Spinons as Composite Fermions

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ABSTRACT

We give an explicit holomorphic factorization of $SU(N)_1$ WZW primaries in terms of gauge invariant composite fermions. In the $N = 2$ case, we show that these composites realize the spinon algebra. Both in this and in the general case, the underlying Yangian symmetry implies that these operators span the whole Fock space.

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i) Introduction

In the last ten years, the notion of generalized statistics has been the subject of active research. Indeed, it has been recognized that particles obeying fractional statistics, intermediate between bosons and fermions, can exist in $2+1$ dimensional [1] and in $1+1$ dimensional [2, 3] systems.

A general formulation of quantum statistical mechanics for systems whose excitations obey fractional statistics, as spinons in 2D or anyons in 3D, has not been fully developed, although some progress has been made in this direction [3, 4].

In the $1+1$ dimensional case, the study of Field Theories whose excitations obey generalized commutation relations has received much attention in connection with the so-called spinon representation of Conformal Field Theories (CFT) [2, 5, 6, 7].

For the reasons above, it would be very important to have as much information as possible concerning systems manifesting these properties. In particular, an explicit Lagrangian formulation of such systems would be useful for a thorough comprehension of its thermodynamical behavior.

A physical manifestation of $1+1$ fractional statistics behavior has arised in the study of incompressible quantum Hall systems. Indeed, it appears in connection with the two-dimensional modes in the boundary whose dynamics is governed by a CFT [8].

The purpose of the present note is to give a first-principle formulation of a two-dimensional field theory in which one can explicitly identify operators creating the Fock space through excitations which exhibit $1+1$ generalized statistics [3]. In the $SU(2)_1$ theory, these excitations are known as spinons. We start by studying the BRST invariant fields in a $SU(N)_1$-WZW theory formulated as a fermionic coset model. As we will see, the fermionic representation is at the root of an explicit holomorphic factorization of the primary fields since their holomorphic factors are indeed given by the gauge-invariant fermions. Moreover, in the $N=2$ case we show that they are nothing but the spinon fields of ref. [6, 7]. Our construction should be compared with what happens in the bosonic theory where a heuristic holomorphic factorization of the WZW fields is usually assumed.

As discussed in refs. [5, 6, 7, 9], the underlying Yangian symmetry ensures that the modes of these holomorphic factors span the whole Fock space of the theory.

ii) Yangian structure of the Fock space of $SU(N)_1$ conformal field theories

We first review the main aspects of a novel representation of the spectrum of the chiral part of $SU(N)_1$ CFTs, first introduced by Haldane et al.[2, 5] and further developed in [6, 7, 9, 10]. It was originally presented in [2] as an extrapolation of the exact solution of long range interaction 1D spin chains [11, 12] and the fact that this chain is described in the low energy limit by the level one $SU(2)$ conformal field theory [13].

In the $SU(2)_1$ case, the new representation, known as spinon representation, allows for a quasi-particle interpretation: in this sense, each state corresponds to a given number of spin one-half, neutral, semionic, free quasi-particles (spinons).

The quantum numbers that characterize the representation correspond to the behavior of the states under the action of the generators Q_n^a of an infinite algebra, known as Yangian algebra [14] (where $n = 0, 1, \ldots$ and a is a Lie algebra index). The Yangian $Y(\hat{g})$ associated to a Lie algebra $\hat{g}$ is a Hopf algebra that is neither commutative nor cocommutative, being a simple non-trivial example of a quantum group.

The Yangian was defined in [14] by starting with the generators Q_0^a and Q_1^a, which for $\hat{g} = sl_N$ (the Lie algebra of $SU(N)$) are defined by the relations

$$[Q_0^a, Q_0^b] = f^{abc}Q_0^c,$$
$$[Q_0^a, Q_1^b] = f^{abc}Q_1^c,$$
$$[Q_1^a, [Q_1^b, Q_0^c]] + (cyclic\ in\ a, b, c) = A^{abcdef}\{Q_0^d, Q_0^e, Q_0^f\}$$
$$[[Q_1^a, Q_1^b], [Q_0^c, Q_1^d]] + [[Q_1^c, Q_1^d], [Q_0^a, Q_1^b]] = (A^{abpqrs}f^{cdp} + A^{cdpqrs}f^{abp})\{Q_0^q, Q_0^r, Q_1^s\}, \qquad (1)$$

