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Path-integral formulation of backward and umklapp scattering for 1d spinless fermions

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Abstract

We present a (1+1)-dimensional fermionic QFT with non-local couplings between currents. This model describes an ensemble of spinless fermions interacting through forward, backward and umklapp scattering processes. We express the vacuum to vacuum functional in terms of a non trivial fermionic determinant. Using path-integral methods we find a bosonic representation for this determinant. Thus we obtain an effective action depending on three scalar fields, two of which correspond to the physical collective excitations whereas the third one is an auxiliary field that is left to be integrated by means of an approximate technique.

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1 Introduction

In recent years there has been a renewed interest in the study of low-dimensional field theories. In particular, research on the one-dimensional (1d) fermionic gas has been very active, mainly due to the actual fabrication of the so called quantum wires [1]. One of the most interesting aspects of these systems is the possibility of having a deviation from the usual Fermi-liquid behavior. This phenomenon was systematically examined by Haldane [2] who coined the term Luttinger-liquid behavior to name this new physical situation in which the Fermi surface disappears and the spectrum contains only collective modes. Perhaps the simplest theoretical framework that presents this feature is the Tomonaga-Luttinger (TL) model [3], a many-body system of right and left-moving particles interacting through their charge densities. In a recent series of papers [4] an alternative, field-theoretical approach was developed to consider this problem. In these works a non-local and non-covariant version of the Thirring model [5] [6] was introduced, in which the fermionic densities and currents are coupled through bilocal, distance-dependent potentials. This non-local Thirring model (NLT) contains the TL model as a particular case. Although it constitutes an elegant framework to analyze the 1d many-body problem, one seriuos limitation appears if one tries to make contact with quantum wires phenomenology. Indeed, one has to recall that, from a perturbative point of view, the building blocks of the NLT are the forward-scattering (fs) processes which are supposed to dominate the scene only in the low transferred momentum limit. This means that in its present form it can only provide a very crude description of the Luttinger liquid equilibrium and transport properties. The main goal of this paper is to start developing an improved version of the NLT in which larger momentum transfers are taken into account. Although the most interesting effects of these contributions, such as the occurrence of mass-gaps in the normal modes spectra, are known to take place for spin-$\frac{1}{2}$ particles, here, for illustrative purposes we shall focus our attention on the spinless case. In Section 2 we show that the manipulations used to write the vacuum functional of the fs problem in terms of a fermionic determinant, also work for the NLT, even when backward (bs) [7] and umklapp (us) [8] scattering are considered. We get a quite peculiar fermionic determinant which resembles the one obtained in the path-integral study [9] of massive fermions in (1+1) dimensions [10]. In Section 3 we obtain a bosonic representation for this non-trivial determinant. This, in turn, allowed us to find a completely bosonized effective action for the model under consideration. Finally, in Section 4, we summarize the main points of our investigation.

2 The Model

In this section we begin the study of an extended version of the NLT which includes the contribution of forward, backward and umklapp scattering. Following the formulation proposed in [4] we shall attempt to describe these interactions by means of a fermionic QFT with Euclidean action given by

$$S = S_0 + S_{fs} + S_{bs} + S_{us} \tag{2.1}$$

where

$$S_0 = \int d^2x \; \bar{\Psi} i\partial\!\!\!/ \Psi \tag{2.2}$$

is the unperturbed action associated to a linearized free dispersion relation. The contributions of the different scattering processes will be written as

$$S_{fs} = -\frac{g^2}{2} \int d^2x \; d^2y \; (\bar{\Psi}\gamma_\mu\Psi)(x) \; V_{(\mu)}(x,y) \; (\bar{\Psi}\gamma_\mu\Psi)(y) \tag{2.3}$$

and

$$S_{bs} + S_{us} \;\; = \;\; -\frac{g'^2}{2} \int d^2x \; d^2y (\bar{\Psi} \; \Gamma_\mu\Psi)(x) \; U_{(\mu)}(x,y) (\bar{\Psi} \; \Gamma_\mu\Psi)(y) \tag{2.4}$$

where the $\gamma_\mu' s$ are the usual two-dimensional Dirac matrices and $\Gamma_0 = 1$, $\Gamma_1 = \gamma_5$. Please keep in mind that no sum over repeated indices is implied when a subindex (μ) is involved. Let us also mention that the coupling potentials $V_{(\mu)}$ and $U_{(\mu)}$ are assumed to depend on the distance $\mid x - y \mid$ and can be expressed in terms of Solyom's "g-ology" [11] as

$$\begin{aligned} V_{(0)}(x,y) &= \frac{1}{g^2}(g_2 + g_4)(x,y) \\ V_{(1)}(x,y) &= \frac{1}{g^2}(g_2 - g_4)(x,y) \end{aligned} \tag{2.5}$$

$$U_{(0)}(x,y) = \frac{1}{g'^2}(g_3+g_1)(x,y)$$

$$U_{(1)}(x,y) = \frac{1}{g'^2}(g_3-g_1)(x,y) \qquad (2.6)$$

In the above equations g and g' are just numerical constants that could be set equal to one. We keep them to facilitate comparison of our results with those corresponding to the usual Thirring model. Indeed, this case is obtained by choosing $g' = 0$ and $V_{(0)}(x,y) = V_{(1)}(x,y) = \delta^2(x-y)$. On the other hand, the non-covariant limit $g' = 0$, $V_{(1)}(x,y) = 0$ gives the TL model [3].

