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The Fermion-Boson Mapping in Three Dimensional Quantum Field Theory

Eduardo Fradkin

Department of Physics, University of Illinois at Urbana-Champaign
1110 W.Green St., Urbana, Illinois 61801-3080, USA

and

Fidel A. Schaposnik*

Departamento de Física, Universidad Nacional de La Plata
C.C. 67, (1900) La Plata, Argentina

Abstract

We discuss bosonization in three dimensions by establishing a connection between the massive Thirring model and the Maxwell-Chern-Simons theory. We show, to lowest order in inverse fermion mass, the identity between the corresponding partition functions; from this, a bosonization identity for the fermion current, valid for length scales long compared with the Compton wavelength of the fermion, is inferred. We present a non-local operator in the Thirring model which exhibits fractional statistics.

*Investigador CICBA, Argentina

In this paper we investigate the problem of the mapping of quantum field theories of interacting fermions in $2+1$ dimensions onto an equivalent theory of interacting bosons. These mappings, commonly called *bosonization*, are well established in the context of $1+1$ dimensional theories. There, bosonization constitutes one of the main tools available for the study of the non-perturbative behavior of both quantum field theories [1] and of condensed matter systems [2]. The bosonization identities, which relate the fermionic current with the topological current of a bosonic theory, can be viewed as a consequence of a non-trivial current algebra. However, in dimensions other than $1+1$, much less is known. Although current algebras do exist in all dimesions, the Schwinger terms have a much more complex structure in higher dimensions. Also, a simple counting of degrees of freedom shows that a simple minded mapping between fermions and *scalars* can only hold in $1+1$ dimensions. In this work we show that the *low energy sector* of a theory of massive self-interacting fermions, the massive Thirring Model in $2+1$ dimeensions, can be bosonized. However, the bosonized theory is a *gauge theory*, the Maxwell-Chern-Simons gauge theory.

Some time ago Deser and Redlich [3] discussed the equivalence of the three dimensional effective electromagnetic action of the CP^1 model and of a charged massive fermion to lowest order in inverse (fermion) mass, following the ideas of Refs. [4]-[5]. This issue is relevant in the context of transmutation of spin and statistics in three dimensions, with applications to interesting problems both in Quantum Field Theory and Condensed Matter physics [6]. The mapping, first discussed by Polyakov[4] and extended by Deser and Redlich[3], shows that a massive scalar particle coupled to a Chern-Simons gauge field becomes a massive Dirac fermion for a properly chosen value of the Chern-Simons coupling. As such, this Bose-Fermi transmutation is a property which holds only at very long distances, *i.e.* at scales long compared with the Compton wavelength of the particle. In terms of the path-integral picture, the transmutation is a property of very large, smooth, paths. Hence, these results hold to lowest order in an expansion in powers of the mass of the particle. Other approaches to bosonization in $2+1$ dimensions have been developed in Refs. [7]-[8].

In the same vein as in [3], we establish here, to leading order in the inverse fermionic mass, an identity between the partition functions for the three-dimensional Thirring model and the topologically massive $U(1)$ gauge theory, the Maxwell-Chern-Simons theory (MCS). This result enlarges the boson-fermion correspondence by connecting a self-interacting fermion model and a Chern-Simons system.

We start from the three-dimensional (Euclidean) massive Thirring model Lagrangian:

$$\mathcal{L}_{Th} = \bar{\psi}^i (i\partial\!\!\!/ + m)\psi^i - \frac{g^2}{2N} j^\mu j_\mu \tag{1}$$

where ψ^i are N two-component Dirac spinors and J^μ the $U(1)$ current,

$$j^\mu = \bar{\psi}^i \gamma^\mu \psi^i. \tag{2}$$

The coupling constant g^2 has dimensions of inverse mass. (Although non-renormalizable by power counting, four fermion interaction models in $2+1$ dimensions are known to be renormalizable in the $1/N$ expansion [9].)