where $A^{abpdef} = \frac{1}{4}f^{adp}f^{beq}f^{cfr}f^{pqr}$ and $\{\ldots, \ldots, \ldots\}$ denotes a completely symmetrized product. Higher level generators are recursively defined for $n \geq 2$ by

$$c_v Q_n^a = f^{abc}[Q_1^c, Q_{n-1}^b], \qquad (2)$$

where c_v is the Casimir in the adjoint representation.

Another central property of this algebra is the comultiplication rule that can be viewed as a generalization of angular momentum addition:

$$\Delta_\pm(Q_0^a) = Q_0^a \otimes 1 + 1 \otimes Q_0^a$$
$$\Delta_\pm(Q_1^a) = Q_1^a \otimes 1 + 1 \otimes Q_1^a \pm \frac{1}{2}f^{abc}Q_0^b \otimes Q_0^c. \qquad (3)$$

With the help of this structure the whole Hilbert space can be decomposed into a sum of multiplets corresponding to the irreducible representations of the Yangian algebra. Each multiplet is characterized by a highest weight state defined by

$$Q_n^\alpha|\psi\rangle = 0 \qquad (4)$$

where α corresponds to a positive root in the Cartan basis. Its vectors are eigenstates of Q_n^i, where i corresponds to Cartan subalgebra generators. Each highest weight state is characterized by a finite set of non-decreasing integers $\{n_i\}_{i=1,\ldots,p}$. In the $SU(2)_1$ quasi-particle language each one of these integers represents one spinon in the n_i-th orbital.

One can explicitly construct the highest weight states in the context of a field theory by studying the $SU(2)_1$ WZW theory [5, 7]. In this context not only the highest weight states but the full spinon space is constructed by repeatedly acting on the vacuum with the modes of the spin 1/2 primary field. Rigorously speaking, one must assume a heuristic holomorphic factorization of the WZW primary fields $g(z, \bar{z}) = g(z)\bar{g}(\bar{z})$, as was done in [7]. An alternative construction was given in [6] in terms of chiral vertex operators motivated by the bosonization of two complex Dirac fermions with an $SU(2)$ charge degree of freedom wiped out.

iii) Spinons as composite fermions

We will now show how spinons arise as gauge-invariant composite fermions in the fermionic coset formulation of an $SU(2)$ level one conformal field theory.

It is known that the $SU(N)_1$ CFT can be formulated as a constrained fermionic model, that is as a $U(N)/U(1)$ fermionic coset theory [15]. The constraint is imposed on a system of N free Dirac fermions by requiring that physical states $|phys\rangle$ are singlet under the $U(1)$ current,

$$J_\mu|phys\rangle = 0. \tag{5}$$

This is achieved in the path integral formulation by introducing a Lagrange multiplier a_μ which plays the rôle of a $U(1)$ gauge field. The partition function for the constrained model in Euclidean space reads

$$\mathcal{Z} = \int \mathcal{D}\psi^\dagger \mathcal{D}\psi \mathcal{D}a_\mu \exp(-\int \mathcal{L}d^2x) \tag{6}$$

where the fermionic Lagrangian $\mathcal{L}$ is given by

$$\mathcal{L} = \frac{1}{\sqrt{2\pi}}\psi^{\dagger i}(\slashed{\partial} + i\,\slashed{a})\psi^i \tag{7}$$

Here ψ^i, $i = 1, \ldots, N$ are Dirac fermions and a_μ is the abelian gauge field implementing the constraint.

This procedure removes the charge degree of freedom from the fermionic system, this situation being reminiscent of what happens in the Haldane-Shastry spin chain, where the charge degree of freedom is fixed.

