Modified version
February 1, 2008

Non Abelian Bosonization in Two and Three Dimensions

J.C. Le Guillou[a*], E. Moreno[b], C. Núñez[c]
and
F.A. Schaposnik[c†]

[a]*Laboratoire de Physique Théorique ENSLAPP* [‡]
LAPP, B.P. 110, F-74941 Annecy-le-Vieux Cedex, France

[b] *Physics Department, City College of the City University of New York*
New York NY 10031, USA

[c]*Departamento de Física, Universidad Nacional de La Plata*
C.C. 67, 1900 La Plata, Argentina

Abstract

We discuss non-Abelian bosonization of two and three dimensional fermions using a path-integral framework in which the bosonic action follows from the evaluation of the fermion determinant for the Dirac operator in the presence of a vector field. This naturally leads to the Wess-Zumino-Witten action for massless two-dimensional fermions and to a Chern-Simons action for very massive three dimensional fermions. One advantage of our approach is that it allows to derive the exact bosonization recipe for fermion currents in a systematic way.

*Also at *Université de Savoie* and at *Institut Universitaire de France*
†Investigador CICBA, Argentina
‡URA 1436 du CNRS associée à l'Ecole Normale Supérieure de Lyon et à l'Université de Savoie

1 Introduction and results

In this paper we investigate the problem of finding the mapping of a three-dimensional free fermion theory with non-abelian symmetry onto an equivalent bosonic quantum field theory. This mapping, commonly called bosonization, has been already established along the lines of the present investigation for the case of abelian symmetry [1]-[3] and it has been also discussed in the non-abelian case, using a related method, in [4]. In all these investigations we employ an approach close to that put forward in a series of very interesting works on smooth bosonization and duality bosonization [5]-[12]. Other related or alternative approaches to bosonization in $d > 2$ dimensions have been also developed [13]-[22].

The advantage of the bosonization method that we employ here lies on the fact that it provides a systematic procedure for deriving $d \geq 2$ bosonization recipes both for abelian and non-abelian symmetries. In this way, it gives an adequate framework for obtaining the bosonic equivalent of the original fermionic theory, the recipe for mapping fermionic and bosonic currents as well as the current commutation relations which are at the basis of bosonization.

The approach we follow starts from the path-integral defining the generating functional for a theory of free fermions (including sources for fermionic currents) and ends with the generating functional for an equivalent bosonic theory. This allows to identify, *exactly*, the bosonization recipe for fermion currents independently of the number of space-time dimensions. Interestingly enough, one follows a series of steps which are the same for any space-time dimension and both for abelian and non-abelian symmetries. Of course, apart from the two-dimensional case and from the large fermion mass limit in the three dimensional case, one only achieves in general a partial bosonization in the sense that one cannot compute exactly the fermion path-integral in order to derive a local bosonic Lagrangian.

Extending the three dimensional abelian bosonization approach discused in [1]-[3] we derive in the present paper three dimensional non-abelian bosonization. Concerning the fermion current, our method allows to derive the exact bosonization recipe

$$\bar{\psi}^i \gamma_\mu t^a_{ij} \psi^j \to \pm \frac{i}{8\pi} \varepsilon_{\mu\nu\alpha} \partial_\nu A^a_\alpha \ , \tag{1}$$

where $i, j = 1, \cdots, N$, $a = 1, \cdots, \mathrm{dim}G$, t^a are the generators and f^{abc} the structure constants of the symmetry group G. Finally A_μ is a vector field taking values in the Lie algebra of G.

The knowledge of the bosonic action accompanying this bosonization rule is necessarily approximate since it implies the evaluation of the $d = 3$ determinant for fermions coupled to a vector field. We then consider the case of very massive free fermions showing that, within this approximation, the fermion lagrangian bosonizes to the non-abelian Chern-Simons term,

$$\bar{\psi}^i \left(i\gamma^\mu \partial_\mu + m\right) \psi^i \to \mp \frac{i}{8\pi} \epsilon_{\mu\nu\alpha} \left(A^a_\mu \partial_\nu A^a_\alpha + 1/3 f^{abc} A^a_\mu A^b_\nu A^c_\alpha\right) \ , \tag{2}$$

This result, advanced in [4] using a completely different approach, is the natural extension of the abelian result [1]-[3]. In this last case the bosonic theory corresponds, in the large fermion mass limit, to a Chern-Simons theory while in the massless case it coincides with the abelian non-local action discussed in [20],[23]. In fact one should expect that an analysis similar to that in [23] can be carried out leading to an explicit (although complicated) bosonic action valid in all fermion mass regimes.

The results above are derived in section 3. As a warming up exercise, we rederive in section 2 the bosonization rules for two-dimensional non-abelian models. Indeed, following an approach related to that developed in [10] for two-dimensional bosonization we arrive to the Wess-Zumino-Witten action and the well-known bosonization recipe for fermion currents

$$j_+ \to \frac{i}{4\pi} a^{-1} \partial_+ a \tag{3}$$

$$j_- \to \frac{i}{4\pi} a \partial_- a^{-1} \tag{4}$$

with a the group-valued bosonic field with dynamics governed by the Wess-Zumino-Witten action. Now, these rules, as well as the Polyakov-Wiegmann identity that we repeatedly use in its derivation, deeply relay on the holomorphic properties of two-dimensional theories [24][26]. Strikingly, we found that in the three dimensional case, a BRST symmetry structure underlying the bosonic version of the fermionic generating functional plays a similar role and allows to end, at least in the large mass limit, with a simple bosonization rule. This BRST symmetry is highly related to that used in [5]-[7], [11]-[13], and is analogous to that arising in topological field theories [27]-[29], its origin being related to the way the originally "trivial" bosonic field enters into play.

