}{2}\right)} \left[ r\left(-\frac{1}{2}\right) + a_{d+1} (\Psi(1) - 2) + 2 \sum_{j \neq d+1} \frac{a_j}{j-d-1} \right] + \frac{\mu}{2} \frac{a_{d+1}}{\Gamma\left(\frac{1}{2}\right)} \ln\left(\frac{t}{2}\right) \end{aligned} \quad (20)$$

As concerns the Casimir energy regularized via zeta function, from (9) it can be seen to be given by

$$\begin{aligned}
E_\zeta &= \left. \frac{\mu}{2} \zeta \left( \frac{s}{2}, \frac{D_B}{\mu^2} \right) \right|_{s=-1} = \frac{\mu}{2\Gamma(-\frac{1}{2})} \sum_{j \neq d+1} \frac{2a_j}{j-d-1} + \frac{\mu}{2\Gamma(-\frac{1}{2})} r \left( -\frac{1}{2} \right) \\
&\quad + \left. \frac{\mu}{\Gamma(\frac{s}{2})} \frac{a_{d+1}}{s+1} \right|_{s=-1} = \\
&= -\frac{\mu}{2\Gamma(\frac{1}{2})} \sum_{j \neq d+1} \frac{a_j}{j-d-1} - \frac{\mu}{4\Gamma(\frac{1}{2})} r \left( -\frac{1}{2} \right) + \frac{\mu}{2\Gamma(\frac{1}{2})} a_{d+1} \left( \frac{\Psi(1)}{2} + 1 - \ln 2 \right) \\
&\quad - \left. \frac{\mu}{2\Gamma(\frac{1}{2})} \frac{a_{d+1}}{s+1} \right|_{s=-1}
\end{aligned} \tag{21}$$

From (20) and (21) the following conclusions concerning Casimir energies for scalar fields in a  $d+1$ -dimensional space-time can be drawn :

1) Both regularization methods do, in principle, give rise to divergent contributions. If the coefficient  $a_{d+1}$  vanishes, the zeta function regularization gives a finite result, which coincides with the finite part of the energy obtained through exponential regularization. On the other hand, this last regularization method presents poles of order 2, 3,  $\dots$ ,  $d+1$ , the coefficient of the pole of order  $k+1$  being  $\Gamma(k+1)$  times the residue of  $\frac{\mu}{2} \zeta \left( \frac{s}{2}, \frac{D_B}{\mu^2} \right)$  at  $s=k$  ( $k=1, \dots, d$ ).

2) In the general case ( $a_{d+1} \neq 0$ ), the exponential regularization shows - apart from polar singularities - a logarithmic divergence, with a coefficient that equals minus the residue of  $\frac{\mu}{2} \zeta \left( \frac{s}{2}, \frac{D_B}{\mu^2} \right)$  at  $s=-1$ . Moreover, the finite parts appearing in one and the other regularization scheme then differ by terms proportional to  $a_{d+1}$ . The difference between the finite part obtained through exponential regularization and the one obtained via  $\zeta$  is given by

$$-\frac{\mu}{2} \frac{a_{d+1}}{\sqrt{\pi}} \Psi(1) = \frac{\mu}{2} \frac{a_{d+1}}{\sqrt{\pi}} \gamma \tag{22}$$

where  $\gamma$  is the Euler-Mascheroni constant.

Before ending this section, it is worth pointing out that our results are, of course, valid in the case of a boundaryless manifold  $M$ . In this case, the conditions on the boundary problem reduce to the requirement that the operator  $D$  be self adjoint, with a positive definite principal symbol. The coefficients  $a_j$  then include only volume contributions, which vanish for  $j$  odd [6], [8].



## 4 A simple example: massive scalar field in a $d$ -dimensional box

As a simple example of the results just obtained we study, in this section, the Casimir energy of a massive scalar field through  $\zeta$  and exponential regularizations, and compare the results with the general predictions just made. We will consider the field to satisfy the Klein-Gordon equation

$$(\partial^2 + m^2) \varphi(x) = 0 \quad (23)$$

inside a  $d$ -dimensional spatial box of finite dimensions  $L_1, L_2, \dots, L_d$  ( $d \geq 1$ ). Moreover, periodic boundary conditions

$$\varphi(t, L_i) = \varphi(t, 0) \quad , \quad i = 1, \dots, d \quad (24)$$

will be imposed in each spatial direction. (Notice that this problem is equivalent to a boundaryless one).