The partition function for the theory is defined as

$$\mathcal{Z}_{Th} = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp[-\int \left(\bar{\psi}^i (i\partial\!\!\!/ + m)\psi^i - \frac{g^2}{2N} J^\mu J_\mu \right) d^3x] \tag{3}$$

We now eliminate the quartic interaction by introducing a vector field a_μ through the identity

$$\exp(\int \frac{g^2}{2N} J^\mu J_\mu d^3x) = \int \mathcal{D}a_\mu \exp[-\int (\frac{1}{2} a^\mu a_\mu + \frac{g}{\sqrt{N}} J^\mu a_\mu) d^3x] \tag{4}$$

(up to a multiplicative normalization constant) so that the partition function becomes

$$\mathcal{Z}_{Th} = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}a_\mu \exp[-\int (\bar{\psi}^i (i\partial\!\!\!/ + m + \frac{g}{\sqrt{N}}\ a\!\!\!/)\psi^i + \frac{1}{2} a^\mu a_\mu) d^3x]. \tag{5}$$

We are now going to perform the fermionic path-integral which gives as usual the Dirac operator determinant:

$$\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp(-\int \bar{\psi}^i (i\partial\!\!\!/ + m + \frac{g}{\sqrt{N}}\ a\!\!\!/)\psi^i d^3x) = \det(i\partial\!\!\!/ + m + \frac{g}{\sqrt{N}}\ a\!\!\!/) \tag{6}$$

Being the Dirac operator unbounded, its determinant requires regularization. Any sensible regularization approach (for example, ζ-function or Pauli-Villars approaches) gives a parity violating contribution [12]-[14]. There are also parity conserving terms which have been computed as an expansion in inverse powers of the fermion mass:

$$\ln \det(i\partial\!\!\!/ + m + \frac{g}{\sqrt{N}}\ a\!\!\!/) = \pm \frac{ig^2}{16\pi} \int \epsilon_{\mu\nu\alpha} f^{\mu\nu} a^{\alpha} d^3x + I_{PC}[a_\mu] + O(\partial^2/m^2), \quad (7)$$

$$I_{PC}[a_\mu] = -\frac{g^2}{24\pi m} \int d^3x f^{\mu\nu} f_{\mu\nu} + \dots \quad (8)$$

Using this result we can write Z_{Th} in the form:

$$Z_{Th} = \int Da_\mu \exp(-S_{eff}[a_\mu]) \quad (9)$$

where $S_{eff}[a_\mu]$ is given by

$$\begin{aligned} S_{eff}[a_\mu] &= \frac{1}{2} \int d^3x (a_\mu a^\mu \mp \frac{ig^2}{4\pi} \epsilon^{\mu\alpha\nu} a_\mu \partial_\alpha a_\nu) + \\ &+ \frac{g^2}{24\pi m} \int d^3x f^{\mu\nu} f_{\mu\nu} + O(\partial^2/m^2) \end{aligned} \quad (10)$$

Up to corrections of order $1/m$, we recognize in S_{eff} the self-dual action S_{SD} introduced some time ago by Townsend, Pilch and van Nieuwenhuizen [10],

$$S_{SD} = \frac{1}{2} \int d^3x (a_\mu a^\mu \mp \frac{ig^2}{4\pi} \epsilon^{\mu\alpha\nu} a_\mu \partial_\alpha a_\nu) \quad (11)$$