2 Warming up: $d = 2$ non-abelian bosonization

Non-abelian bosonization in two dimensional space time was formulated first by Witten [25] by comparing the current algebra for free fermions and for a bosonic sigma model with a Wess-Zumino term. Afterwards, different approaches rederived and discussed the bosonization recipe [30]-[35] but in general they were not constructive in the sense that the bosonic theory was not obtained from the fermionic one by following a series of steps that could be generalized to other cases, in particular to a possible higher-dimensional bosonization. More recently, using the the duality technique [8]-[9] which has as starting point the smooth bosonization approach [5]-[7], [11]-[12] the recipe for non-abelian bosonization was obtained by Burgess and Quevedo [10] in a way which is more adaptable to generalizations to higher dimensions. The approach to two-dimensional bosonization that we present in this section is related to that in [10] and we think that it is worthwhile to describe it here in detail since it provides many of the clues which allow us to derive the non-abelian bosonization recipe for 3-dimensional fermions.

Following Witten [25] one can see that the bosonic picture for a theory of N free massless Dirac fermions corresponds to a bosonic field $a \in SU(N)$ with a Wess-Zumino-Witten action and a real bosonic field ϕ with a free scalar field action [30]-[32]. Since we shall be mainly interested in the specific non-abelian aspect of bosonization, we will not discuss the ϕ sector of the corresponding bosonic theory (although the method can trivially take into account the $U(1)$ sector associated with it).

We start from the (Euclidean) Lagrangian for free massless Dirac fermions in 2 dimensions

$$L = \bar{\psi}(i\partial\!\!\!/)\psi \tag{5}$$

where fermions are in the fundamental representation of some group G. The corresponding generating functional reads

$$Z_{fer}[s] = \int D\bar{\psi}D\psi \exp[-\int d^2x\bar{\psi}(i\partial\!\!\!/ + s\!\!\!/)\psi] \tag{6}$$

with $s_\mu = s^a_\mu t^a$ an external source taking values in the Lie algebra of $SU(N)$.

Our derivation of bosonization rules both in 2 and 3 dimensions heavily relies on the invariance of the measure under local transformations of the fermion variables, $\psi \to h(x)\psi$, $\bar{\psi} \to \bar{\psi}h(x)^{-1}$ with $h \in G$. As a consequence the generating functional (6) is automatically invariant under local transformations of the source

$$s_\mu \to s^h_\mu = h^{-1}s_\mu h + ih^{-1}\partial_\mu h \tag{7}$$

$$Z[s^h] = Z[s] \tag{8}$$

In view of this, if we perform the change of variables

$$\begin{aligned} \psi &= g(x)\psi' \\ \bar{\psi} &= \bar{\psi}'g^{-1}(x). \end{aligned} \tag{9}$$

$Z_{fer}[s]$ becomes

$$Z_{fer}[s] = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}g \exp[-\int d^2x\bar{\psi}(i\partial\!\!\!/ + s\!\!\!/^g)\psi] \tag{10}$$

where an integration over g has been included since it just amounts to a change of normalization. Integrating out fermions we have

$$Z_{fer}[s] = \int \mathcal{D}g \ \det(i\partial\!\!\!/ + s\!\!\!/^g) \tag{11}$$

Now, posing

$$s^g_\mu = b_\mu \tag{12}$$

and using

$$f_{\mu\nu}[b] = g^{-1}f_{\mu\nu}[s]g \tag{13}$$

we shall trade the g integration for an integration over connections b satisfying condition (13). To this end we shall use the $d = 2$ identity (proven in the Appendix)

$$\int \mathcal{D}b_\mu \mathcal{H}[b]\delta[\varepsilon_{\mu\nu}(f_{\mu\nu}[b] - f_{\mu\nu}[s])] = \int \mathcal{D}g\mathcal{H}[s^g] \tag{14}$$

Here $\mathcal{H}$ is a gauge invariant function. Identity (14) allows us to write eq.(11) in the form

$$Z_{fer} = \int \mathcal{D}b_\mu \Delta \delta(b_+ - s_+)\delta\left[\varepsilon_{\mu\nu}(f_{\mu\nu}[b] - f_{\mu\nu}[s])\right]\det(i\partial\!\!\!/ + b\!\!\!/)\,. \qquad (15)$$

For convenience, we have chosen to fix the gauge using the condition $b_+ = s_+$ being Δ the corresponding Faddeev-Popov determinant.

We now introduce a Lagrange multiplier $\hat{a}$ (taking values in the Lie algebra of G) to enforce the delta function condition

$$\begin{aligned} Z_{fer}[s] &= \int \mathcal{D}\hat{a}\mathcal{D}b_\mu\, \Delta\delta(b_+ - s_+)\det[i\partial\!\!\!/ + b\!\!\!/] \times \\ & \exp\left(-\frac{C}{8\pi} tr \int d^2x \hat{a}\, \varepsilon_{\mu\nu}(f_{\mu\nu}[b] - f_{\mu\nu}[s])\right) \end{aligned} \qquad (16)$$

with C a constant to be conveniently adjusted. We now write sources and the b_μ field in terms of group-valued variables,

$$s_+ = i\tilde{s}^{-1}\partial_+\tilde{s} \qquad (17)$$

$$s_- = is\partial_- s^{-1} \qquad (18)$$

$$b_+ = i(\tilde{b}\tilde{s})^{-1}\partial_+(\tilde{b}\tilde{s}) \qquad (19)$$

$$b_- = i(sb)\partial_-(sb)^{-1} \qquad (20)$$

so that the fermion determinant can be related to the Wess-Zumino-Witten action [24],

$$\det[i\partial\!\!\!/ + b\!\!\!/] = \exp(W[\tilde{b}\tilde{s}sb]) \qquad (21)$$