The theory described by the Lagrangian (7) has a local $U(1)$ invariance, thus the physical operators will be those invariant under $U(1)$ gauge transformations, which in the present case are constructed as [16]

$$\begin{aligned}
\hat{\psi}^i &= \exp(-i\int_x^\infty a_\mu dx^\mu)\psi^i, \\
\hat{\psi}^{i\dagger} &= \psi^{i\dagger}\exp(i\int_x^\infty a_\mu dx^\mu).
\end{aligned} \tag{8}$$

These fields are independent of the path chosen for the integration in the exponential due to the a_μ equations of motion.

In order to study this theory it is convenient to reformulate it in a decoupled form through the following change of variables

$$\begin{aligned}
a &= i(\bar{\partial}h)h^{-1} & \bar{a} &= i(\partial\bar{h})\bar{h}^{-1}, \\
\psi_1 &= h\chi_1 & \psi_2^\dagger &= \chi_2^\dagger h^{-1}, \\
\psi_2 &= \bar{h}\chi_2 & \psi_1^\dagger &= \chi_1^\dagger \bar{h}^{-1}.
\end{aligned} \tag{9}$$

where $\partial \equiv \frac{\partial}{\partial z}$, $\bar{\partial} \equiv \frac{\partial}{\partial \bar{z}}$, $a = (a_0 + ia_1)/\sqrt{2}$, $\bar{a} = (a_0 - ia_1)/\sqrt{2}$, $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ and $h = \exp(-\phi - i\eta)$, $\bar{h} = \exp(\phi - i\eta)$.

Taking into account the gauge fixing procedure and the Jacobians associated to (9) [17] one arrives at the desired decoupled form for the partition function (6):

$$\mathcal{Z} = \mathcal{Z}_{ff}\mathcal{Z}_{fb}\mathcal{Z}_{gh}, \tag{10}$$

where

$$
\begin{aligned}
\mathcal{Z}_{ff} &= \int \mathcal{D}\chi^\dagger \mathcal{D}\chi \exp(-\frac{1}{\pi}\int (\chi_2^\dagger \bar{\partial}\chi_1 + \chi_1^\dagger \partial\chi_2)d^2 x), \\
\mathcal{Z}_{fb} &= \int \mathcal{D}\phi \exp(\frac{1}{\pi}\int \phi\Delta\phi d^2 x), \\
\mathcal{Z}_{gh} &= \int \mathcal{D}\bar{c}\mathcal{D}c\mathcal{D}\bar{b}\mathcal{D}b \exp(-\int (b\partial c + \bar{b}\bar{\partial}\bar{c})d^2 x).
\end{aligned}
\tag{11}
$$

Once we have obtained the desired decoupled expression for the partition function of the $SU(N)_1$ model we proceed further to study its spectrum.

In the decoupled picture, the components of the gauge invariant fields (8) can be written as

$$
\begin{aligned}
\hat{\psi}_1^{\,i}(z) &= e^{\varphi(z)}\chi_1^i(z) & \hat{\psi}_1^{\,i\dagger}(\bar{z}) &= \chi_1^{i\dagger}(\bar{z})e^{\bar{\varphi}(\bar{z})} \\
\hat{\psi}_2^i(\bar{z}) &= e^{-\bar{\varphi}(\bar{z})}\chi_2^i(\bar{z}) & \hat{\psi}_2^{\,i\dagger}(z) &= \chi_2^{i\dagger}(z)e^{\varphi(z)}
\end{aligned}
\tag{12}
$$

where

$$
\begin{aligned}
\varphi(z) &= \phi + i\int_x^\infty dz_\mu \epsilon_{\mu\nu}\partial_\nu\phi \\
\bar{\varphi}(\bar{z}) &= \phi - i\int_x^\infty dz_\mu \epsilon_{\mu\nu}\partial_\nu\phi
\end{aligned}
\tag{13}
$$

are the chiral (holomorphic and anti-holomorphic) components of the free boson ϕ, satisfying

$$\bar{\partial}\varphi = \partial\bar{\varphi} = 0. \tag{14}$$

This fact together with the equation of motion of the free fermions χ ensures that $\hat{\psi}_1^i$ and $\hat{\psi}_2^{i\dagger}$ ($\hat{\psi}_2^i$ and $\hat{\psi}_1^{i\dagger}$) are holomorphic (anti-holomorphic).