After separation of variables, the modes of the field are easily seen to be given by the square roots of the eigenvalues of the  $d$ -dimensional Laplacian ( $D_{per}$ ).

$$\omega_{n_1 \dots n_d} = \left[ m^2 + \left( \frac{2n_1\pi}{L_1} \right)^2 + \dots + \left( \frac{2n_d\pi}{L_d} \right)^2 \right]^{1/2} \quad , \quad n_1, \dots, n_d \in Z \quad (25)$$

(In the massless case, the mode  $n_1 = \dots = n_d = 0$  must be excluded, since it gives no contribution to the Casimir energy).

A meromorphic extension for  $\Gamma\left(\frac{s}{2}\right) \zeta\left(\frac{s}{2}, \frac{D_{per}}{\mu^2}\right)$  can be obtained through Jacobi's inversion formula. Such extension is given by [3]

$$\begin{aligned} \Gamma\left(\frac{s}{2}\right) \zeta\left(\frac{s}{2}, \frac{D_{per}}{\mu^2}\right) &= \frac{L_1 \dots L_d}{\pi^{\frac{d}{2}}} \left(\frac{\mu}{2}\right)^d \left[ \left(\frac{m}{\mu}\right)^{-(s-d)} \Gamma\left(\frac{s-d}{2}\right) \right. \\ &+ 2 \left(\frac{\mu}{m}\right)^{\frac{s-d}{2}} \sum_{n_1=-\infty}^{\infty} \dots \sum_{n_d=-\infty}^{\infty} \left. \left[ \left(\frac{n_1 L_1 \mu}{2}\right)^2 + \dots + \left(\frac{n_d L_d \mu}{2}\right)^2 \right]^{\frac{s-d}{4}} \right] \\ &K_{\frac{d-s}{2}} \left( 2m \left( \left(\frac{n_1 L_1}{2}\right)^2 + \dots + \left(\frac{n_d L_d}{2}\right)^2 \right)^{\frac{1}{2}} \right) \end{aligned} \quad (26)$$

where the prime indicates that the term where all  $n_i = 0$  is to be omitted.

The last term in (26) is analytic in the hole  $s$  plane. The first one has poles at  $s = d - 2k$  ( $k = 0, 1, \dots$ ).

Thus, comparison with (9) shows that, in this case,  $a_j = 0$  for  $j$  odd, which is consistent with our comment at the end of the previous section. As regards  $a_{2k}$ , they can be easily seen to be given by

$$a_{j=2k} = \frac{(-1)^k}{k!} \left(\frac{\mu}{2}\right)^d \frac{L_1 \dots L_d}{\pi^{\frac{d}{2}}} \left(\frac{m}{\mu}\right)^{2k} \quad , \quad k = 0, 1, \dots \quad (27)$$

At this stage, some general conclusions can be drawn (for  $m \neq 0$ ), from our result in Section 3 :

- If the space is even dimensional, then  $a_{d+1} = 0$ ; the Casimir energy will be finite when calculated through  $\zeta$  function regularization, while the exponential regularization will show poles. Finite parts will then coincide.

- On the other hand, if  $d$  is odd,  $a_{d+1} \neq 0$ . Both regularizations will in this case present divergencies. These will show up as a pole at  $s = -1$  in the  $\zeta$ -regularized version, and a logarithmic singularity as well as poles in the exponential one. Finite parts will differ by (22). However, being the divergent terms proportional to the volume of the box, they can be subtracted through a

physically meaningful prescription ( $E \xrightarrow{L_1 \dots L_d \rightarrow \infty} 0$ ) [1],[3][13]. This same

prescription leaves finite results which are coincident; it is in this sense that equivalence between both regularizations is to be understood in this particular example.

It should be remarked here that, being all  $a_j$  ( $j \neq 0$ ) proportional to positive powers of the mass, the massless case is particular : in such case, the  $\zeta$  function will only present a pole at  $s = d$ , and the  $\zeta$ -regularized Casimir energy will thus be finite in any dimension, while the exponential regularization will only show a pole of order  $d + 1$ , both finite parts being coincident.

#### 4.1 Casimir energy for $d = 1$

From (21) and (26), the  $\zeta$ -regularized Casimir energy turns out to be

$$E_{\zeta}^{(1)} = -\frac{m}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} K_1(nmL) + \frac{L\mu^2}{4\sqrt{\pi}} \left( \frac{m}{\mu} \right)^{1-s} \frac{\Gamma\left(\frac{s-1}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \Bigg|_{s=-1} \quad (28)$$

which is divergent at  $s = -1$  as already discussed.