In writing eq.(21) a gauge-invariant regularization is assumed so that the left and right-handed sectors enter in gauge invariant combinations. In this way, gauge transformations of the source s_μ, which as stated before, should leave the generating functional invariant, do not change the determinant. This will be the criterion we shall adopt each time determinants (always needing a regularization) has to be computed. Concerning the Jacobian for passing from the b_μ variable to the $b, \tilde{b}$ one can easily show that

$$\int \Delta\delta(b_+ - s_+)\mathcal{D}b_\mu = \int \exp(\kappa W[\tilde{b}\tilde{s}sb])\delta(\tilde{b} - I)\mathcal{D}\tilde{b}\mathcal{D}b \qquad (22)$$

so that $Z_{fer}[s]$ becomes

$$Z_{fer}[s] = \int \mathcal{D}\hat{a}\mathcal{D}b \exp\left(i\frac{C}{4\pi}tr \int d^2x (D_+[\tilde{s}]\hat{a})\, sb(\partial_- b^{-1})s^{-1}\right) \times \exp\left((1+\kappa)W[\tilde{s}sb]\right) . \tag{23}$$

A convenient change of variables to pass from integration over the algebra valued Lagrange multiplier $\hat{a}$ to a group valued variable a is the one defined through

$$D_+[\tilde{s}]\hat{a} = \tilde{s}^{-1}(a^{-1}\partial_+ a)\tilde{s} \tag{24}$$

Calculation of the corresponding jacobian J_L

$$\mathcal{D}\hat{a} = J_L \mathcal{D}a \tag{25}$$

should be carefully done. Indeed, in the present approach to bosonization, it is the group valued variable a who plays the role of the boson field equivalent to the original fermion field. Since for the latter (a free fermi field) there was no local symmetry, the former should not be endowed with this symmetry. With this in mind, we shall maintain a unchanged under local transformations g while transforming $\hat{a}$, $\hat{a} \to g^{-1}\hat{a}g$, so that eq.(24) changes covariantly when one simultaneously changes $\tilde{s} \to \tilde{s}g$. One can easily prove that

$$J_L = \exp(\kappa W[a\tilde{s}s] - \kappa W[\tilde{s}s]) \tag{26}$$

so that the generating functional reads

$$Z_{fer}[s] = \int \mathcal{D}a\mathcal{D}b \exp((1+\kappa)W[\tilde{s}sb]) \times \exp\left(\kappa(W[a\tilde{s}s] - W[\tilde{s}s])\right) \times \exp(-\tfrac{C}{4\pi}tr \int d^2x \tilde{s}^{-1}(a^{-1}\partial_+ a)\tilde{s}s(b\partial_- b^{-1})s^{-1}) . \tag{27}$$

If one repeatedly uses Polyakov-Wiegmann identity and chooses the up to now arbitrary constant C as

$$-C = 1 + \kappa \tag{28}$$

one can write $Z_{fer}[s]$ in the form

$$Z_{fer}[s] = \int \mathcal{D}a\mathcal{D}b\ \exp(W[\tilde{s}s] - W[a\tilde{s}s]) \exp\left((1+\kappa)W[a\tilde{s}sb]\right) \tag{29}$$

Now, the b integration can be trivially factorized this leading to

$$Z_{fer}[s] = \int \mathcal{D}a \ \exp(-W[a\tilde{s}s] + W[\tilde{s}s])\,. \tag{30}$$

or, after the shift $a\tilde{s}s \to \tilde{s}as$

$$\begin{aligned} Z_{fer}[s_+] &= \int \mathcal{D}a \ \exp(-W[a] + \frac{i}{4\pi}tr\int d^2x(s_+a\partial_-a^{-1} + \\ & s_-a^{-1}\partial_+a)) \times \exp(\frac{1}{4\pi}tr\int d^2x(a^{-1}s_+as_- - s_+s_-)) \end{aligned} \tag{31}$$

We have then arrived to the identity

$$Z_{fer}[s] = Z_{bos}[s] \tag{32}$$

where $Z_{bos}[s]$ is the generating function for a Wess-Zumino-Witten model. Differentiation with respect to any one of the two sources gives correlation functions in a given chirality sector. The answer corresponds to Witten's bosonization recipe [25]

$$\bar{\psi}t^a\gamma_+\psi \to \frac{i}{4\pi}a^{-1}\partial_+a \tag{33}$$

$$\bar{\psi}t^a\gamma_-\psi \to \frac{i}{4\pi}a\partial_-a^{-1}\,, \tag{34}$$

the l.h.s. to be computed in a free fermionic model, the r.h.s. in a Wess-Zumino-Witten model.

3 $d = 3$ non-abelian bosonization

Contrary to the case of two-dimensional massless fermions, one cannot compute exactly the Dirac operator determinant for $d > 2$ in the presence of an arbitrary gauge field b_μ, neither in the massless nor in the massive case. This implies the necessity of making approximations at some stage of our bosonization procedure to render calculations feasible. In the $d = 3$ Abelian case one can handle these approximations in a very general framework [23],[3]. Being the non-Abelian case far more complicated than the Abelian one, we shall only discuss the limiting case of very massive fermions, for which the fermion determinant is related to the Chern-Simons (CS) action [36]-[37].