Eq.(12) makes evident the relevance of the gauge invariance requirement for physical operators: physical excitations created by $\hat{\psi}_1^i$, $\hat{\psi}_2^{i\dagger}$ are automatically holomorphic, while those created by $\hat{\psi}_2^i$, $\hat{\psi}_1^{i\dagger}$ are anti-holomorphic .

Since the composite fermions in (8) are now completely given in terms of free fields, their OPE can be easily evaluated.

In fact, being the composite fermions primary fields, the leading singularity in the OPE is governed by their conformal dimension. This in turn is simply given by the sum of the dimensions of their decoupled constituents (see eqs.(12) and (13)). For the holomorphic composites $\hat{\psi}_1^i$ and $\hat{\psi}_2^{i\dagger}$ one has

$$h = \frac{1}{2} - \frac{1}{2N} = \frac{N-1}{2N}, \quad \bar{h} = 0. \tag{15}$$

(Similarly, $h = 0$ and $\bar{h} = \frac{N-1}{2N}$ for the anti-holomorphic composites $\hat{\psi}_2^i$ and $\hat{\psi}_1^{i\dagger}$).

This altogether allows one to write

$$\hat{\psi}_2^{i\dagger}(z)\hat{\psi}_1^j(w) = \frac{\delta^{ij}}{(z-w)^{(N-1)/N}} + \cdots \tag{16}$$

(Analogously $\hat{\psi}_1^{i\dagger}(\bar{z})\hat{\psi}_2^j(\bar{w}) = \frac{\delta^{ij}}{(\bar{z}-\bar{w})^{(N-1)/N}} + \cdots$).

These composite fields are nothing but the holomorphic factors of the WZW primaries g, $g^\dagger$ in the fundamental representation. This fact is seen by considering the bilinears

$$\hat{\psi}_2^i(\bar{z})\hat{\psi}_2^{j\dagger}(z) = \psi_2^i\psi_2^{j\dagger} = g^{ij}(z,\bar{z}) \tag{17}$$

The first equality stresses again the holomorphic factorization of the fermi bilinears $\psi_2^i\psi_2^{i\dagger}$, while the second one provides, by means of the bosonization dictionary [15], the explicit holomorphic factorization of the WZW primary field g. In the same way one gets

$$\hat{\psi}_1^i(z)\hat{\psi}_1^{j\dagger}(\bar{z}) = \psi_1^i\psi_1^{j\dagger} = (g^\dagger)^{ij}(z,\bar{z}). \tag{18}$$

Eqs. (17) and (18) define the holomorphic factors of g and $g^\dagger$ up to an arbitrary constant phase.

Notice that the factors $\hat{\psi}$ and $\hat{\psi}^\dagger$ independently create physical excitations in contrast with the original fermions which do not. This result is one of the main points in our work, that is the explicit factorization of the g WZW primary field in terms of gauge invariant fermions.

In order to make contact with the spinon formulation discussed in ref.[7] we are going to first study some details of the $SU(2)_1$ case. First of all, the matrix elements of g and $g^\dagger$ are related by

$$g^{ij} = -\epsilon^{jl}(g^\dagger)^{lk}\epsilon^{ki}. \tag{19}$$

This relation does not trivially apply to the fermi bilinears (17) and (18), since it leads to inconsistencies in correlators.

As an example, consider the two-point correlator [18, 19]

$$\langle g^{ij}(g^\dagger)^{kl}\rangle = -\epsilon^{kr}\langle g^{ij}g^{rs}\rangle\epsilon^{sl} \neq 0. \tag{20}$$

The inconsistency is apparent when one uses the bosonization rules to evaluate $\langle g^{ij}g^{rs}\rangle$, thus obtaining a vanishing result. Notice that a similar problem is already present in Witten's original bosonization rules for the $U(N)$ case [20]. To circumvent this problem one can use the bosonization rules with the proviso that the fields g and $g^\dagger$ must appear pairwised.

