Renormalization of the Hamiltonian and a geometric interpretation of asymptotic freedom

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Using a novel approach to renormalization in the Hamiltonian formalism, we study the connection between asymptotic freedom and the renormalization group flow of the configuration space metric. It is argued that in asymptotically free theories the effective distance between configuration decreases as high momentum modes are integrated out. [S0556-2821(99)00320-3]

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I. INTRODUCTION

Looking at the great success that the standard model has had since it was introduced more than 20 years ago, it is quite striking that we still lack understanding of the strong interaction part in the low energy regime. In spite of excellent results in the perturbative QCD, we are unable to produce any analytical computation of quantities such as the magnetic moment of the proton which is known with great accuracy. Some important puzzles, such as where Λ_{QCD} comes from, still need to be addressed. Apart from the numerous lattice results (which reinforce our belief that QCD is the right theory for the strong interactions), the situation remains largely the same more than 20 years later. Many attempts to apply a variational approach have not yet produced any effective calculational methods for the solution of the problems mentioned before.

It was suggested in [1] that perhaps we need an alternative way to look at the Yang-Mills (YM) theory based on a more geometrical point of view. Namely, one tries to study quantum mechanics of the fields in the space of gauge-inequivalent configurations. The following analogy with quantum mechanics is used. Consider a free particle inside a box of size L. The lowest eigenvalue of the Hamiltonian (which is just a Laplacian) is of the order of $\sim 1/L^2$ This is realized by the state of the longest possible wavelength, $\lambda \sim L$. It is clear therefore, that the spectrum of such a system will have a gap due to the fact that L is finite. In other words, the spectrum is going to be discrete as long as distances in the configuration space cannot become arbitrarily large. Feynman's suggestion was to try to adapt this idea to Yang-Mills theory.

In general, in order to determine the distance between field configurations we need to know the metric of the configuration space. The geometry of the configuration space of non-Abelian gauge theories was considered by many authors [2–4] and, recently, in [5]. The major problem is to extract the metric on a space of the gauge-inequivalent configurations $\mathcal{A}/\mathcal{G}$.

A natural distance between two arbitrary gauge configurations A_1 and A_2 (that is gauge connections *modulo* gauge transformations) is given by [5]

$$\|A_1 - A_2\|^2 = inf\left(\int_x \mathrm{Tr}[A_1^g(x) - A_2(x)]^2 \right)\Bigg|_g , \qquad (1)$$

where A_1^g is a gauge-transformed A_1. However the extremum solution of this expression is highly nonlocal and very difficult to work with. Nevertheless one can try to see some qualitative features of this distance. In [1] it was argued that due to the non-Abelian nature of the gauge group, the low-potential-energy configurations are in a space of finite diameter for (2+1)-dimensional Yang-Mills theory. According to our very crude analogy with the quantum mechanics it would mean that the kinetic energy operator will have a discrete spectrum. Of course, these are only qualitative arguments that require a rigorous mathematical formulation if one is to draw any conclusion on the existence and value of the mass gap. Also, as a word of caution we must say that some properties of the configuration space obtained in 2+1 dimensions may not necessarily be the same as in the (3+1)-dimensional case [such as the statement above in the case of YM theory in (3+1)- and (1+1)-dimensional sigma models [5,6]].

Another reason to believe that this approach could lead us to a better understanding is the recent progress in Hamiltonian formulation of the (2+1)-dimensional Yang-Mills theory. In a series of papers [7,8] it was shown that by introducing special gauge-invariant variables one can prove that the (properly regularized) volume of the configuration space for the non-Abelian theory is finite, while the corresponding quantity for the Abelian field is infinite. The discreteness of the spectrum of the kinetic term, E^2, has been also shown explicitly and the string tension computed in [8] is in remarkable agreement with the recent Monte Carlo simulations [9].

Behind all this discussion a natural question arises: What is the behavior of the configuration space as we integrate out high momenta degrees of freedom? Of course we cannot

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answer this question completely, and even a partial response deserves a profound, probably nonperturbative, analysis. However we still can say something, and answer specific matters as to the behavior of the configuration space *metric* under the renormalization group, which encodes much of the properties of the space. Notice that those issues acquire a special significance in Hamiltonian formalism: there the kinetic term is essentially a Laplacian in configuration space and the energy spectrum, at least in the strong coupling regime, is dominated by it.

The choice of this topic was not fortuitous but we have an idea in mind. In the standard understanding of asymptotic freedom all the significance is put in the interaction potential through the statement that the couplings decrease to zero as the energy at which the theory is tested increases to infinity. However in the Hamiltonian picture we can state the problem in a different way. In the Hamiltonian there is an obvious competition between the kinetic energy and the potential energy. So it is natural to analyze the asymptotic freedom in its ''dual'' form, i.e., the variation of the kinetic energy (with respect to potential energy) as the renormalization scale changes. In particular we conjectured the following, *in asymptotically free theories the effective distance between configurations decreases as the momentum is lowered.*

Notice that this question does only makes sense in a Hamiltonian formalism, since it is only in this case that all the geometrical properties of the configuration space can be precisely defined (the Hamiltonian is an operator that acts on the Hilbert space of functionals defined on the configuration space). It can be argued that one can always work in a Lagrangian formalism and at the end construct the respective Hamiltonian by the standard Legendre transformation. However this *cannot* be done if renormalization is involved at any moment. In the Lagrangian formalism, even in the framework of perturbation theory, the process of renormalization generates high time-derivative terms in the Lagrangian, making impossible even the very definition of a Hamiltonian, as the system is governed by high order time-derivatives equations of motion. Even an application of the Ostrogradsky method can in general lead to negative norm states if higher time derivatives are involved in the Lagrangian. Thus, a prescription for renormalization within the Hamiltonian formalism is indispensable. One must not leave the phase space, thereby maintaining the first order time derivative nature of the equations of the system.

Therefore we will use a novel procedure of renormalization of Hamiltonians, already introduced in Ref. [10], where a successful renormalization of the Hamiltonian of $\lambda\phi^4$ was performed. This method relies on the successive diagonalization of the Hamiltonian by performing iterative unitary transformations and subsequent projection onto the Hilbert space of low energy modes. It resembles, in spirit, the renormalization approaches of Glazek and Wilson [11], Wegner [12] and several other authors [13–16], though in practice appears very different. To support our conjecture we have done a detailed analysis of two particular examples: quantum electrodynamics and Yang-Mills theory in 3+1 dimensions, where we constructed the renormalized Hamiltonian up to one loop.

The paper is organized as follows: In Sec. II we briefly present our renormalization group technique for Hamiltonians. In Secs. III, IV, and V we compute the one-loop renormalization of the SU(N) Yang-Mills Hamiltonian. In Sec. VI we did the same with quantum electrodynamics. In Sec. VII we discuss the relation between renormalization of the metric and asymptotic freedom. Finally, Sec. VIII contains some conclusions.

II. RENORMALIZATION IN THE HAMILTONIAN FORMALISM

Let us review in this section the formalism of Hamiltonian renormalization introduced in Ref. [10]. Consider a system described by Hamiltonian H which has already some large cutoff Λ built into it; that is the system is ultraviolet-finite from the very beginning. We are assuming that the Hamiltonian is written in terms of renormalized fields and couplings up to the scale Λ and incorporates all the renormalization $Z(\Lambda)$ factors.

Now if μ is some intermediate scale, $\mu<\Lambda$, we want to find the effective Hamiltonian $H(\mu)$ that has the same low energy spectrum as the original Hamiltonian. More precisely, we want to find the operator that has the same spectrum as the original Hamiltonian when projected onto the Hilbert subspace generated by the excitations with frequency less than μ:

$$H(\mu)=\mathcal{P}_{low}H(\Lambda)\mathcal{P}_{low}, \tag{2}$$

where $\mathcal{P}_{low}$ is the projector onto the low frequency subspace.

We will show that, in the framework of perturbation theory, it is possible to partially diagonalize the Hamiltonian $H(\Lambda)$ and construct the vacuum state for the high frequency modes. Thus, the low energy effective Hamiltonian would take the form

$$H(\mu)=\langle 0_{high}|U^{\dagger}(\Lambda,\mu)H(\Lambda)U(\Lambda,\mu)|0_{high}\rangle. \tag{3}$$

Notice that it is not necessary to diagonalize completely the Hamiltonian as we only need to identify the low energy states and not the whole spectrum.