A second problem arising when one tries to extend the non-Abelian boson-fermion mapping from $d = 2$ to $d = 3$ concerns the central role that plays the Polyakov-Wiegman identity, related to the holomorphic properties of the two-dimensional model [38]. In principle, a 3-dimensional analogue of this identity is not available and this forbids a trivial extension to $d = 3$ of the procedure described in the precedent section for two-dimensional bosonization. However, as we shall see, once one introduces the auxiliary field b_μ, a BRST invariance of the kind arising in topological field theories [27]-[29] can be unraveled. The use of BRST technique for bosonization of fermion models was initiated in the developement of the smooth bosonization approach [5]-[7], [11]-[12], closely related to bosonization duality and to the present treatment. In the present case, it allows to factor out the auxiliary field in the same way Polyakov-Wiegmann identity did the job in $d = 2$.

The resulting bosonization action coincides with that obtained using a completely different approach [4], based in the use of an interpolating Lagrangian [39]-[41]. The advantage of the present method lies in the fact that the BRST symmetry can be formulated in arbitrary dimensions while the interpolating Lagrangian, which replaces the role of this symmetry in decoupling auxiliary and bosonic fields is in principle applicable only in odd-dimensional spaces.

We consider N massive Dirac fermions in $d = 3$ Euclidean dimensions with Lagrangian

$$L = \bar{\psi}(i\partial\!\!\!/ + m)\psi \tag{35}$$

The corresponding generating functional reads

$$Z_{fer}[s] = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp[-\int d^3x \bar{\psi}(i\partial\!\!\!/ + s\!\!\!/ + m)\psi] \tag{36}$$

Again, we introduce an auxiliary vector field b_μ and use the $d = 3$ identity (proven in the Appendix)

$$Z_{fer}[s] = X[s]^{-1}\int Db_\mu X[b] \det(2\varepsilon_{\mu\nu\alpha}D_\nu[b])\delta({}^*\!f_\mu[b] - {}^*\!f_\mu[s]) \det(i\partial\!\!\!/ + m + b\!\!\!/) \tag{37}$$

Here

$${}^*\!f_\mu = \varepsilon_{\mu\nu\alpha}f_{\nu\alpha} \tag{38}$$

Concerning $X[b]$, it is an arbitrary functional which can be introduced in order to control the issue of symmetries at each stage of our derivation. Indeed, bosonization of three dimensional very massive fermions ends with a bosonic field with dynamics governed by a Chern-Simons action. As explained in [2], an appropriate choice of X allows to end with the natural gauge connection transformation law for this bosonic field. Following [2], we choose X in the form

$$X[b] = \exp(\mp \frac{i}{24\pi} \varepsilon_{\mu\nu\alpha} tr \int d^3x b_\mu b_\nu b_\alpha) \tag{39}$$

We can see at this point how an exact bosonization rule for the fermion current can be derived independently of the fact that one cannot calculate exactly the fermion determinant for $d > 2$. Indeed, if we introduce a Lagrange multiplier A_μ to represent the delta function, we can write Z_{fer} in the form

$$Z_{fer}[s] = X[s]^{-1} \int \mathcal{D}A_\mu \exp\left(\mp \frac{i}{16\pi} tr \int d^3x A_\mu {}^*\!f_\mu[s]\right) \times \exp(-S_{bos}[A]) \tag{40}$$

where we have defined the bosonic action $S_{bos}[A]$ as

$$\begin{aligned} \exp(-S_{bos}[A]) &= \int \mathcal{D}b_\mu \det(i\partial\!\!\!/ + m + b\!\!\!/) X[b] \times \\ & \det(2\varepsilon_{\mu\nu\alpha} D_\nu[b]) \exp\left(\pm \frac{i}{16\pi} tr \int d^3x A_\mu {}^*\!f_\mu[b]\right) \end{aligned} \tag{41}$$

With the choice (39) one indeed has gauge invariance of $S_{bos}[A]$, A_μ and b_μ both transforming as gauge fields, and one also explicitely verifies eq.(8).

Then, from eq.(40) we have

$$\bar\psi \gamma_\mu t^a \psi \to \pm \frac{i}{8\pi} \varepsilon_{\mu\nu\alpha} \partial_\nu A^a_\alpha \ . \tag{42}$$

In writing eq.(42) we have ignored terms quadratic and cubic in the source which, as in $d = 2$, are irrelevant for the current algebra. Correlation functions of currents pick a contribution from these terms, as already discussed in other approaches to bosonization [30]-[31]. Having these terms local support, they do not contribute to the current commutator algebra. (That this is so can be easily seen using for example the Bjorken-Johnson-Low method).

We insist that our result (42) does not imply any kind of approximation. However, to achieve a complete bosonization, one needs an explicit local form for the bosonic action and it is at this point where approximations have to be envisaged so as to evaluate the fermion determinant. In $d = 3$ dimensions this determinant cannot be computed exactly. However, all approximation approaches and regularization schemes have shown the occurrence of a parity violating Chern-Simons term together with parity conserving terms which can be computed approximately. We shall use the result obtained by making an expansion in inverse powers of the fermion mass [37],

$$\ln\det(i\partial\!\!\!/ + m + b\!\!\!/) = \pm\frac{i}{16\pi}S_{CS}[b] + I_{PC}[b] + O(\partial^2/m^2), \tag{43}$$

where the Chern-Simons action S_{CS} is given by

$$S_{CS}[b] = \int d^3x\,\varepsilon_{\mu\nu\lambda}\,\mathrm{tr}\int d^3x\,(f_{\mu\nu}b_\lambda - \frac{2}{3}b_\mu b_\nu b_\lambda). \tag{44}$$

Concerning the parity conserving contributions, one has

$$I_{PC}[b] = -\frac{1}{24\pi m}\,\mathrm{tr}\int d^3x\, f^{\mu\nu}f_{\mu\nu} + \cdots, \tag{45}$$