Then, to leading order in $1/m$ we have established the following identification:

$$Z_{Th} \approx \int Da_\mu \exp(-S_{SD}) \quad (12)$$

Now, Deser and Jackiw [11] have proven the equivalence between the model with dynamics defined by S_{SD} and the Maxwell-Chern-Simons theory . In what follows, we shall adapt Deser-Jackiw arguments to the path-integral framework showing that to the leading order in $1/m$ expansion the Thirring model partition function coincides with that of the MCS theory. To this end, let us introduce an "interpolating action" $S_I[a_\mu, A_\mu]$

$$S_I[a_\mu, A_\mu] = \int d^3x(\frac{1}{2}a_\mu a^\mu - i\epsilon^{\mu\alpha\nu}a_\mu\partial_\alpha A_\nu \mp i\frac{2\pi}{g^2}\epsilon^{\mu\alpha\nu}A_\mu\partial_\alpha A_\nu) \qquad (13)$$

and the corresponding partition function Z_I, a path-integral over both A_μ and a_μ,

$$Z_I = \int DA_\mu Da_\mu \exp(-S_I) \qquad (14)$$

The theory with action S_I is invariant under the local gauge transformation

$$\delta A_\mu = \partial_\mu\omega, \quad \delta a_\mu = 0 \qquad (15)$$

To see the connection between Z_I and Z_{Th} when written as in (10) and (11), let us perform the path-integral over A_μ in (13),

$$\begin{aligned} I[a_\mu] &\equiv \int DA_\mu \exp[\int d^3x(\pm i\frac{2\pi}{g^2}\epsilon^{\mu\alpha\nu}A_\mu\partial_\alpha A_\nu - i\epsilon^{\mu\alpha\nu}a_\mu\partial_\alpha A_\nu)] \\ &= \int DA_\mu \exp[-\int d^3x\frac{1}{2}A_\mu S^{\mu\nu}A_\nu + A_\mu J^\mu] \qquad (16) \end{aligned}$$

where we have scaled $A_\mu \to \sqrt{4\pi/g^2}A_\mu$ and defined

$$J^\mu = i\sqrt{\frac{g^2}{4\pi}}\epsilon^{\mu\rho\sigma}\partial_\rho a_\sigma \qquad (17)$$

$$S^{\mu\nu} = \mp i\epsilon^{\mu\alpha\nu}\partial_\alpha \qquad (18)$$

Being $S^{\mu\nu}$ non-invertible we shall define a regulated operator $S^{\mu\nu}[\Lambda]$,

$$S^{\mu\nu}[\Lambda] = \mp i\epsilon^{\mu\alpha\nu}\partial_\alpha + \frac{1}{\Lambda}\partial^\mu\partial\nu \qquad (19)$$

so that $I[a_\mu]$ can be calculated from the identity

$$I[a_\mu] = \lim_{\Lambda\to\infty} I^\Lambda[a_\mu] \qquad (20)$$

$$I^\Lambda[a_\mu] = \int DA_\mu \exp\left(-\int d^3x(\frac{1}{2}A_\mu S^{\mu\nu}[\Lambda]A_\nu + A_\mu J^\mu)\right) \qquad (21)$$

Now, $I^\Lambda[a_\mu]$ can be easily calculated,

$$I^{\Lambda}[a_\mu] = \exp(-\int d^3x d^3y J^\mu(x) S^{-1}_{\mu\nu}[\Lambda] J^\nu(y) \qquad (22)$$

with

$$S^{-1}_{\mu\nu}[\Lambda] = \pm\frac{i}{4\pi}\epsilon_{\mu\nu\alpha}\partial^\alpha \frac{1}{|x-y|} - \frac{\Lambda}{8\pi}\partial_\mu\partial_\nu |x-y| \qquad (23)$$

so that we finally have

$$I[a_\mu] = \exp(\pm i \int d^3x \frac{g^2}{8\pi}\epsilon^{\mu\alpha\nu} a_\mu \partial_\alpha a_\nu) \qquad (24)$$

and with this we see that

$$Z_I = \int Da_\mu \exp(-S_{SD}) \qquad (25)$$

Let us point that the non invertibility of the operator $S_{\mu\nu}$ is a consequence of the gauge invariance of the action S_I. The extra regulating term in (19) can be interpreted as originating from a gauge fixing term in the action of the form $(\Lambda/2)(\partial_\mu A^\mu)^2$ with the limit $\Lambda \to \infty$ enforcing the Lorentz gauge $\partial_\mu A^\mu = 0$.