Suppose that after a unitary transformation we bring a Hamiltonian to the form

$$H(\text{low})+H_{\text{free}}(\text{high})+V, \tag{4}$$

where $H(\text{low})$ contains only low frequency operators, $H_{\text{free}}(\text{high})$ is a free Hamiltonian for the high frequency modes and V has the special form

$$V=\sum_{k,p>\mu} a_k^{\dagger}S_{kp}(\mu,a^{\dagger},a)a_p \tag{5}$$

with $S_{kp}(\mu,a^{\dagger},a)$ an arbitrary operator of low and high frequency modes. Thus, using standard Rayleigh-Schroedinger perturbation theory it follows that the correction to an arbitrary state $|n\rangle$ is given by

$$|\delta n\rangle = \sum_{l\neq n} \frac{\langle l|V|n\rangle}{(E_l - E_n)}|l\rangle + \sum_{l,n\neq m} \frac{\langle l|V|m\rangle\langle m|V|n\rangle}{(E_l - E_m)(E_m - E_n)}|l\rangle + \ldots . \quad (6)$$

Therefore, if $|n\rangle$ is the vacuum state of $H_{\text{free}}(\text{high})$, it is annihilated by V and there is no correction to it at any order in perturbation theory.

Now we will show how to find the unitary transformation $U(\mu,\Lambda)$ that transforms the Hamiltonian into the form (4). First we split the Hamiltonian in four pieces:

$$H = H_1 + H_2 + V_A + V_B . \quad (7)$$

Here H_1 contains only the modes with $k<\mu$, H_2 is the *free* part for all the modes with $k>\mu$, V_A contains the ''pure'' terms that have only high frequency creation operators or high frequency annihilation operators, but not both, and V_B the remaining terms containing at least one annihilation *and* one creation operator of high energy modes (we assume here that V_A and V_B are normal-ordered with respect to the free high frequency vacuum). Notice that the term V_B is already in the form (5). Then, according to the preceding discussion, the objective of the renormalization procedure is to ''eliminate'' the terms in V_A that spoils the form (4), (5).

We will proceed iteratively: we write the unitary operator in the form $U = U_0 U_1 U_2 \ldots$, and compute each U_n perturbatively. Each factor U_n will bring the Hamiltonian to the form (4) at a given order in μ/Λ. Therefore, there will be two expansion parameters in this procedure: one is the coupling constant of the theory λ and the other is the ratio μ/Λ (effectively $\omega_{low}/\omega_{high}$).

Let us parametrize U_n as $\exp(i\Omega_n)$, where Ω_n is a Hermitian operator to be determined. We start from Eq. (7) and perform a first unitary transformation expanding in powers of Ω (in the general case Ω is at least of order λ so at a given order only a finite number of terms are needed):

$$e^{-i\Omega_0}(H_1 + H_2 + V_A + V_B)e^{i\Omega_0} = H_1 + H_2 + V_A + V_B + i[H_1,\Omega_0] + i[H_2,\Omega_0] + i[V_A,\Omega_0] + i[V_B,\Omega_0]\cdots . \quad (8)$$

We want to eliminate the term V_A which is of the first order in λ, so Ω_0 has to be of the same order. Furthermore we want to generate an expansion in μ/Λ, so we impose the following condition on Ω_0:

$$i[H_2,\Omega_0] + V_A = 0. \quad (9)$$

This equation can be solved perturbatively and since commutators with H_2 generate time derivatives we have the desired expansion.

Notice that Eq. (6) defines Ω_0 up to the terms that commute with H_2. As it is shown in Appendix A this freedom corresponds to the arbitrariness of the definition of the low energy Hamiltonian up to a unitary transformation. Therefore we will assume some kind of ''minimal'' scheme; namely, that Ω_0, after normal ordering, does not have a part that commutes with H_2 (terms that are functions of $a_k^\dagger a_k$).

After Ω_0 is chosen to cancel V_{12} in the effective Hamiltonian a new mixing term of order λ has appeared from $i[H_1,\Omega_0]$. Notice however that this new term is of higher order in μ/Λ and will be eliminated by the next unitary transformation $U_1 = e^{i\Omega_1}$. Explicitly,

$$e^{-i\Omega_1}e^{-i\Omega_0}(H_1 + H_2 + V_A + V_B)e^{i\Omega_0}e^{i\Omega_1} = H_1 + H_2 + V_B + i[H_1,\Omega_0] + i[H_2,\Omega_1] + i[H_1,\Omega_1] + \text{second order terms} \ldots \quad (10)$$

and we now choose Ω_1 so that

$$i[H_1,\Omega_0] + i[H_2,\Omega_1] = 0. \quad (11)$$

Using Eqs. (10) and (11) we obtain

$$e^{-i\Omega_1}e^{-i\Omega_0}(H_1 + H_2 + V_{12})e^{i\Omega_0}e^{i\Omega_1} = H_1 + H_2 + V_B + \frac{i}{2}[V_A,\Omega_0] + i[V_B,\Omega_0] + i[V_B,\Omega_1] + i[H_1,\Omega_1] - \frac{1}{2}[[H_1,\Omega_0],\Omega_0] - \frac{1}{2}[[H_1,\Omega_0],\Omega_1] - \frac{1}{2}[[H_1,\Omega_1],\Omega_1] + \text{higher order terms} \ldots . \quad (12)$$

The obvious next step is to introduce Ω_2 in order to cancel $i[H_1,\Omega_1]$ and continue with the same process. Then a simple question emerges: Where should we stop? To answer this question we have to consider the divergence properties of the terms introduced by each new Ω into the H_{eff}. These contributions like, for example, $-\frac{1}{2}[[H_1,\Omega_0],\Omega_0]$ may diverge as $\Lambda\to\infty$. Nevertheless, the degree of divergence of each new term will be smaller as we introduce more and more Ω_n factors. In general, the following power counting can be used: since Ω_n is determined from

$$[H_2,\Omega_{n+1}] = -[H_1,\Omega_n] \quad (13)$$

and $H_2 \approx \Lambda, H_1 \approx \mu$, then

$$\Omega_{n+1} \approx \frac{\mu}{\Lambda}\Omega_n . \quad (14)$$

Therefore the next Ω will be less divergent and eventually all the new terms introduced by this prescription will be convergent starting from certain n. At this point we will stop since for our purposes we are only interested in divergent contributions.

Of course so far we have only eliminated the high momentum degrees of freedom up to the first order in the coupling constant. Requiring the absence of the λ^2-order mixing terms will lead to the introduction of a whole new series of unitary transformations, and the same arguments can be applied to them.

Finally, the actual process of renormalization is performed by choosing the renormalization $Z(\Lambda)$ factors of the original Hamiltonian to cancel the divergent contributions coming from evaluation of Eq. (3). We now turn to the explicit computation for QED and QCD Hamiltonians.

III. SU(N) YANG MILLS THEORY: PRELIMINARIES

The kinetic energy term for the Hamiltonian of Yang-Mills theory is essentially a Laplace operator on the configuration space. However, due to the gauge invariance, the physical degrees of freedom belong to the space of gauge connections modulo the group of gauge transformations, i.e, the space of nonequivalent gauge potentials, and a satisfactory parametrization of this space is needed. In this section we find a perturbatively adequate coordinate system of the configuration space (a similar analysis was done in Ref. [17]) and compute the associated metric.

We consider a SU(N) Yang-Mills theory in the temporal gauge, $A_0=0$. The canonical variables are the vector potential $A_i^a(x)$ and the electric field $E^{ai}(x)$. They satisfy canonical commutation relations

$$[A_i^a(x),E^{bj}(y)]=i\delta^{ab}\delta_i^j\delta^{(3)}(x-y) \tag{15}$$

which permit the representation of the electric field as

$$E^{ai}(x)=-i\frac{\delta}{\delta A_i^a(x)}. \tag{16}$$

The Hilbert space of the theory is supplemented by the Gauss law that enforces a constraint on the wave functionals, essentially only allowing gauge invariant configurations,

$$\mathcal{G}^a(x)\Psi[A]=-iD[A]_i^{ab}\frac{\delta}{\delta A_i^b(x)}\Psi[A]=0, \tag{17}$$

where $D[A]_i^{ab}=\nabla_i\delta^{ab}-ef^{abc}A_i^c(x)$ is the covariant derivative.