By developing the last term around  $s = -1$ , we get

$$E_{\zeta}^{(1)} = -\frac{m}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} K_1(nmL) + \frac{m^2 L}{4\pi} \left( \frac{1}{s+1} \Bigg|_{s=-1} - \ln\left(\frac{m}{2\mu}\right) - \frac{1}{2} \right) \quad (29)$$

As regards exponential regularization

$$E_{\text{exp}}^{(1)} = -\frac{\mu}{2} \frac{d}{dt} \left( \sum_{n=-\infty}^{\infty} e^{-t\left(\left(\frac{2n\pi}{L\mu}\right)^2 + \left(\frac{m}{\mu}\right)^2\right)^{\frac{1}{2}}} \right) \Bigg|_{t=0} \quad (30)$$

The series can be evaluated making use of Poisson's summation formula (see

Appendix 1) and, after differentiating, one gets

$$E_{\text{exp}}^{(1)} = -\frac{m}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} K_1(nmL) + \frac{m^2 L}{4\pi} \left( -\ln(t) - \ln\left(\frac{m}{2\mu}\right) + 2\left(\frac{mt}{\mu}\right)^{-2} - \gamma - \frac{1}{2} \right) \Big|_{t=0} \quad (31)$$

Comparison among the various coefficients (both in divergent as well as in finite parts) in (29) and (31) shows a complete agreement with our results in

Section 3. When the prescription  $E^{(1)} \xrightarrow{L \rightarrow \infty} 0$  is imposed, the physically meaningful Casimir energy turns out to be

$$E_{\text{Cas}}^{(1)} = -\frac{m}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} K_1(nmL) \quad (32)$$

in the framework of both regularization schemes. This remaining finite result can be easily seen to decay exponentially with  $L$ . On the other hand, divergences as well as finite parts proportional to  $L$  have been subtracted as a consequence of the forementioned prescription, which amounts to adding a "constant" to the energy density.

## 4.2 Casimir energy for $d = 2$

Again, from (21) and (26), the  $\zeta$ -regularized Casimir energy is given by

$$\begin{aligned} E_{\zeta}^{(2)} &= -\frac{L_1 L_2}{2^3 \pi^{\frac{3}{2}}} \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} ' \left( \sqrt{\left(\frac{n_1 L_1}{2}\right)^2 + \left(\frac{n_2 L_2}{2}\right)^2} \right)^{-\frac{3}{2}} \\ &\quad K_{\frac{3}{2}} \left( 2m \sqrt{\left(\frac{n_1 L_1}{2}\right)^2 + \left(\frac{n_2 L_2}{2}\right)^2} \right) \\ &\quad + \frac{\mu^3}{2^3} \frac{L_1 L_2}{\pi} \left( \frac{m}{\mu} \right)^{-(s-2)} \frac{\Gamma\left(\frac{s-2}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \Big|_{s=-1} = \\ &= -\frac{L_1 L_2}{2^3 \pi^{\frac{3}{2}}} \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} ' \left( \sqrt{\left(\frac{n_1 L_1}{2}\right)^2 + \left(\frac{n_2 L_2}{2}\right)^2} \right)^{-\frac{3}{2}} K_{\frac{3}{2}} \left( 2m \sqrt{\left(\frac{n_1 L_1}{2}\right)^2 + \left(\frac{n_2 L_2}{2}\right)^2} \right) \\ &\quad - \frac{L_1 L_2 m^3}{12\pi} \end{aligned} \quad (33)$$

The exponentially regularized Casimir energy is

$$E_{\text{exp}}^{(2)} = -\frac{\mu}{2} \frac{d}{dt} \left( \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} e^{-t \left( \left( \frac{2n_1\pi}{L_1\mu} \right)^2 + \left( \frac{2n_2\pi}{L_2\mu} \right)^2 + \left( \frac{m}{\mu} \right)^2 \right)^{\frac{1}{2}}} \right) \Big|_{t=0} \quad (34)$$