The terms in the action containing g_2 and g_4 represent forward scattering events, in which the associated momentum transfer is small. In the g_2 processes the two branches (left and right-moving particles) are coupled, whereas in the g_4 processes all four participating electrons belong to the same branch. On the other hand g_1 and g_3 are related to scattering diagrams with larger momentum transfers of the order of $2k_F$ (bs) and $4k_F$ (us) respectively (This last contribution is important only if the band is half-filled). Let us now turn to the treatment of the partition function. At this point we recall that in ref.[4] we wrote the fs piece of the action in a localized way :

$$S_{fs} = -\frac{g^2}{2}\int d^2x\ J_\mu K_\mu. \qquad (2.7)$$

where J_μ is the usual fermionic current, and K_μ is a new current defined as

$$K_\mu(x) = \int d^2y\ V_{(\mu)}(x,y)J_\mu(y). \qquad (2.8)$$

Using a functional delta and introducing auxiliary bosonic fields in the path-integral representation of the partition function Z we were able to write (see [4] for details):

$$Z = N\int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi\mathcal{D}\tilde{A}_\mu\mathcal{D}\tilde{B}_\mu\ exp\{-\int d^2x[\bar{\Psi}i\partial\!\!\!/\Psi + \tilde{A}_\mu\tilde{B}_\mu + \frac{g}{\sqrt{2}}(\tilde{A}_\mu J_\mu + \tilde{B}_\mu K_\mu)]\} \qquad (2.9)$$

If we define

$$\bar{B}_\mu(x) = \int d^2y\ V_{(\mu)}(y,x)\tilde{B}_\mu(y), \qquad (2.10)$$

$$\tilde{B}_\mu(x) = \int d^2y\ b_{(\mu)}(y,x)\bar{B}_\mu(y), \qquad (2.11)$$

with $b_{(\mu)}(y,x)$ satisfying

$$\int d^2y\ b_{(\mu)}(y,x)V_{(\mu)}(z,y) = \delta^2(x-z), \tag{2.12}$$

and change auxiliary variables in the form

$$A_\mu = \frac{1}{\sqrt{2}}(\tilde{A}_\mu + \bar{B}_\mu), \tag{2.13}$$

$$B_\mu = \frac{1}{\sqrt{2}}(\tilde{A}_\mu - \bar{B}_\mu), \tag{2.14}$$

we obtain

$$Z = N \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi\mathcal{D}A_\mu\mathcal{D}B_\mu\ e^{-S(A,B)-S_{bs}-S_{us}}\ exp\{-\int d^2x\ \bar{\Psi}(i\partial\!\!\!/ - gA\!\!\!/)\Psi\} \tag{2.15}$$

where

$$S(A,B) = \frac{1}{2}\int d^2x\ d^2y\ b_{(\mu)}(x,y)[A_\mu(x)A_\mu(y) - B_\mu(x)B_\mu(y)] \tag{2.16}$$

The Jacobian associated with the change $(\tilde{A},\tilde{B}) \to (A,B)$ is field- independent and can then be absorbed in the normalization constant N. Moreover, we see that the B-field is completely decoupled from both the A-field and the fermion field. Keeping this in mind, it is instructive to try to recover the partition function corresponding to the usual covariant Thirring model ($b_{(0)}(y,x) = b_{(1)}(x,y) = \delta^2(x-y)$, and $g' = 0$), starting from (2.15). In doing so one readily discovers that B_μ describes a negative-metric state whose contribution must be factorized and absorbed in N in order to get a sensible answer for Z. This procedure paralells, in the path-integral framework, the operator approach of Klaiber [6], which precludes the use of an indefinite-metric Hilbert space. Consequently, from now on we shall only consider the A contribution.

At this stage we see that when bs and us processes are disregarded the procedure we have just sketched allows us to express Z in terms of a fermionic determinant. Now we will show that this goal can also be achieved when the larger momentum transfers are taken into account. To this end we write:

$$S_{bs} + S_{us} = -\frac{g'^2}{2}\int d^2x\ L_\mu M_\mu \tag{2.17}$$

where L_μ and M_μ are fermionic bilinears defined as

$$L_\mu(x) = \bar{\Psi}(x)\Gamma_\mu \Psi(x), \tag{2.18}$$

$$M_\mu(x) = \int d^2y\ U_{(\mu)}(x,y) L_\mu(y). \tag{2.19}$$

Thus it is evident that we can follow the same prescriptions as above, with L_μ and M_μ playing the same roles as J_μ and K_μ, respectively. After the elimination of a new negative metric state whose decoupled partition function is absorbed in the normalization factor, as before, one obtains

$$Z = N \int \mathcal{D}A_\mu \mathcal{D}C_\mu\ \det(i\not\partial - g\not A - g'\Gamma_\mu C_\mu)\ \exp\{-S[A] - S[C]\} \tag{2.20}$$