The gauge potential can be written also as a Lie algebra valued field $A_i(x)=t^aA_i^a(x)$ where t^a are the Hermitian generators of the Lie algebra of SU(N) in the fundamental representation, normalized to $\mathrm{tr}(t^at^b)=\frac{1}{2}\delta^{ab}$.

The Hamiltonian can be written as

$$H=\frac{1}{2}\int d^3x[E^{ai}(x)E^{ai}(x)+B^{ai}(x)B^{ai}(x)], \tag{18}$$

where we used the magnetic field B_i^a:

$$B^{ai}(x)=\epsilon_{ijk}[\partial_jA_k^a(x)+\tfrac{1}{2}ef^{abc}A_j^b(x)A_k^c(x)]. \tag{19}$$

Note that due to the constraint (17), not all the degrees of freedom in the Hamiltonian (18) are ''physical.'' In order to isolate the physical degrees of freedom we will perform a change of coordinates in such a way that the Gauss law takes its simplest form. We parametrize an arbitrary configuration in terms of a ''gauge fixed'' configuration plus gauge transformations. The former will define a coordinate system of the orbit space. We write

$$A_i(x)=g^{-1}(x)\mathcal{A}_i(x)g(x)+ig^{-1}(x)\partial_ig(x), \tag{20}$$

where g is a SU(N)-valued matrix and $\mathcal{A}_i$ is a configuration satisfying the Coulomb gauge condition

$$\nabla_i\mathcal{A}_i=0. \tag{21}$$

It is well known that the parametrization (20) with condition (21) is not well defined globally due to the Gribov ambiguity problem. However, we will work in the framework of perturbation theory where the parametrization (20) and (21) defines an acceptable isomorphism in the configuration space.

From Eq. (20) we deduce,

$$\delta A_i=g^{-1}(\delta\mathcal{A}_i+[D[\mathcal{A}]_i,i\delta gg^{-1}])g \tag{22}$$

where $D[\mathcal{A}_i]=\partial_i-ie\mathcal{A}_i$.

Thus we can decompose the metric of vector valued configurations in''gauge fixed'' and ''pure gauge'' parts:

$$\begin{aligned}\delta s^2&=\int d^dx\,\delta A_i^a(x)\,\delta A_i^a(x)\\&=\int d^3x[\delta\mathcal{A}_i^a(x)\,\delta\mathcal{A}_i^a(x)\\&\quad+D[\mathcal{A}]_i^{ab}(i\delta gg^{-1})^bD[\mathcal{A}]_i^{ac}(i\delta gg^{-1})^c].\end{aligned} \tag{23}$$

The normalization of the wave functionals is then given by

$$\begin{aligned}\langle\Psi_1|\Psi_2\rangle&=\frac{1}{Vol\,G}\int[DA_i^a]\Psi_1^*[A]\Psi_2[A]\\&=\int[D\mathcal{A}_i^a]\sqrt{G}\Psi_1^*[\mathcal{A}]\Psi_2[\mathcal{A}]\end{aligned} \tag{24}$$

where G is the matrix metric defined by Eq. (23).

Using the Coulomb gauge condition Eq. (21) we can invert Eq. (22) and write

$$i(\delta gg^{-1})^a=(\vec{\nabla}\cdot\vec{D}[\mathcal{A}])_b^{-1a}\vec{\nabla}\cdot(\delta\vec{A}R^{-1})^b \tag{25}$$

and

$$\delta\mathcal{A}_i^a=[\delta_{ij}\delta^{ab}-D_i^{ac}[\mathcal{A}](\vec{\nabla}\cdot\vec{D}[\mathcal{A}])_a^{-1c}\nabla_j](\delta A_jR^{-1})^b, \tag{26}$$

where $R\equiv R(g)$ is the adjoint representation representative of g.

Notice that not all the components of $\mathcal{A}_i$ are independent as they are subject to condition (21), so we can parametrize the space of gauge configurations modulo gauge transformations with the $2(N^2-1)$ functions $\mathcal{A}_{\hat{\imath}}^a$, $\hat{\imath}=1,2$.

From Eqs. (25) and (26) we can write the functional derivative as

$$\frac{\delta\Psi}{\delta A_i^a(x)} = \int R^{-1\,ad}(R_y^{-1}\nabla_i^x(\vec{D}\cdot\vec{\nabla})^{-1}_{(y,x)})^{bd} \times \frac{\delta\Psi}{i(g^{-1}\delta g)^b(y)} + \int R^{-1\,ad}\{\delta_{ji}\delta^{db}\delta(x,y) + [D[\mathcal{A}]_j^y\nabla_i^x(\vec{D}\cdot\vec{\nabla})^{-1}_{(y,x)}]^{bd}\}P_{ji}(y,z)\frac{\delta\Psi}{\delta\mathcal{A}_i^b(z)}, \tag{27}$$

where $P_{ij}(x,y)=\delta_{ij}\delta(x,y)-\nabla_i\nabla_j\triangle^{-1}(x,y)$ is the projector on the transverse modes.

As previously announced, in these variables the Gauss law has the simple form

$$\frac{\delta\Psi[A]}{(i\,\delta g g^{-1})^b}=0 \tag{28}$$

so it is trivially imposed by demanding that the wave functionals be independent of g.

It is useful to define a ''transverse'' functional derivative as

$$\frac{\delta^T}{\delta\mathcal{A}_i^a(x)} = \int d^3z P_{ij}(x,z)\frac{\delta}{\delta\mathcal{A}_j^a(z)} \tag{29}$$

restoring rotational invariance at the expense of modifying the canonical commutation relations:

$$\left[\mathcal{A}_i^a(x), \frac{\delta^T}{\delta\mathcal{A}_j^b(y)}\right] = -P_{ij}(x,y)\,\delta^{ab}. \tag{30}$$

Now we can write the kinetic energy term in terms of the variables $\mathcal{A}_i^a$ [taking into account Eq. (29)],

$$\begin{aligned}\langle\Psi_1|T|\Psi_2\rangle &= \frac{1}{Vol\,G}\int[D\mathcal{A}_i^a]\frac{1}{2}\int d^3x\frac{\delta\Psi_1^*}{\delta\mathcal{A}_i^a(x)}\frac{\delta\Psi_2}{\delta\mathcal{A}_i^a(x)}\\ &= \int[D\mathcal{A}_i^a]\sqrt{\det G}\,\frac{1}{2}\\ &\quad\times\int d^3x d^3y d^3z\{\delta^{ij}\delta^{ab}\delta(x,y)\\ &\quad+[D[\mathcal{A}]_j^y\nabla_i^x(\vec{D}\cdot\vec{\nabla})^{-1}_{(y,x)}]^{ba}\}\\ &\quad\times\frac{\delta^T\Psi_1^*}{\delta\mathcal{A}_j^b(y)}\{\delta^{ik}\delta^{ac}\delta(x,z)\\ &\quad+[D[\mathcal{A}]_k^z\nabla_i^x(\vec{D}\cdot\vec{\nabla})^{-1}_{(z,x)}]^{ca}\}\frac{\delta^T\Psi_2}{\delta\mathcal{A}_k^c(z)}.\end{aligned} \tag{31}$$

That is, the kinetic energy density term takes the form

$$\mathcal{T}=-\frac{1}{2\sqrt{G}}\mathcal{E}^{ia}(x)\sqrt{G}G^{(ia)(jb)}_{\mathcal{A}/\mathcal{G}}(x,y)\mathcal{E}^{jb}(y), \tag{32}$$

where $\mathcal{E}^{ia}=-i(\delta^T/\delta\mathcal{A}_i^a)$ and $G_{\mathcal{A}/\mathcal{G}}$ is the effective metric of the space of gauge configurations modulo gauge transformations,

$$\begin{aligned}G^{(ia)(jb)}_{\mathcal{A}/\mathcal{G}}(x,y) &= \int d^3z\{\delta^{ik}\delta^{ac}\delta(x,z)\\ &\quad+[D[\mathcal{A}]_i^x\nabla_k^z(\vec{D}\cdot\vec{\nabla})^{-1}_{(x,z)}]^{ac}\}\{\delta^{jk}\delta^{bc}\delta(y,z)\\ &\quad+[D[\mathcal{A}]_j^y\nabla_k^z(\vec{D}\cdot\vec{\nabla})^{-1}_{(y,z)}]^{bc}\}.\end{aligned} \tag{33}$$

