## Brans-Dicke wormholes in nonvacuum spacetime

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## Abstract

Analytical wormhole solutions in Brans-Dicke theory in the presence of matter are presented. It is shown that the wormhole throat must not be necessarily threaded with exotic matter.

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The field equations of general relativity, being local in character, admit solutions with nontrivial topology. Among these, wormholes have been extensively studied [1]. Their most salient feature is that an embedding of one of their spacelike sections in Euclidean space displays two asymptotically flat regions joined by a throat.

The interest on wormholes is twofold. From the point of view of the Euclidean path integral formulation of quantum gravity, Coleman [2] and Giddings and Strominger [3], among others, have shown that the effect of wormholes is to modify low energy coupling constants and to provide probability distributions for them. In particular, Coleman [4] showed that, in the dilute wormhole approximation, the probability distribution for universes is infinitely peaked at  $\Lambda = 0$ , rendering all other values of the cosmological constant improbable.

On the purely gravitational side, the interest has been recently focused on traversable wormhole [1,5–8]. Most of the efforts are directed to study static configurations [9] that must have a number of specific properties in order to be traversable. The most striking of these properties is the violation of the energy conditions [10]. It implies that the matter that generates the wormhole is exotic [1], viz. its energy density is negative, as seen by static observers. Geometrically, this is a direct consequence of the singularity theorems of Hawking and Penrose [11]. Although we do not know of any such exotic material to date, quantum field theory might come to the rescue [12].

Finally, we should mention yet another proposal related to wormholes. It has been shown [5,13] that a nonstatic wormhole's throat can be transformed into a time tunnel. Physical effects in this type of spacetimes have been studied in [14].

Wormhole solutions have also been discussed in alternative theories of gravity, such as  $R + R^2$  theories [15], Moffat's nonsymmetric theory [16], Einstein-Gauss-Bonnet theory [17], and Brans-Dicke (BD) theory [18]. In the last case, static wormhole solutions were found in vacuum, the source of gravity being the scalar field. Dynamical solutions are discussed in [19]. The aim of this paper is to look for static wormhole solutions of Brans-Dicke theory in a general setting, *i.e.* in the presence of matter that obeys a generic equation of state [20]. We shall also discuss whether the BD scalar can be the "carrier" of exoticity, as was shown

in [18] for the vacuum case.

Following the conventions of [21], the field equations of Brans-Dicke theory are

$$R_{\mu\nu} = \frac{8\pi}{\Phi} \left( T_{\mu\nu} - \frac{\omega + 1}{2\omega + 3} T g_{\mu\nu} \right) + \omega \frac{\Phi_{;\mu}\Phi_{;\nu}}{\Phi^2} + \frac{\Phi_{;\mu;\nu}}{\Phi}$$
 (1)

$$\Phi^{;\mu}_{;\mu} = \frac{8\pi}{2\omega + 3} T \tag{2}$$

The assumption of a static spacetime entails that it is possible to choose a metric and a scalar field such that

$$g_{\mu\nu,t} = 0$$
  $\Phi_{,t} = 0$   $g_{ti} = 0$  (3)

 $(i=r,\theta,\phi)$ . We further require spherical symmetry, so that the line element can be written in Schwarzschild form:

$$ds^{2} = -e^{2\psi}dt^{2} + e^{2\lambda}dr^{2} + r^{2}(d\theta^{2} + \sin^{2}\theta d\phi^{2})$$
(4)

For the stress-energy tensor of matter we choose

$$T_t^t = -\rho(r) \qquad T_r^r = -\tau(r) \qquad T_\theta^\theta = T_\phi^\phi = p(r) \tag{5}$$

and zero otherwise. Finally, we adopt the following equation of state for matter:

$$-\tau + 2p = \epsilon \rho \tag{6}$$

where  $\epsilon$  is a constant. Now, the trace of the stress-energy tensor can be written as  $T=-\tau+2p-\rho=\rho(\epsilon-1)$ . The field equations take the form

$$-\psi'' - (\psi')^2 + \lambda'\psi' + 2\frac{\lambda'}{r} = -\frac{8\pi}{\Phi} \left[ \tau + \frac{\omega + 1}{2\omega + 3} T \right] e^{2\lambda} + (\omega + 1)(\ln \Phi)'^2 + (\ln \Phi)'' - \lambda'(\ln \Phi)'$$
(7a)

$$1 - re^{-2\lambda} \left[ \psi' - \lambda' + \frac{1}{r} \right] = \frac{8\pi}{\Phi} \left[ p - \frac{\omega + 1}{2\omega + 3} T \right] r^2 + re^{-2\lambda} (\ln \Phi)'$$
 (7b)

$$e^{2(\psi-\lambda)} \left[ \psi'' + (\psi')^2 - \lambda' \psi' + 2 \frac{\psi'}{r} \right] = \frac{8\pi}{\Phi} \left[ \rho + \frac{\omega+1}{2\omega+3} T \right] e^{2\psi} - \psi' e^{2(\psi-\lambda)} (\ln \Phi)'$$
 (7c)

$$\Phi'' - \Phi' \left( \lambda' - \psi' - \frac{2}{r} \right) = \frac{8\pi}{2\omega + 3} T e^{2\lambda}$$
 (7d)