Using eq.(12) we can then establish the following relation:

$$Z_{Th} \approx Z_I \qquad (26)$$

Now, if instead of integrating out A_μ in Z_I we integrate over a_μ, we easily find

$$Z_L = \int DA_\mu \exp \int d^3x (\frac{1}{4}F^2_{\mu\nu} \pm i\frac{2\pi}{g^2}\epsilon^{\mu\alpha\nu} A_\mu \partial_\alpha A_\nu) \qquad (27)$$

which is nothing but the partition function Z_{MCS} for the Maxwell-Chern-Simons theory [15]-[16]. Then, using eq.(26) one finally proves the equivalence, to leading order in $1/m$, of the partition functions for Thirring model and the MCS theory:

$$Z_{Th} \approx Z_{MCS} \qquad (28)$$

(As stressed above, $\approx$ means that the identification has been proven to leading order in $1/m$)

Equation (28) is one of the main results in our work. It expresses the equivalence between the low energy sector of a theory of 3-dimensional fermions (interacting via a current-current term) and (gauge) bosons (with Maxwell-Chern-Simons action). The Thirring coupling constant g^2/N in the fermionic model enters as $2\pi/g^2$ in the CS term. This means that the Thirring spin 1/2 fermion in 2+1 dimensions is equivalent to a spin 1 massive excitation, with mass π/g^2 [15]-[16].

The equivalence has been established at lowest order in $1/m$. However, if we follow Deser and Redlich [3] and consider that $g^2 = c/m$, with c a dimensionless coupling constant, the first term in I_{PC} becomes of order $1/m^2$ (see eq.(8)) and the equivalence is then extended to the next order.

In order to infer the bosonization recipe deriving from the equivalence, we add a source for the Thirring current:

$$L_{Th}[b_\mu] = L_{Th} + \int d^3x j^\mu b_\mu \tag{29}$$

Then, instead of (5), the partition function now reads:

$$\mathcal{Z}_{Th}[b_\mu] = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}a_\mu \exp[-\int (\bar{\psi}^i(i\partial\!\!\!/ + m + \frac{g}{\sqrt{N}} \; a\!\!\!/ + b\!\!\!/)\psi^i + \frac{1}{2}a^\mu a_\mu) d^3x]. \tag{30}$$

Or, after shifting $a_\mu \to a_\mu - (g/\sqrt{N})b_\mu$,

$$Z_{Th}[b_\mu] = \exp\left(-\frac{N}{2g^2}\int d^3x b_\mu b^\mu\right) \times \int Da_\mu \exp(-S_{eff}[a_\mu] + \frac{\sqrt{N}}{g^2}\int d^3x b_\mu a^\mu) \tag{31}$$

with $S_{eff}[a_\mu]$ still given by eq.(10). Then, we can again establish to order $1/m$ the connection between the Thirring and self-dual models, in the presence of sources:

$$Z_{Th}[b_\mu] = \exp\left(-\frac{N}{2g^2}\int d^3x b_\mu b^\mu\right) \times \int Da_\mu \exp(-S_{SD} + \frac{\sqrt{N}}{g}\int d^3x b_\mu a^\mu) \tag{32}$$

or

$$Z_{Th}[b_\mu] = \exp(-\frac{N}{2g^2}\int d^3x b_\mu b^\mu) \times Z_{SD}[b_\mu] \tag{33}$$

In order to connect this with the Maxwell-Chern-Simons system, let us again consider Jackiw-Deser [11] interpolating action S_I (eq.13) but now in the presence of sources:

$$S_I[a_\mu, A_\mu; b_\mu] = S_I[a_\mu, A_\mu] + \frac{\sqrt{N}}{g} \int d^3x a_\mu b^\mu \tag{34}$$