As we mentioned above, the previous analysis was only valid in the framework of perturbation theory where condition (21) defines a local system of coordinates on the orbit space. So it is consistent with this approach to compute all the elements of $\mathcal{T}$ (the metric of the space of gauge transformations G and the metric of the true configuration space $G_{\mathcal{A}/\mathcal{G}}$) in powers of the coupling constant e. In fact, after a straightforward computation, the kinetic energy, up to order e^2, can be shown to be

$$\begin{aligned}T &= \frac{1}{2}\int d^3x d^3y G^{(ia)(jb)}_{\mathcal{A}/\mathcal{G}}(x,y)\mathcal{E}^{ai}(x)\mathcal{E}^{bj}(y)\\ &\quad-\frac{i}{2}e^2c_A\triangle^{-1}(0)\int d^3x\mathcal{A}_i^a\mathcal{E}^{ai}(x),\end{aligned} \tag{34}$$

where c_A is the Casimir of G in the adjoint representation,

$$\begin{aligned}G^{(ia)(jb)}_{\mathcal{A}/\mathcal{G}}(x,y) &= \delta^{ij}\delta^{ab}\delta(x,y)\\ &\quad-e^2f^{acd}f^{bce}\mathcal{A}_i^d(x)\mathcal{A}_j^e(y)\triangle^{-1}(x,y)\\ &\quad+O(e^3)\end{aligned} \tag{35}$$

and the transverse variables $\mathcal{A}_i^a$ and $\mathcal{E}_i^a$ satisfy the commutation relations [see Eq. (30)]:

$$[\mathcal{A}_i^a(x),\mathcal{E}_j^b(y)]=i\,\delta^{ab}P_{ij}(x,y). \tag{36}$$

IV. RENORMALIZATION OF YANG-MILLS HAMILTONIAN

In this section we will compute the renormalization contribution to the E^2 and B^2 terms in the effective Hamiltonian. In what follows, we will assume the ''gauge-fixed'' variables from the previous section and will use A and E instead of $\mathcal{A}$ and $\mathcal{E}$. Using expression (34) the gauge-fixed Yang-Mills Hamiltonian can be written as follows:

$$\begin{aligned}H &= \frac{1}{2}\int_{x,y}\alpha_{ij}^{ab}(x,y)E_i^a(x)E_j^b(y)\\ &\quad+\frac{1}{2}ic_AA_i^a(x)E_i^a(x)G(x,x)+\frac{1}{2}B_i^a(x)B_i^a(x),\end{aligned} \tag{37}$$

where c_A is N for SU(N) and

$$\alpha_{ij}^{ab}(x,y)=\delta_{ij}\delta^{ab}\delta^3(x-y)+e^2f^{adc}f^{bde}A_i^c(x)A_j^e(y)G(x,y)+\cdots, \tag{38}$$

$$G(x,y)=\int\frac{1}{q^2}e^{-iq(x-y)}\frac{d^3q}{(2\pi)^3}. \tag{39}$$

This is the so called ''bare'' Hamiltonian. For the loop calculations we have to introduce the Z factors by the following procedure. Using (38) let us rewrite expression (37) as

$$\begin{aligned}H=&\frac{1}{2}Z_{E^2}(\Lambda)E^2+\frac{1}{2}Z_{AAEE}(\Lambda)e^2f^{adc}f^{bde}A_i^c(x)A_j^e(y)\\&\times G(x,y)E_i^a(x)E_j^b(y)+\frac{1}{2}Z_{B^2}(\Lambda)B^2\\&+eZ_1(\Lambda)f^{abc}\partial_iA_j^aA_i^bA_j^c\\&+\frac{e^2}{4}Z_4(\Lambda)f^{abc}f^{dec}A_i^aA_j^bA_i^dA_j^e+\cdots.\end{aligned} \tag{40}$$

Each of the Z factors will have the following form

$$Z=1+ef_1(\Lambda)+e^2f_2(\Lambda)+\cdots. \tag{41}$$

The functions f_n will be chosen order-by-order from the requirement that after integration of the modes from μ to Λ all the corrections sum up in such a way that $f_n(\Lambda)\to f_n(\mu)$ and $Z(\Lambda)\to Z(\mu)$, accordingly. When doing the one-loop corrections one can therefore assume that all the Z's are initially 1 and choose the corresponding f's from the condition that high-cutoff dependence be canceled after computing the H_{eff} to one loop. Since one-loop wave function renormalization in QCD is of the second order in coupling constant it is easy to see that we need Ω only up to the first order in e. Then there is only one term of V that is relevant:

$$V^{(3)}=ef^{abc}\partial_iA_j^aA_i^bA_j^c. \tag{42}$$

To compute the renormalization of the quadratic terms in the Hamiltonian we have to assign, according to the general procedure of Sec. II, two A's to be ''high'' and one ''low.'' Therefore, the relevant part of $V^{(3)}$ looks like[1]

$$V^{(3)}=ef^{abc}(2\partial_iA_{1j}^aA_{2i}^bA_{2j}^c+\partial_iA_{2j}^aA_{1i}^bA_{2j}^c). \tag{43}$$

We now write A_2 and E_2 in second-quantized form:

$$A_{2i}^a(x)=\sum_k\frac{1}{(2\omega_k)^{1/2}}(a_{ki}^ae^{ikx}+a_{ki}^{a\dagger}e^{-ikx}), \tag{44}$$

$$E_{2i}^a(x)=\sum_k i\left(\frac{\omega_k}{2}\right)^{1/2}(a_{ki}^{a\dagger}e^{-ikx}-a_{ki}^ae^{ikx}), \tag{45}$$

where the creation and annihilation operators satisfy

$$[a_{ki}^a,a_{pj}^{b\dagger}]=\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)\delta^{ab}\delta_{kp}. \tag{46}$$

After normal ordering, $V^{(3)}$ can have three kind of terms: $a_k^\dagger a_p^\dagger$, a_ka_p, and $a_k^\dagger a_p$. As was explained in Sec. II, only the first two types will be used to in order to determine Ω, so the final form of $V^{(3)}$ is therefore

$$\begin{aligned}V^{(3)}=&ef^{abc}\sum_{p,q}\int d^3x\{2(a_{pi}^{a\dagger}a_{qj}^{c\dagger}e^{-i(p+q)x}\\&+a_{pi}^ac_{qj}^ce^{i(p+q)x})\partial_jA_i^b(x)-iq_j(a_{pi}^{a\dagger}a_{qi}^{b\dagger}e^{-i(p+q)x}\\&-a_{pi}^ab_{qi}^be^{i(p+q)x})A_j^c(x)\}\frac{1}{2\sqrt{\omega_p\omega_q}}.\end{aligned} \tag{47}$$

In order so solve Eq. (9) with the interaction given by (43) we use

$$H_2=\sum_k\omega(k)a_{ki}^{b\dagger}a_{ki}^b,\quad [H_2,a_{pi}^{a\dagger}]=\omega_pa_{pi}^{a\dagger},$$

$$[H_2,a_{pi}^a]=-\omega_pa_{pi}^a. \tag{48}$$

Then,

$$\begin{aligned}\Omega_0=&ief^{abc}\sum_{p,q}\int d^3x\{2(a_{pi}^{a\dagger}a_{qj}^{c\dagger}e^{-i(p+q)x}\\&-a_{pi}^ac_{qj}^ce^{i(p+q)x})\partial_jA_i^b(x)-iq_j(a_{pi}^{a\dagger}a_{qi}^{b\dagger}e^{-i(p+q)x}\\&+a_{pi}^ab_{qi}^be^{i(p+q)x})A_j^c(x)\}\frac{1}{2\sqrt{\omega_p\omega_q}(\omega_p+\omega_q)}.\end{aligned} \tag{49}$$