As a general rule, we will use eq.(19) in order to pair the g-fields up before using the bosonization rules (17) and (18). This procedure, in terms of fermions, is summarized by the additional rules

$$\hat{\psi}_2^{i\dagger} \quad \leftrightarrow \quad -\epsilon^{ij}\hat{\psi}_1^j, \tag{21}$$

$$\hat{\psi}_1^{i\dagger} \quad \leftrightarrow \quad \epsilon^{ij}\hat{\psi}_2^j, \tag{22}$$

that should be used together with the bosonization rules whenever the g-fields are not pairwised. (The phase for each holomorphic factor is chosen for later convenience; it is of course irrelevant while considering g correlators).

We will now discuss the construction of the holomorphic part of the Fock space by the action of the fermion composites on the vacuum. In view of the discussion above, one has to alternate the action of $\hat{\psi}_1^i$ and $\hat{\psi}_2^{j\dagger}$ in order to compare with the bosonic construction.

In this respect, however, the literature about the spinon formulation of $SU(2)_1$ WZW theory [7] only makes reference to the holomorphic part of the g-field and makes implicit use of the relation (19) (or equivalently (22)). In order to conform to the usual notation, we will then adopt the following consistent notation:

$$\sigma^j(z) = \begin{cases} \epsilon^{jk}\hat{\psi}_2^{k\dagger}(z) & \text{when acting on the spin 1/2 sector} \\ \hat{\psi}_1^j(z) & \text{when acting on the vacuum sector} \end{cases} \tag{23}$$

It means that when acting on the vacuum one must apply $\sigma^i = \hat{\psi}_1^i$ and on such states, as one should only apply $\hat{\psi}_2^{j\dagger}$, an operator σ^j will mean $\epsilon^{jl}\hat{\psi}_2^{l\dagger}$. In general there will be two sectors: the vacuum sector and the spin 1/2 sector. In the vacuum sector one must always apply σ^i in its form $\hat{\psi}_1^i$ and in the other sector σ^j will mean $\epsilon^{jl}\hat{\psi}_2^{l\dagger}$.

In this notation and using (16) the σ-σ OPE is given by

$$\sigma^i(z)\sigma^j(w) = (-1)^q \frac{\epsilon^{ij}}{(z-w)^{1/2}} + \dots \tag{24}$$

where $q = 0$ when this product acts on the vacuum sector and $q = 1$ otherwise. As pointed out in [7] this OPE characterizes the σ's as spinon fields.

Once this identification is made, one can follow the same procedure as in [6, 7] to construct the full Hilbert space of the $SU(2)_1$ CFT from the action of the modes of the spinon field. Due to the square root branch cut in the OPE singularity (24) the excitations created by the spinon fields are semionic. This in turn implies that their mode expansion will involve integer (half-integer) powers of z when acting on the vaccum (spin 1/2) sector:

$$\sigma^i(z) = \sum_{m=0}^{\infty} z^{m+\frac{q}{2}} \sigma^i_{-m-\frac{1}{4}-\frac{q}{2}}, \tag{25}$$

where

$$\sigma^i_{-m+\frac{3}{4}-\frac{q}{2}} = \oint \frac{dz}{2\pi i} z^{-m-\frac{q}{2}} \sigma^i(z). \tag{26}$$

These modes satisfy generalized commutation relations which can be obtained from (24), giving

$$\sum_{l=0}^{\infty} C_l^{(-1/2)} \left(\sigma^i_{-m-\frac{q+1}{2}-l+\frac{3}{4}} \sigma^j_{-n-\frac{q}{2}+l+\frac{3}{4}} - \sigma^j_{-n-\frac{q+1}{2}-l+\frac{3}{4}} \sigma^i_{-m-\frac{q}{2}+l+\frac{3}{4}} \right) = (-1)^q \epsilon^{ij} \delta_{m+n+q-1,0} \tag{27}$$

where $C_l^{(-1/2)}$ are the coefficients of the Taylor expansion of $(1-z)^{-1/2}$ [19].