To solve the system made up of Eqs. (7) we shall follow the philosophy sketched in [22]. We shall look for a differential equation relating  $\psi$  and  $\lambda$ , starting from the equations of motion and the equation of state. The equation we shall obtain is second order and nonlinear in  $\psi$  but, after a change of variables, first order and linear in  $\lambda$ . We shall then make a specific choice for  $\psi$  consistent with asymptotic flatness and nonexistence of horizons and singularities. We shall finally substitute this  $\psi$  into the linear equation and solve for  $\lambda$ .

As explained in [21], from Eqs. (6), (7c), and (7d), it can be shown that  $\Phi = \Phi_0 e^{c\psi}$  where  $c = (\epsilon - 1)/[2\omega + 3 + (\omega + 1)(\epsilon - 1)]$ , and  $\Phi_0$  is related to the value of the gravitational coupling constant when  $r \to \infty$ . In the case  $\omega \to \infty$  or  $\epsilon \to 1$ , we get general relativity back (although in the latter case, other solutions different from  $\Phi = \text{const}$  might exist).

After a bit of algebra, we get the equation:

$$A\psi'' + B(\psi')^2 + 2A\psi' - A\lambda'\psi' + \frac{2}{r^2}(e^{2\lambda} - 1) = 0$$
(8)

where

$$A = -2 \frac{2 + \epsilon + 2\omega}{2 + \epsilon + \omega(1 + \epsilon)}$$

$$B = -\frac{8 + \epsilon^2(\omega + 2) + 4\omega^2(1 + \epsilon) + 8\epsilon + 11\omega + 12\omega\epsilon}{[2 + \epsilon + \omega(\epsilon + 1)]^2}$$

In the spirit of [22], we make the ansatz  $\psi = -\alpha/r$ , where  $\alpha$  is a positive constant. With this election, which guarantees that the gravitational constant takes the correct value at  $r \to \infty$ , Eq. (8) takes the form

$$h(r) + f(r) e^{2\lambda} + g(r) \lambda' = 0$$
(9)

where

$$h(r) = B\left(\frac{\alpha}{r^2}\right)^2 - \frac{2}{r^2} \qquad \qquad f(r) = \frac{2}{r^2} \qquad \qquad g(r) = -\frac{A\alpha}{r^2} + \frac{4}{r}$$

A suitable change of variables transforms Eq. (9) into a Bernoulli equation, and afterwards into a linear equation. Its general solution is given by

$$e^{-2\lambda} = \frac{e^{2s/\varphi}}{\varphi} \left( 1 + \frac{R}{\varphi} \right)^{-(8l+1)} \{ I + \mathcal{K} \}$$
 (10)

where

$$\varphi = \frac{r}{\alpha}$$
  $s = \frac{B}{A}$   $l = -\frac{A}{4}$   $l = -\frac{B}{A^2}$ 

$$I \equiv \int e^{-2s/\varphi} \left( 1 + \frac{R}{\varphi} \right)^{8l} d\varphi$$

and K is a constant. It is not valid when  $A \to 0$ , *i.e.* for  $\omega = -1 - \epsilon/2$ . The binomial  $(1 + R/\varphi)^{8l}$  is related to the hypergeometric function  ${}_2F_1$  [23]. Using the relation [23]

$$e^{t} {}_{p}F_{q}(\alpha_{1}, \dots \alpha_{p}; \beta_{1} \dots \beta_{q}; -xt) = \sum_{n=0}^{\infty} {}_{p+1}F_{q}(-n, \alpha_{1}, \dots \alpha_{p}; \beta_{1}, \dots \beta_{q}; x) \frac{t^{n}}{n!}, \qquad (11)$$

the integral I can be written

$$I = 2s \sum_{n=0}^{\infty} \int_{3} F_{1}(-n, -8l, b; b; R/2s) \left(\frac{-2s}{\varphi}\right)^{n}$$
 (12)

Integrating out the terms corresponding to n = 0 and n = 1, we finally get

$$I = \varphi - 8 l R \ln \varphi + \varphi \sum_{n=2}^{\infty} {}_{3}F_{1}(-n, 8l, b; b; R/2s)(-1)^{n} \left(\frac{2s}{\varphi}\right)^{n} \frac{1}{n!(n-1)}$$
(13)

It is easily seen that  $e^{2\lambda} \to 1$  when  $\varphi \to \infty$ .