According to Eq. (11), the next Ω will be

$$\begin{aligned}\Omega_1=&-ief^{abc}\sum_{p,q}\int d^3x\{2(a_{pi}^{a\dagger}a_{qj}^{c\dagger}e^{-i(p+q)x}\\&+a_{pi}^ac_{qj}^ce^{i(p+q)x})[H_1,\partial_jA_i^b(x)]\\&-iq_j(a_{pi}^{a\dagger}a_{qi}^{b\dagger}e^{-i(p+q)x}-a_{pi}^ab_{qi}^be^{i(p+q)x})\\&\times[H_1,A_j^c(x)]\}\frac{1}{2\sqrt{\omega_p\omega_q}(\omega_p+\omega_q)^2}.\end{aligned} \tag{50}$$

To study the renormalization of the metric we have to determine the corrections to the E^2 term in the Hamiltonian. There are only two possibilities. The first one is $-\frac{1}{2}[[H_1,\Omega_0],\Omega_1]$, since $\Omega_0\sim A$ and $\Omega_1\sim E$. The other comes from the normal-ordering of the quadratic part of the α term in Eq. (38). When computing the double-commutator, there is only one divergent term:

[1]We use the following convention: fields with subscript 1 indicate ''low momenta'' fields and fields with subscript 2 indicate ''high momenta'' fields.

$$-\frac{1}{2}[[H_1,\Omega_0],\Omega_1]$$

$$=e^2\sum_{k,p}\frac{1}{2}[H_1,A_i^b(x)]\times[H_1,A_\alpha^n(y)]f^{abc}f^{mnl}k_iq_\alpha(\delta_{kq}^{am}\delta_{ps}^{cl}+\delta_{ks}^{al}\delta_{pq}^{cm})P_{j\gamma}(k)P_{j\gamma}(p)(e^{-i(k+p)(x-y)}+e^{i(k+p)(x-y)})\times\frac{1}{4(\omega_k+\omega_p)(\omega_q+\omega_s)^2\sqrt{\omega_k\omega_p\omega_q\omega_s}}. \quad (51)$$

At this point we can say that momenta $(k+p)$ and $-(k+p)$ are essentially the momenta of the ''low'' fields. Therefore $|k+p|<\mu$. We can now change the summation by the following trick: say $k+p=r$ and $|r|<\mu$. Then $k=r-p$ and the summation goes over k and r. Any possible divergence can only come from the summation over k. Using $f^{adc}f^{bdc}=c_A\delta^{ab}$ the expression (27) can be simplified as follows:

$$\sum_{r,k}\frac{c_A}{4\omega_k\omega_p}\frac{[H_1,A_i^c(r)][H_1,A_\alpha^c(-r)]}{(\omega_k+\omega_p)^3}\times(k_ik_\alpha-k_ip_\alpha)P_{j\gamma}(k)P_{j\gamma}(p). \quad (52)$$

Noticing that the leading divergence of the expression is logarithmic, which means that we can neglect the difference between k and p (for the divergent contribution only) and using

$$\sum_k=\int\frac{d^3k}{(2\pi)^3},\quad \omega_k=|k|,\quad k\sim -p, \quad (53)$$

$$P_{j\gamma}(k)P_{j\gamma}(p)=\left(1+\frac{(k\cdot p)^2}{k^2p^2}\right)\sim 2, \quad (54)$$

we obtain

$$\frac{c_A}{24}\sum_r[H_1,A_i^c(r)][H_1,A_i^c(-r)]\sum_k\frac{1}{\omega_k^3}=-\frac{c_A}{24}\frac{1}{2\pi^2}\ln\left(\frac{\Lambda}{\mu}\right)\int d^3xE^2. \quad (55)$$

As we mentioned earlier, there is another term that can correct the E^2 term of the effective Hamiltonian. It is the ''Coulomb interaction'' term from the kinetic term in (38). Looking at the e^2 order correction part of (38),(42) and choosing the A's to be ''high'' and the E's to be ''low'' we have

$$\frac{e^2}{2}f^{adc}f^{bde}\int_{x,y}A_{2i}^c(x)A_{2j}^e(y)G(x,y)E_{1i}^a(x)E_{1j}^b(y). \quad (56)$$

As written, this term is not normal ordered with respect to the high-energy vacuum. Using (44),(45), and (46) we obtain

$$\frac{e^2}{2}f^{adc}f^{bde}\int_x\int_y\sum_k\frac{\delta^{ce}P_{ij}(k)}{2\omega_k}e^{ik(x-y)}G(x,y)E_{1i}^a(x)E_{1j}^b(y). \quad (57)$$

Upon using the definition (39) of $G(x,y)$ one can see that the leading divergence is logarithmic and that the final expression reads

$$\frac{e^2}{6}c_A\sum_k\frac{1}{\omega_k^3}\int d^3xE^2=\frac{e^2}{6}\frac{c_A}{2\pi^2}\ln\left(\frac{\Lambda}{\mu}\right)\int d^3xE^2. \quad (58)$$

The sum of the terms (55) and (58) gives the total correction to the kinetic energy at one loop:

$$\delta H_{E^2}=\frac{c_A}{8}\frac{e^2}{2\pi^2}\ln\left(\frac{\Lambda}{\mu}\right)\int d^3xE_i^2(x). \quad (59)$$

At this point we can say that the Z_{E^2} factor is therefore,

$$Z_{E^2}=1-\frac{c_A}{4}\frac{e^2}{2\pi^2}\ln\left(\frac{\Lambda}{m}\right)+\cdots. \quad (60)$$

Since the operator E_i is represented by a variational derivative with respect to gauge field, $\delta/\delta A_i$, one would naturally expect that Z_{E^2} should be equal to the $Z_{B^2}^{-1}$. It is therefore an important check on our method to show that it is indeed so. To compute $Z_{B^2}=Z_3$ we need to find out B^2 correction to the effective H. In comparison with the computation of the E^2 correction it is much more involved due to the fact that for most of the terms B^2 comes as a subleading divergence. We will not present detailed computation but sketch the main steps and give the final result. B^2 contributions can arise from the following terms: $(i/2)[V,\Omega_0]$ and normal ordering of the α term from Eq. (38) again. This term is similar to (56), the only difference being how the ''high'' and ''low'' components are assigned:

$$\frac{e^2}{2}f^{adc}f^{bde}A_{1i}^c(x)A_{1j}^e(y)G(x,y)E_{2i}^a(x)E_{2j}^b(y). \quad (61)$$

Leading divergence for both terms is quadratic and gives correction of the form A^2. The appearance of this term is related to our choice of the cutoff procedure as a way of regulating the theory; it can be dealt with by introducing $A^2\Lambda^2$ counterterms in the bare Hamiltonian and defining appropriate boundary conditions at the ends of the renormalization group flow trajectory [18]. To capture the logarithmic contribution one has to expand the denominators of the Ω_0 up to the second order in the momenta of the ''low'' fields. Tedious but straightforward computation gives

$$-\frac{i}{2}[V,\Omega_0]=-\frac{27}{120}e^2c_A\left(\sum_k\frac{1}{k^3}\right)\int d^3xB^2. \quad (62)$$

The logarithmic divergent part of the normal ordering of the two E's from (61) gives

$$\frac{1}{10}e^2 c_A\left(\sum_k \frac{1}{k^3}\right)\int d^3x B^2. \tag{63}$$

The final result is

$$\delta H_{B^2} = -\frac{c_A}{8}\frac{e^2}{2\pi^2}\ln\left(\frac{\Lambda}{\mu}\right)\int d^3x B^2 \tag{64}$$

which makes the corresponding Z factor

$$Z_{B^2} = Z_3 = 1 + \frac{c_A}{4}\frac{e^2}{2\pi^2}\ln\left(\frac{\Lambda}{m}\right) + \cdots. \tag{65}$$

This coincides with the value of Z_3 for QCD in the Coulomb gauge obtained in the Lagrangian formalism [19].

V. THREE POINT FUNCTION RENORMALIZATION

Let us show briefly the renormalization of the three point function within our formalism. To do that let us recall the general method of Sec. II. We will have to be a little more careful in the analysis of the relevant contributions to the renormalized Hamiltonian.