We can then write, up to corrections of order $1/m$, the bosonic action $S_{bos}[A]$ in the form (From here on we shall omit to indicate the trace *tr* for notation simplicity)

$$\begin{aligned}\exp(-S_{bos}[A]) \;&=\; \int \mathcal{D}b_\mu\, X[b]\exp(\pm\frac{i}{16\pi}S_{CS}[b])\times\\ &\det(2\varepsilon_{\mu\nu\alpha}D_\nu[b])\exp\left(\pm\frac{i}{16\pi}\int d^3x A_\mu {}^*\!f_\mu[b]\right)\end{aligned} \tag{46}$$

We shall now introduce ghost fields $\bar{c}_\alpha$ and c_α to write the determinant in the r.h.s. of eq.(46). With this, $Z_{fer}[s]$ takes the form

$$\begin{aligned}Z_{fer}[s] \;&=\; X[s]^{-1}\int \mathcal{D}b_\mu \mathcal{D}\bar{c}_\alpha \mathcal{D}c_\alpha \mathcal{D}A_\mu \exp\left(\mp\frac{i}{16\pi}\int d^3x A_\mu {}^*\!f_\mu[s]\right)\times\\ &\exp(-S_{eff}[b,A,\bar{c},c])\end{aligned} \tag{47}$$

with

$$\begin{aligned}S_{eff}[b,A,\bar{c},c] \;&=\; \mp\frac{i}{16\pi}S[b]\\ &\mp\frac{i}{8\pi}\varepsilon_{\mu\nu\alpha}\int d^3x(A_\mu(\partial_\nu b_\alpha + b_\nu b_\alpha) - \bar{c}_\mu D_\nu[b]c_\alpha)\end{aligned} \tag{48}$$

and

$$S[b] = 2\varepsilon_{\mu\nu\alpha}\int d^3x b_\mu(\partial_\nu b_\alpha + \frac{1}{3}b_\nu b_\alpha) \tag{49}$$

At this point we have arrived to an exact bosonization recipe for the fermion current, eq.(42), but we still need an explicit formula for the bosonic action as a functional of A_μ. This requires integration over the auxiliary fields b_μ, $\bar{c}_\mu$ and c_μ of the complicated effective action S_{eff} as defined by eq.(48). In the two-dimensional case, this last step was possible because Polyakov-Wiegmann identity allowed us to decouple the auxiliary fields from the bosonic field (A_μ). In the present case, integration will be possible because of the existence of an underlying BRST invariance that can be made apparent in S_{eff}. In order to directly get an *off-shell* nilpotent set of BRST transformations leaving invariant the effective action, we shall introduce additional auxiliary fields [42], thus writing

$$\begin{aligned} Z_{fer}[s] &= X[s]^{-1} \int \mathcal{D}b_\mu \mathcal{D}\bar{c}_\alpha \mathcal{D}c_\alpha \mathcal{D}A_\mu \mathcal{D}h_\mu \mathcal{D}l \mathcal{D}\bar{\chi} \\ & \qquad \exp\left(\mp \frac{i}{16\pi}\int d^3x A_\mu {}^*\!f_\mu[s]\right) \exp(-\tilde{S}_{eff}[b,A,\bar{c},c,h,l,\bar{\chi}]) \end{aligned} \tag{50}$$

with $\tilde{S}_{eff}$ defined as

$$\begin{aligned} &\tilde{S}_{eff}[b,A,\bar{c},c,h,l,\chi] = \mp\frac{i}{16\pi}S[b-h] \mp \frac{i}{16\pi}\int d^3x(l h_\mu h_\mu - 2\bar{\chi}h_\mu c_\mu) \\ &\mp\frac{i}{8\pi}\varepsilon_{\mu\nu\alpha}\int d^3x(A_\mu(\partial_\nu b_\alpha + b_\nu b_\alpha) - \bar{c}_\mu D_\nu[b]c_\alpha) \end{aligned} \tag{51}$$

Integration over the auxiliary field l makes $h_\mu = 0$ this showing the equivalence of eq.(50) and eq.(47). Now, the effective action $\tilde{S}_{eff}$ is invariant under BRST transformations defined as

$$\begin{gathered} \delta\bar{c}_\alpha = A_\alpha \qquad \delta A_\alpha = 0 \\ \delta b_\alpha = c_\alpha \qquad \delta c_\alpha = 0 \\ \delta h_\alpha = c_\alpha \quad \delta\bar{\chi} = l \quad \delta l = 0 \end{gathered} \tag{52}$$

This BRST transformations are related to those employed in the smooth bosonization [5]-[7], [11]-[12] approach and resemblant of those arising in topological field theories [27]-[29]. For example, in $d = 4$ topological Yang-Mills theory the invariance of the starting classical action (the Chern-Pontryagin topological charge) under the most general transformation of the gauge field, $b_\mu \to b_\mu + \epsilon_\mu$, leads to a BRST transformation for b_μ of the form

$\delta b_\mu = c_\mu$, which corresponds to that in formula (52) [26]-[27]. Closer to our model are the so-called Schwartz type topological theories which include the Chern-Simons theory and the BF model analyzed in detail in refs.[28]-[29]. It should be stressed that the topological character of the effective action (48) exclusively concerns the large fermion mass regime where the fermion determinant can be written in terms of the CS action.