The double series can again be calculated by repeated use of Poisson's formula (see Appendix 2). After differentiating, we get

$$E_{\text{exp}}^{(2)} = -\frac{L_1 L_2}{2^3 \pi^{\frac{3}{2}}} \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \left( \sqrt{\left( \frac{n_1 L_1}{2} \right)^2 + \left( \frac{n_2 L_2}{2} \right)^2} \right)^{-\frac{3}{2}} K_{\frac{3}{2}} \left( 2m \sqrt{\left( \frac{n_1 L_1}{2} \right)^2 + \left( \frac{n_2 L_2}{2} \right)^2} \right) - \frac{L_1 L_2 m^3}{12\pi} + \frac{L_1 L_2 \mu^3}{2\pi t^3} \Big|_{t=0} \quad (35)$$

As predicted, the  $\zeta$  regularization gives a finite result, which coincides with the finite part in the exponential regularization. This last regularization presents a single pole of order  $d + 1 = 3$ , whose coefficient coincides with  $\Gamma(3)$  times the residue of  $E_{\zeta}^{(2)}$  at  $s = 2$  as expected. After applying the prescription

$E^{(2)} \rightarrow 0$  the divergence is eliminated, and a finite piece -proportional  $L_1 L_2 \rightarrow \infty$  to the volume- is also discarded; we thus get

$$E_{\text{Cas}}^{(2)} = -\frac{L_1 L_2}{2^3 \pi^{\frac{3}{2}}} \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \left( \sqrt{\left( \frac{n_1 L_1}{2} \right)^2 + \left( \frac{n_2 L_2}{2} \right)^2} \right)^{-\frac{3}{2}} K_{\frac{3}{2}} \left( 2m \sqrt{\left( \frac{n_1 L_1}{2} \right)^2 + \left( \frac{n_2 L_2}{2} \right)^2} \right) \quad (36)$$

which, as in the  $d = 1$  case, decays exponentially when the volume increases.

## 5 Conclusions

In this paper, we have studied the connection between  $\zeta$  and cutoff regularizations of Casimir energies for scalar fields in a space-time  $R \times M$ , with  $M$  a  $d$ -dimensional manifold with or without boundary.

Under fairly general conditions on the associated boundary problem (which are those of physical interest), we have shown that, in general, both regularizations lead to divergent terms. These divergencies appear as a simple pole when regularizing via  $\zeta$ , and are logarithmic as well as polar in the exponential

regularization. Moreover, finite parts do not in general coincide. We have also determined the precise relationship among the various coefficients appearing in one case and the other.

As an example of application, we have evaluated Casimir energies for a scalar field in a  $d$ -dimensional box, under periodic boundary conditions. In this particular example, the  $\zeta$  function turns out to be finite, in the massive case, for  $d$  even and, in the massless case, for any dimension. For whatever dimension, both regularizations have been shown to be equivalent once the same prescription

( $E_C \rightarrow 0$ , with  $V$  the volume of the box) is imposed to eliminate  $V \rightarrow \infty$

infinities. We have performed the calculation of the energy with exponential regularization, and we have verified the agreement with our general result in the cases  $d = 1$  and  $2$ . Although we haven't found the energy via Poisson's formula for  $d > 2$  (the process becomes increasingly tedious as the space dimension grows up), it is possible, by using the relationships among coefficients determined in Section 3, to obtain the exact result of the exponential regularization for whatever dimension, from the energy obtained via zeta function.

The extension of these results to fields with other spins is at present under study.

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## Appendix 1- Poisson sum for $d = 1$

In this appendix, we derive (31) for the exponentially regularized Casimir energy, making use of Poisson's formula :

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{p=-\infty}^{\infty} c_p \quad (37)$$

with

$$c_p = \int_{-\infty}^{\infty} dx e^{2\pi i p x} f(x) \quad (38)$$

When applied to the calculation of  $h\left(t, \frac{D_B}{\mu^2}\right)$ , it gives

$$h\left(t, \frac{D_B}{\mu^2}\right) = \sum_{n=-\infty}^{\infty} e^{-t\left(\left(\frac{2n\pi}{L\mu}\right)^2 + \left(\frac{m}{\mu}\right)^2\right)^{\frac{1}{2}}} = \sum_{p=-\infty}^{\infty} c_p(t) \quad (39)$$

where

$$\begin{aligned}
c_p(t) &= \int_{-\infty}^{\infty} dx e^{2\pi i p x - t \left( \left( \frac{2n\pi}{L\mu} \right)^2 + \left( \frac{m}{\mu} \right)^2 \right)^{\frac{1}{2}}} = \\
&= \frac{L\mu}{\pi} \int_0^{\infty} dx \cos(L\mu p x) e^{-t \left( x^2 + \left( \frac{m}{\mu} \right)^2 \right)^{\frac{1}{2}}} = \\
&= \frac{mL}{\pi \sqrt{1 + \left( \frac{L\mu p}{t} \right)^2}} K_1 \left( \frac{m}{\mu} \sqrt{t^2 + (L\mu p)^2} \right)
\end{aligned} \tag{40}$$