According to the notation of Sec. II, we write the Yang-Mills Hamiltonian in terms of the low momentum and high momentum fields as

$$H = H_1 + H_2 + (V_A^{(3)} + V_B^{(3)} + V^{(4)} + \cdots), \tag{66}$$

where H_1 is the part of the Hamiltonian that only contains low momentum fields, H_2 is the part that only contains high momentum fields, and the V's are the ''mixing'' terms. For convenience we have separated these last terms according to the vertex number and the high momentum creation-annihilation operators structure:

$$\begin{aligned} V_A^{(3)} = ef^{abc}\sum_{p,q}\int d^3x\{2(&a_{pi}^a a_{qj}^c e^{i(p+q)x} \\ &+ a_{pi}^{a\dagger} a_{qj}^{c\dagger} e^{-i(p+q)x})\partial_j A_i^b(x) + iq_j(a_{pi}^a a_{qi}^b e^{i(p+q)x} \\ &- a_{pi}^{a\dagger} a_{qi}^{b\dagger} e^{-i(p+q)x})A_j^c(x)\}\frac{1}{2\sqrt{\omega_p\omega_q}}, \end{aligned} \tag{67}$$

$$\begin{aligned} V_B^{(3)} = ef^{abc}\sum_{p,q}\int d^3x\{2(&a_{qj}^{c\dagger} a_{pi}^a e^{i(p-q)x} \\ &+ a_{pi}^{a\dagger} a_{qj}^c e^{-i(p-q)x})\partial_j A_i^b(x) + iq_j(a_{pi}^{a\dagger} a_{qi}^b e^{-i(p-q)x} \\ &- a_{qi}^{b\dagger} a_{pi}^{a\dagger} e^{i(p-q)x})A_j^c(x)\}\frac{1}{2\sqrt{\omega_p\omega_q}}, \end{aligned} \tag{68}$$

$$\begin{aligned} V^{(4)} = \frac{e}{2}\sum_{p,q}\int d^3x\{&(a_{pi}^a a_{qi}^c e^{i(p+q)x} \\ &+ a_{pi}^{a\dagger} a_{qi}^{c\dagger} e^{-i(p+q)x})f^{abe}f^{cde}A_j^b(x)A_j^d(x) \\ &+ (a_{pi}^a a_{qj}^b e^{i(p+q)x} + a_{pi}^{a\dagger} a_{qj}^{b\dagger} e^{-i(p+q)x})(f^{abe}f^{cde} \\ &+ f^{ade}f^{cbe})A_i^c(x)A_j^d(x)\}\frac{1}{2\sqrt{\omega_p\omega_q}}. \end{aligned} \tag{69}$$

Now we have to analyze which terms contribute to the three point vertices. At this order only Ω at order e is needed, moreover, one can convince oneself that only Ω_0, i.e., the first iteration of the unitary transformation leads to divergent contributions. Then Eq. (8) reads in this case:

$$\begin{aligned} H' = {} & H_1 + H_2 + V_A^{(3)} + V_B^{(3)} + V^{(4)} + \cdots + i[H_1,\Omega_0] \\ & + i[H_2,\Omega_0] + i[V_A^{(3)},\Omega_0] + i[V_B^{(3)},\Omega_0] + i[V^{(4)},\Omega_0] \\ & - \frac{1}{2}[[H_1,\Omega_0],\Omega_0] - \frac{1}{2}[[H_2,\Omega_0],\Omega_0] \\ & - \frac{1}{2}[[V_A^{(3)},\Omega_0],\Omega_0] - \frac{1}{2}[[V_B^{(3)},\Omega_0],\Omega_0] \\ & - \frac{1}{2}[[V^{(4)},\Omega_0],\Omega_0] - \frac{i}{6}[[[H_1,\Omega_0],\Omega_0],\Omega_0] \\ & - \frac{i}{6}[[[H_2,\Omega_0],\Omega_0],\Omega_0] + \cdots. \end{aligned} \tag{70}$$

As explained in Sec. II, we choose Ω_0 in such a way that its commutator with H_2 cancels the mixing terms that contain high momentum annihilation operators or high momentum creation operators, but not both. Also, we note that up to this order, only $V^{(3)}$ is of order e. Then Ω_0 satisfies

$$[H_2,\Omega_0] = iV_A^{(3)} \tag{71}$$

[which is precisely the equation that gives Eq. (49)].

Now it is not difficult to individualize the only terms that contribute to the three point vertex:

$$\begin{aligned} \delta H^{(3)} = {} & \Delta H_1 + i[H_1,\Omega_0]^{(3)} + i[V_A^{(4)},\Omega_0]^{(3)} \\ & - \frac{1}{2}[[V_B^{(3)},\Omega_0],\Omega_0]^{(3)}. \end{aligned} \tag{72}$$

Here ΔH_1 stands for the normal ordering contribution (tadpole diagram) of order e^3 in the kinetic energy. In fact, at this order, the kinetic energy has a term of the form

$$\begin{aligned} T_{e^3} = {} & e^3 f^{abc}f^{ade}f^{emn} \\ & \times \int_{xyz} A_i^b(x)G(x,y)A_j^d(y)\partial_j G(y,z)A_k^m(z)E_i^c(x)E_k^n(z) \end{aligned} \tag{73}$$

which, when properly contracted, generates the contribution

$$\Delta H_1 = \frac{1}{6} c_A e^3 \ln\left(\frac{\Lambda}{\mu}\right) f^{abc} \int_x \partial_i A_j^a A_i^b A_j^c . \tag{74}$$

The remaining terms in Eq. (72), are computed similarly to the ones of Sec. II. After a very lengthy, but straightforward computation, we get the following results:

$$[H_1, \Omega_0]^{(3)} = -i \frac{5}{24} c_A e^3 \frac{1}{2\pi^2} \ln\left(\frac{\Lambda}{\mu}\right) f^{abc} \times \int_x \partial_i A_j^a A_i^b A_j^c ,$$

$$[V^{(4)}, \Omega_0]^{(3)} = i \frac{3}{8} c_A e^3 \frac{1}{2\pi^2} \ln\left(\frac{\Lambda}{\mu}\right) f^{abc} \int_x \partial_i A_j^a A_i^b A_j^c ,$$

$$[[V_B^{(3)}, \Omega_0], \Omega_0]^{(3)} = -\frac{1}{6} c_A e^3 \frac{1}{2\pi^2} \ln\left(\frac{\Lambda}{\mu}\right) f^{abc} \times \int_x \partial_i A_j^a A_i^b A_j^c . \tag{75}$$

Finally, after adding up all the contributions, we have

$$\delta H^{(3)} = \frac{1}{12} c_A e^3 \frac{1}{2\pi^2} \ln\left(\frac{\Lambda}{\mu}\right) f^{abc} \int_x \partial_i A_j^a A_i^b A_j^c . \tag{76}$$

This result corresponds to a renormalization constant Z_1 equal to

$$Z_1 = 1 - \frac{1}{12} c_A e^2 \frac{1}{2\pi^2} \ln\left(\frac{\Lambda}{m}\right), \tag{77}$$

which is the same as the Coulomb gauge result in the Lagrangian approach as well [19].

VI. QED

In this section we will outline a similar computation for quantum electrodynamics. The QED Hamiltonian can be written as

$$H = \frac{1}{2} \int d^3x (E^2 + B^2) + \int d^3x \{ \bar{\psi} (i \vec{\gamma} \cdot \vec{\partial} + m) \psi + e \bar{\psi} A \cdot \gamma \psi \}. \tag{78}$$

The imposition of the Gauss law constraint,

$$\partial_i E^i = e \bar{\psi} \gamma^0 \psi, \tag{79}$$

generates an ''instantaneous'' Coulomb interaction and H takes the form

$$H = \frac{1}{2} \int d^3x (E^2 + B^2) + \frac{1}{2} \int_x \int_y e^2 \bar{\psi}(x) \gamma^0 \psi(x) G(x,y) \bar{\psi}(y) \gamma^0 \psi(y) + \int d^3x \{ \bar{\psi} (i \vec{\gamma} \cdot \vec{\partial} + m) \psi + e \bar{\psi} A \gamma \psi \}, \tag{80}$$

where $G(x,y)$ is given by (39). According to the general idea we are supposed to split it in to ''high'' and ''low'' energy parts,

$$H_1 = \frac{1}{2} \int d^3x (E_1^2 + B_1^2) + \int d^3x \bar{\psi}_1 (i \vec{\partial} \cdot \vec{\gamma} + m) \psi_1 + e \bar{\psi}_1 A \cdot \gamma \psi_1 , \tag{81}$$

$$H_2 = \sum_k \omega_k b_{k\alpha}^\dagger b_{k\alpha} + \sum_p \omega_p d_{p\beta}^\dagger d_{p\beta} ,$$

$$V^{(3)} = e \int d^3x \{ \bar{\psi}_2 A_1 \gamma \psi_2 + e \bar{\psi}_2 A_2 \gamma \psi_1 + e \bar{\psi}_1 A_2 \gamma \psi_2 \}, \tag{82}$$

$$V^{(4)} = \int_x \int_y e^2 \bar{\psi}_2(x) \gamma^0 \psi_1(x) G(x,y) \bar{\psi}_1(y) \gamma^0 \psi_2(y) + \cdots . \tag{83}$$