Now, using transformations (52), $\tilde{S}_{eff}$ can be compactly written in the form

$$\tilde{S}_{eff}[b, A, \bar{c}, c] = \mp\frac{i}{16\pi} S[b-h] \mp \frac{i}{8\pi}\int d^3x\, \delta\mathcal{F}[\bar{c}, b, h, \bar{\chi}] \tag{53}$$

with

$$\mathcal{F} = \varepsilon_{\mu\nu\alpha}\bar{c}_\mu(\partial_\nu b_\alpha + b_\nu b_\alpha) + \frac{1}{2}\bar{\chi} h_\mu h_\mu \tag{54}$$

At this point, an arbitrary functional $\mathcal{G}$ may be added to $\mathcal{F}$ without changing the partition function since it will enter in Z_{fer} as an exact BRST form. The idea is to choose $\mathcal{G}$ so as to decouple the auxiliary field b_μ (to be integrated out afterwards) from the vector field A_μ which will be the bosonic counterpart of the original fermion field. We shall then consider

$$\mathcal{F} \to \mathcal{F} + \mathcal{G} \tag{55}$$

with

$$\begin{aligned} \mathcal{G} \;=\; & \frac{1}{2}\varepsilon_{\mu\nu\alpha}\bar{c}_\mu\left([b_\nu, A_\alpha] + [A_\nu, A_\alpha] + C[b_\nu, h_\alpha] + (1+C)[A_\nu, h_\alpha]\right. \\ & \left. -(C+1)[h_\nu, h_\alpha] + 2\partial_\nu b_\alpha + 4\partial_\nu A_\alpha + 2C\partial_\nu h_\alpha\right) \end{aligned} \tag{56}$$

Here C is an arbitrary constant.

The addition of $\delta\mathcal{G}$ allows us to make contact at this point with the effective action discussed in refs.[4],[41]. Indeed, after the shift

$$b_\mu \to 2b_\mu - A_\mu + h_\mu \tag{57}$$

(the new b_μ transforms again as a gauge connection, with h_μ transforming covariantly) the Lagrange multiplier A_μ (which will play the role of the bosonic field in our bosonization approach, as identified by the source term) completely decouples for $h_\mu = 0$, so that integrating out auxiliary fields we end with

$$Z_{fer}[s] = \mathcal{N} X[s]^{-1}\int \mathcal{D}A_\mu \exp(\mp\frac{i}{16\pi}\int d^3x A_\mu {}^*\!f_\mu[s]) \times \exp(\pm\frac{i}{16\pi}S_{CS}[A]). \tag{58}$$

Here $\mathcal{N}$ is a constant (i.e. it is independent of the source) resulting from integration of auxiliary, ghosts and the b field,

$$\mathcal{N} = \int \mathcal{D}b_\mu \; det\, (2(2+C)\varepsilon_{\mu\nu\alpha} D_\nu[b]) \exp(\pm \frac{i}{4\pi} S_{CS}[b]) \tag{59}$$

We have then the bosonization result

$$Z_{fer}[s] \approx Z_{CS}[s] \tag{60}$$

where $\approx$ means that our result is valid up to $1/m$ corrections since we used a result for the fermion determinant which is valid up to this order. We then see that we have ended with a Chern-Simons action as the bosonic equivalent of the original free fermion action with a coupling to the external source s_μ of the form $A_\mu {}^*\mathcal{J}_\mu[s]$.

In considering fermion current bosonization within the $1/m$ approximation, the following facts should be taken into account. It is at the lowest order in $1/m$ that the resulting bosonic action is topological and a large BRST invariance is unraveled. Now, using the freedom to modify the action by BRST exact forms, one could think of adding to the topological bosonic action terms of the form $\delta\mathcal{H}$ with

$$\mathcal{H} = \int d^3x \; \varepsilon_{\mu\nu\alpha} \bar{c}_\mu \mathcal{H}_{\nu\alpha}[s]\,, \tag{61}$$

with $\mathcal{H}_{\mu\nu}[s]$ an arbitrary functional of the external source s_μ. In particular, choosing adequately $\mathcal{H}$ one could think of changing or even eliminating, to this order in $1/m$, the source dependence from $Z_{fer}[s]$. Now, this is a characteristic of Schwarz like topological models [29]. In particular, the phase space of the Chern-Simons theory is the moduli space of flat connections on the given space manifold. So, if one looks *up to this order in* $1/m$, at the generating functional of current Green functions, one has, from Eqs.(36) and (43) that the generating functional of connected Green functions is precisely a Chern-Simons action for the source s,

$$W[s] = -logZ[s] = \mp \frac{i}{16\pi} S_{CS}[s]. \tag{62}$$

Thus, making functional derivatives in the above expression with respect to the source and then putting the sources to zero, all current vacuum expectation values vanish identically up to contact terms (these terms, derivatives of the Dirac delta function, also vanish if we regularize appropriately the product of operators at coincident points. Moreover our results are valid in the $m \to \infty$ limit where the deep ultraviolet region is excluded). Non-vanishing observables are in fact topological objets, non local functionals of A_μ (Wilson loops) that are in correspondance to knots polynomial invariants. Hence the bosonization recipe (42) when used to this order in $1/m$ makes sense if one is to calculate vacuum expectation values of fermion objects leading for example to holonomies in terms of the bosonic field A_μ. This calculation was discussed at length in [4].

4 Summary

We have shown in this paper that the path-integral bosonization approach developed in previous investigations [1]-[4] is well-suited to study fermion models in $d \geq 2$ dimensions when a non-Abelian symmetry is present.

We have started by reobtaining in Section 2 the well-honoured non-Abelian bosonization recipe for two dimensional massless fermions. Although well-known, this result allowed us to identify the point in which the non-Abelian character of the symmetry makes difficult the factorization of the path-integral which will represent the partition function of the resulting bosonic model. In two dimensions this factorization can be seen as a result of the existence of the Polyakov-Wiegmann identity for Wess-Zumino-Witten actions, and this can *a priori* put some doubts on the possibility of extending the approach to $d > 2$.