By integrating out A_μ, one easily shows that the corresponding partition function $Z_I[b_\mu]$ coincides with $Z_{SD}[b_\mu]$,

$$Z_I[b_\mu] = Z_{SD}[b_\mu] \tag{35}$$

Now, if one integrates first over a_μ one has

$$\begin{aligned} Z_I[b_\mu] &= \exp(\frac{N}{2g^2} \int d^3x b_\mu b^\mu) \times \int DA_\mu \exp(-\int d^3x \frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \\ & \pm i\frac{2\pi}{g^2} \int d^3x \epsilon_{\mu\nu\alpha}\partial^\mu A^\nu A^\alpha + \frac{\sqrt{N}}{g} \int d^3x \epsilon_{\mu\nu\alpha}\partial^\mu A^\nu b^\alpha) \end{aligned} \tag{36}$$

or

$$Z_I[b_\mu] = \exp(\frac{N}{2g^2} \int d^3x b_\mu b^\mu) \times Z_{MCS}[b_\mu] \tag{37}$$

Finally, using eq.(33) we have the identity between partition functions for the Thirring and Maxwell-Chern-Simons models in the presence of sources

$$Z_{Th}[b_\mu] \approx Z_{MCS}[b_\mu] \tag{38}$$

Here too, it is convenient to rescale the vector potential $A_\mu \to (g/\sqrt{4\pi})A_\mu$, so that the MCS action takes the standard form. With this change, $Z_{MCS}[b_\mu]$ in (38) now reads,

$$\begin{aligned} Z_{MCS}[b_\mu] = \int DA_\mu \exp(& - \int d^3x \frac{1}{4e^2} F_{\mu\nu}F^{\mu\nu} \pm i\frac{1}{2} \int d^3x \epsilon_{\mu\nu\alpha}\partial^\mu A^\nu A^\alpha \\ & + \sqrt{\frac{N}{4\pi}} \int d^3x \epsilon_{\mu\nu\alpha}\partial^\mu A^\nu b^\alpha) \end{aligned} \tag{39}$$

where $e^2 = 4\pi/g^2$.

From eqs.(38)-(39) we see that the bosonization rule for the fermion current reads, to leading order in $1/m$,

$$\bar{\psi}\gamma^\mu\psi \to i\sqrt{\frac{N}{4\pi}}\epsilon^{\mu\nu\alpha}\partial_\nu A_\alpha \tag{40}$$

A few comments are in order. Firstly, the bosonized expression for the fermion current is manifestly conserved. Secondly, this formula is the $2+1$-dimensional analog of the $1+1$-dimensional result $\bar{\psi}\gamma^\mu\psi \to (1/\sqrt{\pi})\epsilon_{\mu\nu}\partial^\nu\phi$. (The factor of i in the expression for the current in eq.(40) appears because, here, we are working in Euclidean space). However, unlike the $1+1$-dimensional formula, which is a short distance identity valid for length scales long compared to a short distance cutoff but small compared with the Compton wavelength of the fermion, the $2+1$-dimensional identity is valid only for length scales long compared with the Compton wavelength of the fermion.

We give now a first application of the bosonization formulas and, in this way, explore their physical content. The effective action of eq.(27) has a Chern-Simons term which controls its long distance behavior. It is well known[4, 17] that the Chern-Simons gauge theory is a theory of knot invariants which realizes the representations of the Braid group. These knot invariants are given by expectation values of Wilson loops in the Chern-Simons gauge theory. In this way, it is found that the expectation values of the Wilson loop operators imply the existence of excitations with fractional statistics. Thus, it is natural to seek the fermionic analogue of the Wilson loop operator W_Γ which, in the Maxwell-Chern-Simons theory is given by

$$W_\Gamma = \langle \exp\{i\frac{\sqrt{N}}{g}\oint_\Gamma A_\mu dx^\mu\}\rangle \tag{41}$$

where Γ is the union of a an arbitrary set of closed curves (loops) in three dimensional euclidean space. Given a closed loop (or union of closed loops) Γ, it is always possible to define a set of open surfaces Σ whose boundary is Γ, *i.e.* $\Gamma = \partial\Sigma$. Stokes' theorem implies that