By replacing (40) into (39) and taking the derivative, we get

$$-\frac{\mu}{2} \frac{d}{dt} h \left( t, \frac{D_B}{\mu^2} \right) = -\frac{\mu}{2} \frac{d}{dt} \left( 2 \sum_{p=1}^{\infty} c_p(t) + c_0(t) \right) \tag{41}$$

Recurrence relations and ascending series for modified Bessel functions then give (31).

## Appendix 2- Poisson sums for $d = 2$

In order to derive (35), we make repeated use of Poisson's formula, as given by (37) and (38).

In this case we have

$$h \left( t, \frac{D_B}{\mu^2} \right) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} e^{-t \left( \left( \frac{2n_1\pi}{L_1\mu} \right)^2 + \left( \frac{2n_2\pi}{L_2\mu} \right)^2 + \left( \frac{m}{\mu} \right)^2 \right)^{\frac{1}{2}}} \tag{42}$$

We first perform the sum over  $n_2$  in the same fashion as in Appendix 1, to obtain

$$\begin{aligned}
&h \left( t, \frac{D_B}{\mu^2} \right) = \\
&\sum_{n_1=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} \frac{L_2\mu t}{\pi} \frac{\sqrt{\left( \frac{2n_1\pi}{L_1\mu} \right)^2 + \left( \frac{m}{\mu} \right)^2}}{\sqrt{t^2 + (L_2\mu p)^2}} K_1 \left( \sqrt{\left[ \left( \frac{2n_1\pi}{L_1\mu} \right)^2 + \left( \frac{m}{\mu} \right)^2 \right] \left[ t^2 + (L_2\mu p)^2 \right]} \right)
\end{aligned} \tag{43}$$

Now, due to the convergence properties of the double sum, the summation order can be interchanged, and Poisson's formula can again be used to obtain

$$h \left( t, \frac{D_B}{\mu^2} \right) = \sum_{p=-\infty}^{\infty} \frac{L_2\mu}{\pi} \frac{t}{\sqrt{t^2 + (L_2\mu p)^2}} \sum_{k=-\infty}^{\infty} c_k(t) \tag{44}$$

with

$$c_k(t) = \frac{L_1\mu}{\sqrt{2\pi}} \sqrt{t^2 + (L_2\mu p)^2} \left(\frac{m}{\mu}\right)^{\frac{3}{2}} \left[ t^2 + (L_2\mu p)^2 + (L_1\mu k)^2 \right]^{-\frac{3}{4}} K_{\frac{3}{2}} \left( \frac{m}{\mu} \sqrt{t^2 + (L_2\mu p)^2 + (L_1\mu k)^2} \right) \quad (45)$$

Again, (44) can be differentiated term by term in a straightforward although tedious calculation, and recurrence formulas for modified Bessel functions can be used to get

$$\begin{aligned} -\frac{\mu}{2} \frac{d}{dt} h \left( t, \frac{D_F}{\mu^2} \right) \Big|_{t=0} &= -\frac{\mu}{2} \frac{d}{dt} \left( \sum_{p=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \frac{L_2\mu}{\pi} \frac{t}{\sqrt{t^2 + (L_2\mu p)^2}} c_k(t) + \frac{L_2\mu}{\pi} c_0(t) \right) \Big|_{t=0} \\ &= -\frac{L_1 L_2}{2^3 \pi^{\frac{3}{2}}} \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \left( \sqrt{\left(\frac{n_1 L_1}{2}\right)^2 + \left(\frac{n_2 L_2}{2}\right)^2} \right)^{-\frac{3}{2}} K_{\frac{3}{2}} \left( 2m \sqrt{\left(\frac{n_1 L_1}{2}\right)^2 + \left(\frac{n_2 L_2}{2}\right)^2} \right) \\ &\quad - \frac{L_1 L_2 m^3}{12\pi} + \frac{L_1 L_2 \mu^3}{2\pi t^3} \Big|_{t=0} \end{aligned} \quad (46)$$

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