In order to fix the constant  $\mathcal{K}$ , we must select a value for the dimensionless radius ( $\varphi_{th}$ ) such that the "flaring out" condition

$$\lim_{\varphi \to \varphi_{\rm sh}^+} e^{-2\lambda} = 0^+ \tag{14}$$

is satisfied. In the case  $R \leq 0$ ,  $\varphi_{\rm th}$  must necessarily be greater than |R|, so that the flaring out condition holds for all values of  $\omega$  and  $\epsilon$  except, obviously, those where R diverges, which are given by  $\omega = -(2+\epsilon)/(1+\epsilon)$ . Nevertheless, the absolute size of the throat also depends

on  $\alpha^{-1}$ . The aforementioned properties of  $\lambda$ , together with the definition of  $\psi$ , bear out that the metric tensor describes two asymptotically flat spacetimes joined by a throat.

Let us now study the issue of weak energy condition (WEC) violation. Using the field equations and the expression for the trace, we easily obtain

$$\frac{2e^{2\lambda}}{r^2} - \frac{4\psi'}{r} - \frac{2}{r^2} = \frac{16\pi}{\Phi}\tau e^{2\lambda} + \frac{4\Phi'}{r\Phi} - \omega\left(\frac{\Phi'}{\Phi}\right)^2 + 2\frac{\Phi'}{\Phi}\psi' \tag{15}$$

At the throat,  $e^{2\lambda} \to \infty$ , and then

$$\tau_{\rm th} \approx \frac{\Phi_{\rm th}}{8\pi r_{\rm th}^2} \tag{16}$$

To calculate  $\rho_{\text{th}}$ , we use the nontrivial component of the equation  $T^{\mu}_{\nu;\mu} = 0$ :

$$\tau' = \psi'(\rho - \tau) - \frac{2\tau}{r} - \frac{\epsilon\rho + \tau}{r} \tag{17}$$

Using Eqs. (16) and (17), and the derivative of Eq. (15),

$$\rho_{\rm th} \approx \tau_{\rm th} \, \frac{c + 1 + \varphi_{\rm th}}{1 - \epsilon \varphi_{\rm th}} \tag{18}$$

And finally, from Eq. (6),

$$p_{\rm th} \approx \frac{\tau_{\rm th}}{2} \, \frac{\epsilon \, (c+1) + 1}{1 - \epsilon \, \varphi_{\rm th}} \tag{19}$$

We shall show now that WEC may be violated (at least near the throat) with nonexotic matter. This means that we shall present the parameters for which a wormhole solution exists whenever the matter content of the theory satisfying the inequalities

$$\rho_{\rm th} \geq 0 \qquad \rho_{\rm th} - \tau_{\rm th} \geq 0 \qquad \rho_{\rm th} + p_{\rm th} \geq 0 \tag{20} \label{eq:20}$$

or equivalently,

<sup>&</sup>lt;sup>1</sup>This situation is analogous to what Kar and Sahdev have found for wormholes in general relativity [22].

$$\frac{c+1+\varphi_{\text{th}}}{1-\epsilon\varphi_{\text{th}}} \ge 1 \tag{21}$$

$$\frac{\epsilon(c+1) + 3 + 2(c+\varphi_{\text{th}})}{1 - \epsilon\varphi_{\text{th}}} \ge 0 \tag{22}$$

In addition, a necessary condition for the violation of the weak energy condition for matter plus Brans-Dicke field at the throat is given by

$$\frac{2(\omega+1)+\epsilon}{2\omega+3}\,\rho_{\rm th} \le 0\tag{23}$$

As an example, let us study the case  $\epsilon = 2$ . From Eqs. (16), (18), and (19), the inequalities (20) will be satisfied if

$$\left(\varphi_{\text{th}} \ge -\frac{1}{9\omega + 12} \text{ and } \varphi_{\text{th}} < \frac{1}{2}\right) \quad \text{or} \quad \left(\varphi_{\text{th}} \le -\frac{1}{9\omega + 12} \text{ and } \varphi_{\text{th}} > \frac{1}{2}\right)$$
 (24)

Inequality (23) will be satisfied for  $\omega \in (-2, -3/2)$ . Finally, we have to impose that  $\varphi_{\text{th}} \ge |A/4|$ , which implies that

$$\varphi_{\text{\tiny th}} \ge \left| \frac{2 + \omega}{4 + 3\omega} \right| \tag{25}$$

These inequalities constrain  $\varphi_{\rm th}$  to an interval in which a nonexotic wormhole can be constructed, for instance, in the case  $\omega = -1.75$ . We should recall that a definite interval for  $\varphi_{\rm th}$  does not determine the radius of the throat, because of the dependence of  $\varphi$  on  $\alpha$ .

Summing up, we showed that Brans-Dicke theory in the presence of matter with a fairly general equation of state admits analytical wormhole solutions. They generalize the vacuum ones presented by Agnese and La Camera [18]. It should be noted that there exists some regions of the parameter space in which the Brans-Dicke field may play the role of exotic matter, implying that it might be possible to build a wormholelike spacetime with the presence of ordinary matter at the throat.

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