$$\begin{aligned}\langle \exp\{i\frac{\sqrt{N}}{g}\oint_\Gamma A_\mu dx^\mu\}\rangle &= \langle \exp\{i\frac{\sqrt{N}}{g}\int_\Sigma dS_\mu \epsilon^{\mu\nu\lambda}\partial_\nu A_\lambda\}\rangle \\ &= \langle \exp\{i\frac{\sqrt{N}}{g}\int d^3x \epsilon^{\mu\nu\lambda}\partial_\nu A_\lambda\, b_\lambda\}\rangle \qquad (42)\end{aligned}$$

is an identity. Here $b_\lambda(x)$ is the vector field

$$b_\lambda(x) = n_\lambda(x)\delta_\Sigma(x) \tag{43}$$

where n_λ is a field of unit vectors normal to the surface Σ and $\delta_\Sigma(x)$ is a delta function with support on Σ. Using eq.(38) we find that this expectation value becomes, in the Thirring Model, equivalent to

$$W_\Gamma = \langle \exp\{i\frac{\sqrt{N}}{g}\int_{\partial\Sigma} dx_\mu A^\mu\}\rangle_{MCS} = \langle \exp\{\int_\Sigma dS_\mu \bar{\psi}\gamma^\mu\psi\}\rangle_{Th} \tag{44}$$

More generally we find that the Thirring operator $\mathcal{W}_\Sigma$

$$\mathcal{W}_\Sigma = \langle \exp\{q\int_\Sigma dS_\mu \bar{\psi}\gamma^\mu\psi\}\rangle_{Th} \tag{45}$$

obeys the identity

$$\langle \exp\{q\int_\Sigma dS_\mu \bar{\psi}\gamma^\mu\psi\}\rangle_{Th} = \langle \exp\{iq\frac{\sqrt{N}}{g}\oint_\Gamma A_\mu dx^\mu\}\rangle MCS \tag{46}$$

for an arbitrary fermionic charge q.

The identity (46) relates the flux of the fermionic current through an open surface Σ with the Wilson loop operator associated with the boundary Γ of the surface. The Wilson loop operator can be trivially calculated in the Maxwell-Chern-Simons theory. For very large and smooth loops the behavior of the Wilson loop operators is dominated by the Chern-Simons term of the action. The result is a topological invariant which depends only on the linking number ν_Γ of the set of curves Γ [4, 17]. By an explict calculation one finds

$$\langle \exp\{q\int_\Sigma dS_\mu \bar{\psi}\gamma^\mu\psi\}\rangle_{Th} = \exp\{\mp i\nu_\Gamma\frac{Nq^2}{8\pi}\} \tag{47}$$

This result implies that the non-local Thirring loop operator $\mathcal{W}_\Sigma$ exhibits fractional statistics with a statistical angle $\delta = Nq^2/8\pi$. The topological significance of this result bears close resemblance with the bosonization identity in $1+1$ dimensions between the circulation of the fermion current on a closed curve and the topological charge (or instanton number) enclosed in the interior of the curve[18]. From the point of view of the Thirring model, this is a most surprising result which reveals the power of the bosonization identities.

To the best of our knowledge, this is the first example of a purely fermionic operator, albeit non-local, which is directly related to a topological invariant.

In summary, in this work we presented a mappping between the low energy sector of a self-interacting fermionic quantum field theory, the massive Thirring model in 2+1 dimensions, and a bosonic theory of a vector field, the Maxwell-Chern-Simons theory. This dual gauge theory has a spectrum which consists of a spin one bosonic excitation of mass π/g^2. We presented a number of identities for the partition functions and for the generating function of the fermion current correlation functions. The Wilson loop operator of the dual gauge theory were found to have a natural expression in terms of the fermion theory. As a byproduct, we found a fermion loop operator which exhibits fractional statistics.

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