Here the ellipses mean that we show only those terms that play a role in one loop effects. Using arguments similar to those in Yang-Mills theory one can see that in order to determine Z factors up to the order e^2 we need Ω only up to the first order. It turns out that only first two iterations in μ/Λ are needed—Ω_0 and Ω_1. In parallel to the previous sections, we determine Z_3 and Z_1 by identifying corrections of the type E^2, B^2, and $\bar{\psi} A \cdot \gamma \psi$. There is only one commutator that can contribute to the B^2 term: $(i/2)[V, \Omega_0]$ where one has to take subleading divergence to identify the ln corrections. E^2 correction is given by two commutators $-\frac{1}{2}[[H_1, \Omega_0], \Omega_1]$ and commutator of the ''Coulomb'' term with Ω_0: $i[V^{(4)}, \Omega_0]$. The final result is

$$\delta H_{E^2} = -\frac{e^2}{6} \frac{1}{2\pi^2} \ln\left(\frac{\Lambda}{\mu}\right) \int d^3x E_1^2 , \tag{84}$$

$$\delta H_{B^2} = \frac{e^2}{6} \frac{1}{2\pi^2} \ln\left(\frac{\Lambda}{\mu}\right) \int d^3x B_1^2 \tag{85}$$

which leads us to the well-known answer for the Z_3 factor in QED:

$$Z_{E^2}^{-1} = Z_{B^2} = Z_3 = 1 - \frac{e^2}{3} \frac{1}{2\pi^2} \ln\left(\frac{\Lambda}{m}\right). \tag{86}$$

Up to this point all our Z's were identical to those known from the covariant calculations in the Coulomb gauge. However, when computing the other two renormalization constants, Z_1 and Z_2, we find results which are different from

the covariant ones. In the case of Z_2-fermion kinetic term renormalization there are three possible contributions arising from $(i/2)[V,\Omega_0]$, $-\frac{1}{2}[[H_1,\Omega_0],\Omega_0]$ and normal-ordering of the four-fermion term in the Hamiltonian (80). Extracting $\bar{\psi}\vec{\gamma}\cdot\vec{\partial}\psi$-type corrections from each of these terms one can see that they cancel. The similar thing happens in case of Z_1 as well: all the $\bar{\psi}\gamma\cdot A\psi$-type terms cancel at the e^3 order. This makes both Z_1 and Z_2 equal to 1 at one loop. Nevertheless there is no contradiction between our result and the conventional one. In covariant formalism there exists a Ward identity $Z_1=Z_2$ which is essential for maintaining the gauge invariance of the effective action. It is satisfied, presumably, in our case as well. But values of Z_1 and Z_2 are gauge dependent and cancel out from the final expression that defines the beta function for QED, which is determined by Z_3 only. There is an obvious reason why, say, Z_2 must be 1 in our case. Since $\bar{\psi}$ and ψ are conjugated variables, in the Hamiltonian formalism one should be represented by the variational derivative with respect to the other. Then, similarly to the E^2 and B^2 terms for the gauge field, they will have inverse renormalization factors which will cancel each other in the final expression for the kinetic term for fermions.

VII. GEOMETRIC INTERPRETATION OF THE ASYMPTOTIC FREEDOM

Let us recall the Yang-Mills Hamiltonian, after integrating the high momenta modes down to the scale μ (at order e^2), incorporating the correct renormalization factors Z_i:

$$H_{YM}=\frac{1}{2}\frac{1}{Z_3(\mu)}E^2+\frac{1}{2}Z_3(\mu)(\partial_iA_j^a-\partial_jA_j^a)^2+Z_1(\mu)e_Rf^{abc}\partial_iA_j^aA_i^bA_j^c+Z_4(\mu)\frac{1}{4}e_R^2f^{abc}f^{dec}A_i^aA_j^bA_i^dA_j^e+\cdots. \quad (87)$$

A similar expression for the QED Hamiltonian will be

$$H_{QED}=\frac{1}{2}\frac{1}{Z_3(\mu)}E^2+\frac{1}{2}Z_3(\mu)B^2+Z_2(\mu)\bar{\psi}i\vec{\gamma}\cdot\vec{\partial}\psi+Z_1(\mu)e_R\bar{\psi}A\gamma\psi\cdot\cdots. \quad (88)$$

Here e_R is a fixed quantity at certain scale and all the dependence on the scale is hidden in the Z factors. The definition of the ''renormalized'' fields through the incorporation of the Z factors was done in analogy with the Lagrangian covariant approach where the renormalized quantities are included in such a way that the ''renormalized'' effective action gives finite results when the cutoff is removed. But in the Schrodinger picture this requirement is not necessary as the fields are only coordinates of the configuration space and do not enter explicitly in the computation of correlation functions. With this fact in mind we will alter this requisite and adapt it to our needs.

The usual covariant renormalization program puts the emphasis on the interactions, and the scaling properties of the theory are extracted from the study of the β functions. However, in a Hamiltonian description the kinetic term plays a significant role since essentially it is nothing but the Laplacian in configuration space. Hence, many of the properties of the quantum-field theory (QFT) can be inferred from the geometrical features of the configuration space (as compactness, boundness, etc.). In particular we are interested in the change of the configuration space metric under the renormalization group flow. We claim that in asymptotically free theories the distance between configurations increases as we move to the UV limit, thus ''flattening'' the potential energy and consequently fading the interaction. We will support our claim with the analysis of the one-loop Yang-Mills theory.

It is clear then, that in the spirit of our work we want to stress the kinetic term (better, the configuration space metric) over the potential energy and try to understand the asymptotic behavior of the theory through the renormalization flow properties of the distance in the configuration space. For this reason we will rescale the fields in such a way to transfer the renormalization scaling properties to the kinetic term.

Then let us rescale the fields as

$$A_i^a\to\frac{1}{e_R}Z_3(\mu)Z_1(\mu)^{-1}A_i^a,\quad E_i^a\to e_RZ_3^{-1}(\mu)Z_1(\mu)E_i^a \quad (89)$$

so the QCD Hamiltonian takes the form

$$H_{QCD}=\frac{1}{2}e_R^2\left(\frac{Z_1^2}{Z_3^3}\right)E_i^{a2}+\frac{1}{e_R^2}\left(\frac{Z_3^3}{Z_1^2}\right)\left\{\frac{1}{2}(\partial_iA_j^a-\partial_jA_i^a)^2+f^{abc}\partial_iA_j^aA_i^bA_j^c+\frac{1}{4}f^{abc}f^{dec}A_i^aA_j^bA_i^dA_j^e\right\}+\cdots, \quad (90)$$

where we have used the Slavnov-Taylor identities (adapted for the Hamiltonian formalism)[2]

$$\frac{Z_1}{Z_3}=\frac{Z_4}{Z_1}. \quad (91)$$

Note that with this normalization we have ''homogenized'' the potential term (up to an overall factor) by transferring all the cutoff dependence to the kinetic term. Now we can read from the kinetic term the cutoff dependence of the (inverse of the) metric. In fact, using the result of Sec. III, we can write the cutoff dependent configuration metric as

$$G_{(ia)(jb)}(x,y;\mu)=\left(1+e_R^2\frac{11N}{12}\frac{1}{2\pi^2}\ln(\mu/m_R)\right)\times G^0_{(ia)(jb)}(x,y)+\mathcal{O}(e_R^3) \quad (92)$$

[2]By writing both the Hamiltonian H and Gauss law $\mathcal{G}^a$ in terms of renormalized fields with the explicit Z factors and requiring $[H,\mathcal{G}^a]=0$ one can get linear relations between Z's that lead to (91).

where $G^0_{(ia)(jb)}$ is the metric defined in Eq. (33) written in terms of the rescaled fields (89) and e_R.