That also in $d = 3$ one can obtain very simple bosonization rules for the non-Abelian case is the main result of section 3. Concerning the fermion current, we obtained an exact bosonization result which is the natural extension of the Abelian case. In respect with the bosonization recipe for the fermion action, we considered the case of very massive fermions for which the fermion determinant is related to the non-Abelian Chern-Simons action. In this case the factorization of the auxiliary and Lagrange multiplier fields is achieved after discovering a BRST invariance reminiscent of that at the root of topological models and related to that exploited in the smooth bosonization approach [5]-[7], [11]-[12]. Addition of BRST exact terms allows us to extract the partition function for the boson counterpart of the original fermion fields.

Our bosonization method starts by introducing in the fermionic generating functional an auxiliary field as it is done in the smooth bosonization and duality approaches to bosonization [5]-[15]. It becomes clear in our approach that, for non-Abelian symmetries, it is crucial to include the "Faddeev-Popov" like determinant which accompanies the delta function imposing a condition on the auxiliary field curvature. In fact, the BRST symmetry which allowed to arrive to the correct bosonic generating functional can be seen as a result of this fact and related to the way in which BRST symmetry can be unraveled by a change of variables as advocated in ref.[42].

It should be stressed that the only approximation in our approach stems from the necessity of evaluating the fermion determinant which, in $d > 2$, implies some kind of expansion. In the present work we have used a result valid for very massive fermions but one can envisage approximations which can cover other regimes, in particular the massless case. This was considered for the abelian case in [23] and the corresponding bosonization analysis thoroughly discussed in [3]. We expect that a similar analysis can be done in the non-abelian case and we hope to report on it in a future paper.

Appendix

$d = 2$

We shall prove identity (14) used in our derivation of $d = 2$ bosonization rules,

$$\int \mathcal{D}b_\mu \mathcal{H}[b]\delta\left[\varepsilon_{\mu\nu}(f_{\mu\nu}[b] - f_{\mu\nu}[s])\right] = \int \mathcal{D}g\mathcal{H}[s^g] \tag{63}$$

where $\mathcal{H}$ is a gauge-invariant functional. Note that in eq.(63) it is implicit that b_μ should be treated as a gauge field and hence a gauge fixing is required. A convenient gauge choice is

$$b_+ = s_+ \tag{64}$$

so that identity (63) takes the form

$$\int \mathcal{D}b_+\mathcal{D}b_- \ \Delta\delta(b_+ - s_+)\mathcal{H}[b_+, b_-]\delta[\varepsilon_{\mu\nu}(f_{\mu\nu}[b] - f_{\mu\nu}[s]] = \int \mathcal{D}g\mathcal{H}[s^g] \tag{65}$$

with Δ the Faddeev-Popov determinant for gauge condition (64),

$$\Delta = detD_+^{Adj}[s_+] \tag{66}$$

We now prove eq.(65). Let us start from the l.h.s. of eq.(65) performing first the b_+ trivial integration and then the b_- one

$$\begin{aligned}
&\int \mathcal{D}b_+\mathcal{D}b_-\Delta\delta(b_+ - s_+)\mathcal{H}[b_+, b_-]\delta(\varepsilon_{\mu\nu}(f_{\mu\nu}[b] - f_{\mu\nu}[s]) = \\
&\Delta[s_+]\int \mathcal{D}b_-\mathcal{H}[s_+, b_-]\delta(D_+[s_+]b_- - D_+[s_+]s_-) = \\
&\frac{\Delta[s_+]}{detD_+^{Adj}[s_+]}\int \mathcal{D}b_-\mathcal{H}[s_+, b_-]\delta(b_- - s_-) = \mathcal{H}[s_+, s_-]
\end{aligned} \tag{67}$$

In the last line we have used the explicit form of the Faddeev-Jacobian to cancel out both determinants. Being $\mathcal{H}[s_+, s_-]$ gauge independent, we can rewrite (67) in the form (appart from a gauge group volume factor)

$$\int \mathcal{D}b_+\mathcal{D}b_-\Delta\delta(b_+ - s_+)\mathcal{H}[b_+, b_-]\delta\left[\varepsilon_{\mu\nu}(f_{\mu\nu}[b] - f_{\mu\nu}[s])\right] = \int \mathcal{D}g\mathcal{H}[s_+^g, s_-^g] \tag{68}$$

Identity (63) is then proven.

$d = 3$

The proof of identity (37), the analogous in the $d = 3$ case of (63), is very simple. One wishes to prove that the generating functional $Z_{fer}[s]$ in the presence of a source s_μ,

$$Z_{fer}[s] = \det(i\partial\!\!\!/ + m + s\!\!\!/) = F[s] \tag{69}$$

can be written in the form

$$Z_{fer}[s] = X[s]^{-1} \int \mathcal{D}b_\mu \, X[b]F[b] \, \det(2\varepsilon_{\mu\nu\alpha}D_\nu[b]) \, \delta\left(\varepsilon_{\mu\nu\alpha}(f_{\nu\alpha}[b] - f_{\nu\alpha}[s])\right) \tag{70}$$

Here $X[b]$ is an arbitrary functional of b_μ satisfying $X[0^g] = 1$. As advocated in [2], its introduction allows to end with a model in which the bosonic field transforms as a connection, this being consistent with the fact its dynamics is governed by a Chern-Simons action.

The proof of eq.(70) is based on the well-known identity

$$\delta(H[b]) = [\det(\delta H/\delta b)]^{-1}\delta(b - b^*) \tag{71}$$

with $H[b^*] = 0$ and the fact that the equation $f_{\mu\nu}[b] = f_{\mu\nu}[s]$ has the unique solution $b_\mu = s_\mu$.