The chiral part of the Fock space of the $SU(2)_1$ conformal field theory can be constructed in terms of these modes. As stated above, this space can be classified into multiplets corresponding to the irreducible representations of the Yangian algebra $Y(sl_2)$. Each multiplet is

constructed from a highest weight state by the action of the Yangian generators. The highest weight states can be constructed in terms of the spinon modes in the following way: one has to take linear combinations of M-spinon states

$$\sigma^1_{-n_1-1/4}\cdots\sigma^1_{-n_M-1/4}|0\rangle \tag{28}$$

that are annihilated by the raising operators Q_0^+ and Q_1^+. These states correspond to the "fully polarized M-spinon sates" of Haldane [2]. The defining level 0 and 1 generators of $Y(sl_2)$ can be written in terms of the modes of the Kac-Moody currents [5] as

$$
\begin{aligned}
Q_0^a &= J_0^a, \\
Q_1^a &= \frac{1}{2}f_{bc}^a \sum_{m=0}^{\infty} : J_{-m}^b J_m^c : .
\end{aligned}
\tag{29}
$$

The irreducible Yangian multiplet corresponding to a given highest weight state is then constructed by repeatedly applying the generators Q_0^- and Q_1^-. Finally, the union of all of these multiplets forms a basis of the full Hilbert space.

In the more general $SU(N)_1$ case the Yangian structure is also present [9], although the existence of more than one primary field prevents us from speaking about "spinon fields" (even about spinon excitations). Nevertheless, the holomorphic factors of the primaries can still be explicitly constructed in terms of the gauge invariant fermions (8).

In fact, in the $SU(N)_1$ WZW theory there are N primary fields corresponding to the integrable representations which can be constructed as suitable symmetrized products of fields in the fundamental representation. The integrable representations for the $SU(N)_1$ case correspond to Young diagrams with one column and at most $N-1$ rows. The primary fields in these representations, Φ_p, are then obtained by taking normal-ordered antisymmetric products of p fields ($p < N$) in the fundamental representation [15].

In terms of the composite fermions (8), the holomorphic part of the primaries is given by:

$$\Phi_p^{i_1,i_2,\ldots,i_p}(z) = \mathcal{A}\left(: \hat{\psi}_1^{i_1}\hat{\psi}_1^{i_2}...\hat{\psi}_1^{i_p} :\right), \tag{30}$$

which in terms of the decoupled fields they can be written as:

$$\Phi_p^{i_1,i_2,\ldots,i_p}(z) = \mathcal{A}\left(: \chi_1^{i_1}\chi_1^{i_2}...\chi_1^{i_p} :\right) : \exp p\varphi :, \tag{31}$$

which is holomorphic.

The conformal dimensions of these fields can be evaluated as the sum of two independent contributions, $p/2$ coming from the p free fermions and $-p^2/2N$ coming from the vertex operator $: \exp p\varphi :$, thus giving

$$h_p = \frac{p}{2} - \frac{p^2}{2N} = \frac{p(N-p)}{2N} \tag{32}$$

This dimension corresponds to the conformal dimension of an $SU(N)_1$ primary field in the representation Λ_p whose Young diagram has p vertical boxes, as given by [18]

$$h_{\Lambda_p} = \frac{c_{\Lambda_p}}{c_v + k} \tag{33}$$

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where $c_v = N$ for $SU(N)$, $k = 1$ and $c_{\Lambda_p} = \frac{p}{2N}(N + 1)(N - p)$, is the Casimir of the representation Λ_p.

As we already mentioned the Yangian structure is present in this case, but the construction of the Fock space in terms of gauge invariant fields (Φ) is out of the scope of the present work.

It will be interesting to see whether an underlying structure similar to the Yangian appearing in $SU(N)_1$ is also present in other CFTs, such as higher level $SU(N)$ CFT or coset models. In fact, the holomorphic factorization of the WZW primaries in the $SU(N)_k$ case ($k > 1$) can be envisaged following the same lines as those presented here. The connection with the spinon basis constructed in ref. [10] will be discussed elsewhere [21].

The present description of the spinon representation of $SU(N)$ CFT in terms of composite fermions in a constrained fermionic formulation seems to be, in our opinion, the most natural framework to investigate in the above mentioned directions. In the same line, the fundamental rôle played by composite gauge invariant fermions has already shown up in the coset formulation of the Ising model [16], where they were used to build up the order and disorder operators. We expect that this guideline should allow for the holomorphic factorization of some primaries in more general coset models.

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