Looking at the QED Hamiltonian and using $Z_1=1$ and $Z_2=1$ one can see that no rescaling is needed at all, since all the cutoff dependence is already shifted to the kinetic term,

$$H_{QED}=\frac{1}{2}\left(\frac{1}{Z_3}\right)E^2+\frac{1}{2}Z_3B^2+\bar{\psi}i\vec{\gamma}\cdot\vec{\partial}\psi+e_R\bar{\psi}A\gamma\psi\cdots. \tag{93}$$

Following our procedure the corresponding metric will be

$$G_{(ij)}(x,y;\mu)=\left(1-e_R^2\frac{1}{3}\frac{1}{2\pi^2}\ln(\mu/m_R)\right)\delta_{(ij)}\delta(x-y)+\mathcal{O}(e_R^3). \tag{94}$$

At this point it is easy to compare relative behavior of the two metrics under the renormalization flow. Equation (92) clearly shows that the distance between configurations decreases as the cutoff is lowered, while the corresponding expression (94) increases, sustaining our claim. Incidentally, it is worthwhile to mention that the combination Z_1^2/Z_3^3 that appears in Eq. (90) is precisely the one that defines the β function of the Yang-Mills theory and, at least in the Lagrangian approach, it has been proved to be independent of the gauge fixing condition. Similarly, QED expression involves only Z_3, since $Z_1=Z_2$ is 1 to this order, and Z_3 is the only constant that determines beta function for QED and it is known to be gauge independent to all loops.

VIII. SUMMARY AND CONCLUSION

There are several issues that are more natural to address in the Hamiltonian picture than in the usual covariant Lagrangian formalism. One of them, which we are interested in, is the relevance of the geometry of the configuration space to the properties of the corresponding quantum field theory. The reason is simple: in the Hamiltonian formalism the kinetic energy term is nothing but a Laplacian operator in the configuration space and its topological and geometrical features determine the nature of its spectrum. Then it is natural to ask what is the behavior of the configuration space as we integrate out high momentum degrees of freedom.

To answer a small part of this question was the aim of this paper. To be precise, we were interested in the following aspect of the problem: the evolution of the distance between field configurations (and more precisely the metric) with the renormalization group in asymptotically free theories. In particular we state the following conjecture: *in asymptotically free theories the effective distance between configurations decreases as high momenta degrees of freedom are integrated out.*

To support this statement we first developed an original renormalization group technique for Hamiltonian formalism in the framework of perturbation theory. This method resembles the Hamiltonian renormalization approaches of Glazek and Wilson [11] and Wegner [12] and operates by a progressive diagonalization of the Hamiltonian by means of a succession of iterative unitary transformations followed by a projection onto the Hilbert space of the low-momentum degrees of freedom. Finally we applied the formalism to two conspicuous QFT's: quantum electrodynamics and Yang-Mills theory in $3+1$ dimensions, where we constructed the renormalized Hamiltonian up to one loop.

Our results were substantially supportive of our conjecture. In the case of Yang-Mills, an asymptotically free theory, the one-loop metric renormalization showed that in fact the distance between configuration increases as the momentum scale increases, and on the contrary for QED, not asymptotically free, the behavior of the metric is the opposite.

We are aware, of course, that our results are not decisive but just consistent with the conjecture. After all we have only studied two examples at one-loop order in perturbation theory. However from the examples considered we can observe a pattern that seems to repeat at any instance: when moving all the weight of the renormalization group onto the configuration space metric, it acquires a renormalization factor which is a function of the same combination of renormalization constants that defines the β function of the theory, and thus, presumably, inherits its asymptotic behavior properties.

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APPENDIX

In this appendix we show that the ambiguity in the definition of the effective Hamiltonian due to the freedom in the solution of Eq. (6) [and generally (10)] is just the standard ambiguity of the Hamiltonian operator, namely the freedom of unitary transformations.

Let us recall that the solution of Eqs. (6), (8), or (10) is not uniquely defined; if Ω_n is a solution so will be Ω_n+O_n with O_n satisfying the homogeneous equation $[H_2,O_n]=0$. Now let us consider two sets of Ω's—say $\Omega_n^{(a)}$ and $\Omega_n^{(b)}$, all of them satisfying the proper equations (6) and (10) but generated from different type of solutions. Then we have

$$H_A=\cdots e^{-i\Omega_1^{(a)}}e^{-i\Omega_0^{(a)}}He^{i\Omega_0^{(a)}}e^{i\Omega_1^{(a)}}\cdots,$$

$$H_B=\cdots e^{-i\Omega_1^{(b)}}e^{-i\Omega_0^{(b)}}He^{i\Omega_0^{(b)}}e^{i\Omega_1^{(b)}}\cdots. \tag{A1}$$

Obviously H_A and H_B are unitarily related: $H_A=U^{-1}H_BU$ and consequently they have identical spectrum. What we have to show is that this property remains *after* projecting onto the ''high'' perturbative vacuum $|0_{high}\rangle$.

Following the prescription given in Sec. II, we have shown that, up to given order n in coupling constant λ and a given order m in μ/Λ we can write

$$H_A = H_1^{(a)}(\text{low}) + H_2^{\text{free}}(\text{high}) + \sum_{k,p>\mu} a_k^\dagger S_{kp}^{(a)} a_p + O(\lambda^n, (\mu/\Lambda)^m),$$

$$H_B = H_1^{(b)}(\text{low}) + H_2^{\text{free}}(\text{high}) + \sum_{k,p>\mu} a_k^\dagger S_{kp}^{(b)} a_p + O(\lambda^n, (\mu/\Lambda)^m), \quad \text{(A2)}$$

where the Hamiltonians $H_1^{(a,b)}$ depend only on the low energy modes, H_2 is the free Hamiltonian for the energy modes and $S^{(a,b)}$ are *arbitrary* operators of low and high frequency modes.

Without losing any generality we can assume that $H_1^{(a)}$ and $H_1^{(b)}$ are diagonal, as they can always be brought to that form with a ''low energy modes''-unitary transformation that respects the structure of (A2).

Now we will show that the low energy Hamiltonians $H_1^{(a)}$ and $H_1^{(b)}$ have the same spectrum and consequently they are unitarily related.

Consider the eigenvalues equations for $H_1^{(a)}$ and $H_1^{(b)}$:

$$H_1^{(a)}\psi_\alpha^a = E_\alpha^a \psi_\alpha^a, \quad H_1^{(b)}\psi_\alpha^b = E_\alpha^b \psi_\alpha^b. \quad \text{(A3)}$$

Using standard perturbation theory we can compute the eigenvalues of the whole Hamiltonians, A and B as an expansion in powers of the matrix elements of the interaction terms

$$V^{(a,b)} = \sum_{k,p>\mu} a_k^\dagger S_{kp}^{(a,b)} a_p. \quad \text{(A4)}$$

We get, for the eigenvalues of the operator H_A,

$$E_{\alpha,n}^{\text{TOT}} = E_\alpha^a + E_n^0 + \langle \alpha, n | V^a | \alpha, n \rangle + \sum_{\gamma,m} \frac{\langle \alpha, n | V^a | \gamma, m \rangle \langle \gamma, m | V^a | \alpha, n \rangle}{E_\gamma^a + E_m^0 - E_\alpha^a - E_n^0} + \cdots \quad \text{(A5)}$$

and a similar equation is valid for the eigenvalues of the operator H_B. But the low energy spectrum corresponds to those states with $n=0$, and in this case, due to the particular form of the interaction, all the perturbative contributions vanish and the eigenvalue E_α^a is the *exact* eigenvalue of the whole Hamiltonian (up to the given order in λ and μ/Λ):

$$E_{\alpha,0}^{\text{TOT}} = E_\alpha^a + O(\lambda^n, (\mu/\Lambda)^m). \quad \text{(A6)}$$

And finally, since both Hamiltonians H_A and H_B have the same spectrum, or since E^{TOT} is the same for both H_A and H_B, we deduce that

$$E_\alpha^a = E_\alpha^b + O(\lambda^n, (\mu/\Lambda)^m). \quad \text{(A7)}$$

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