Let us end by noting that if one compares formula (70) in $d = 3$ dimensions with the corresponding one in $d = 2$ (for example the identity (63)), one sees that a determinant equivalent to that appearing in the former is absent in the latter. This is due to the fact that the curvature condition requires three delta functions in $d = 3$ dimensions but only one in $d = 2$. Handling these delta functions leaves a jacobian in three dimensions while no jacobian remains in two dimensions.

Acknowledgements: The authors wish to thank Matthias Blau, Daniel Cabra, François Delduc, César Fosco, Eric Ragoucy and Frank Thuillier for helpful discussions and comments. F.A.S. and C.N. are partially suported by Fundacion Antorchas, Argentina and a Commission of the European Communities contract No:C11*-CT93-0315.

References

[1] E. Fradkin and F.A. Schaposnik, Phys. Lett. **B338** (1994) 253.

[2] F.A. Schaposnik, Phys. Lett. **B356** (1995) 39.

[3] J.C. Le Guillou, C.Núñez and F.A. Schaposnik, Ann. of Phys. (N.Y.) in press.

[4] N. Bralić, E. Fradkin, M.V. Manías and F.A. Schaposnik, Nucl.Phys. **B446** (1995) 144.

[5] P.H. Damgaard, H.B. Nielsen and R. Sollacher, Nucl. Phys. **B385** (1992) 227.

[6] P.H. Damgaard, H.B. Nielsen and R. Sollacher, Phys. Lett. **B296** (1992) 132.

[7] P.H. Damgaard, H.B. Nielsen and R. Sollacher, Nucl. Phys. **B414** (1994) 541.

[8] C.P. Burgess and F. Quevedo, Nucl. Phys. **B421** (1994) 373.

[9] C.P. Burgess an, C.A. Lütken and F. Quevedo, Phys. Lett. **B326** (1994) 18.

[10] C.P. Burgess and F. Quevedo, Phys. Lett. **B329** (1994) 457.

[11] P.H. Damgaard and R. Sollacher, Nucl. Phys. **B433** (1995) 671.

[12] P.H. Damgaard, F. De Jonghe and R. Sollacher, Nucl. Phys. **B454** (1995) 701.

[13] A.N. Theron, F.A. Schaposnik, F.G. Scholtz and H.B. Geyer, Nucl.Phys. **B437** (1995) 187.

[14] J.L. Cortés, E. Rivas and L. Velázquez, Phys. Rev. **D53** (1996) 5952.

[15] P.A. Marchetti in Common Trends in Condensed Matter and High Enery Physics, Chia Laguna, Italy, Sep 1995. hep-th/9511100 report.

[16] A. Luther, Phys. Rev. **D19** (1979) 320.

[17] M. Luscher, Nucl. Phys. **B326** (1989) 557.

[18] F.D.M. Haldane, Helv. Phys. Acta **65** (1992) 52.

[19] A. Kovner and P.S. Kurzepa, Phys. Lett. **B321** (1994) 129.

[20] E.C. Marino, *Phys. Lett.* **B263** (1991), 63.

[21] J. Frohlich, R. Götschmann and P.A. Marchetti, J.Phys. **A28** (1995) 1169.

[22] N. Banerjee, S. Banerjee and S. Gosh, hep-th/9607065, unpublished report.

[23] D.G. Barci, C.D. Fosco and L.E. Oxman, Phys. Lett. **B375** (1996) 267.

[24] A.M. Polyakov and P.B. Wiegmann, Phys. Lett. **131B** (1983) 121; Phys Lett. **141B** (1984) 223.

[25] E. Witten, Commun. Math. Phys. **92** (1984) 455.

[26] E. Witten, Commun. Math. Phys. **117** (1988) 353.

[27] L. Baulieu and I.M. Singer, Nucl. Phys. (Proc. Suppl.) **5B** (1988) 12.

[28] G. Horowitz, Comm. Math. Phys. **125** (1989) 417.

[29] D. Birmingham, M. Blau, M. Rakowski and G. Thompson, Phys. Rep. **209** (1991) 129.

[30] P. di Vecchia and P. Rossi, Phys. Lett. **140B** (1984) 344.

[31] Y. Frishmann, Phys. Lett. **146B** (1984) 204.

[32] P. Goddard and D. Olive, Int. Jour. Mod. Phys. **A1** (1986) 303.

[33] E. Fradkin, C.M. Naón and F.A. Schaposnik, Phys. Rev. **D36** (1987) 3809.

[34] P. di Vecchia, B. Durhus and J.L. Petersen, Phys. Lett. **144B** (1984) 245.

[35] D. Gonzáles and A.N. Redlich, Phys. Lett. 147B (1984) 150.

[36] A.T. Niemi and G.W. Semenoff, Phys. Rev. Lett. **51** (1983) 2077

[37] A.N. Redlich, Phys. Rev. Lett. **52** (1984) 18; Phys. Rev. **D29** (1984)236.

[38] E. Witten, Commun. Math. Phys. **144** (1992) 189.

[39] P.K. Townsend, K. Pilch and P. van Nieuwenhuizen, Phys. Lett. **B136** (1984) 38; **B137** (1984) 443.

[40] S. Deser and R. Jackiw, Phys.Lett. **B139** (1984) 371.

[41] A. Karlhede, U. Lindström, M. Roček and P. van Nieuwenhuizen, Phys. Lett. **186B** (1987) 96.

[42] F. Bastianelli, Nucl. Phys. **B361** (1991) 555.