where

$$\begin{aligned} S[A_\mu] &= \frac{1}{2}\int d^2x d^2y\ A_\mu(x) b_{(\mu)} A_\mu(y) \\ S[C_\mu] &= \frac{1}{2}\int d^2x d^2y\ C_\mu(x) d_{(\mu)} C_\mu(y) \end{aligned} \tag{2.21}$$

and

$$\int d^2y\ d_{(\mu)}(y,x) U_{(\mu)}(z,y) = \delta^2(x-z), \tag{2.22}$$

This is our first interesting result: we have been able to express Z in terms of a fermionic determinant. Let us stress, however, that this determinant is a highly non trivial one. Indeed, the term in g' is not only a massive-like term (in the sense that it is diagonal in the Dirac matrices space) but it also depends on the auxiliary field $C_\mu(x)$. In the next section we shall show how to deal with this determinant.

3 Boson representation for the determinant

In this section we shall combine a chiral change in the fermionic path-integral measure with a formal expansion in g' in order to get a bosonic representation for the fermionic determinant derived in the previous section. Let us start by performing the following transformation:

$$\Psi(x) = \exp g[\gamma_5 \Phi(x) - i\eta(x)]\ \chi(x)$$
$$\bar{\Psi}(x) = \bar{\chi}(x)\ \exp g[\gamma_5 \Phi(x) + i\eta(x)] \tag{3.1}$$

$$\mathcal{D}\bar{\Psi}\mathcal{D}\Psi = J_F[\Phi,\eta]\mathcal{D}\bar{\chi}\mathcal{D}\chi, \tag{3.2}$$

where Φ and η are scalar fields and $J_F[\Phi,\eta]$ is the Fujikawa Jacobian [12] whose non-triviality is due to the non-invariance of the path-integral measure under chiral transformations. As it is well known, the above transformation permits to decouple the field A_μ from the fermionic fields if one writes

$$A_\mu(x) = \partial_\mu \eta(x) + \epsilon_{\mu\nu}\partial_\nu \Phi(x) \tag{3.3}$$

which can also be considered as a bosonic change of variables with trivial (field independent) Jacobian. As a result we find

$$det(i\partial\!\!\!/ - gA\!\!\!/ - g'\Gamma_\mu C_\mu) = J_F[\Phi,\eta] det(i\partial\!\!\!/ - g' e^{2g\gamma_5\Phi}\Gamma_\mu C_\mu) \tag{3.4}$$

The computation of the fermionic Jacobian requires the choice of a regularization procedure which, in turn , involves certain ambiguity (See for instance [13]). In this paper we follow the same prescription that allowed us to get sensible results when only forward scattering was considered [4]. The result is

$$J_F[\Phi,\eta] = \exp \frac{g^2}{2\pi}\int d^2x\ \Phi\partial_\mu\partial_\mu\Phi \tag{3.5}$$

As a consequence of the above transformations the partition function is now expressed as:

$$Z = N' \int \mathcal{D}\Phi\mathcal{D}\eta\mathcal{D}C_\mu e^{-(S[\Phi,\eta]+S[C_\mu])} J_F[\Phi,\eta] \det(i\partial\!\!\!/ - g' e^{2g\gamma_5\Phi}\Gamma_\mu C_\mu) \tag{3.6}$$

where $S[\Phi,\eta]$ arises when one inserts (3.3) in $S[A_\mu]$ (See equation (2.21)).

The fermionic determinant in the above expression can be analyzed in terms of a perturbative expansion. Indeed, taking g' as perturbative parameter, and using the fermionic fields χ and $\bar{\chi}$ defined in (3.1) one can write

$$Z_F = \sum_{n=0}^{\infty} \frac{g'^n}{n!} \langle \prod_{j=1}^{n} \int d^2x_j \bar{\chi}(x_j)\ \mathsf{C}(x_j)\ \chi(x_j)\rangle_0 \tag{3.7}$$

where, for later convenience we have defined

$$Z_F = \det(i\partial\!\!\!/ - g' e^{2g\gamma_5 \Phi} \Gamma_\mu C_\mu) \tag{3.8}$$

and

$$\mathsf{C} = \begin{pmatrix} C_+ & 0 \\ 0 & C_- \end{pmatrix} \tag{3.9}$$

with

$$\begin{cases} C_+ = (C_0 + C_1)\ e^{2g\Phi(x)} \\ C_- = (C_0 - C_1)\ e^{-2g\Phi(x)} \end{cases} \tag{3.10}$$

By carefully analyzing each term in the series we found a selection rule quite similar to the one obtained in the path-integral treatment of (1+1) massive fermions with local [9] and non-local interactions [14]. Indeed, due to the fact that in (3.7) $\langle\ \rangle_0$ means v.e.v. with respect to free massless fermions, the v.e.v.'s corresponding to $j = 2k+1$ are zero. Thus, we obtain

$$\begin{aligned} Z_F &= \sum_{k=0}^{\infty} \frac{(g' c\rho)^{2k}}{(k!)^2 (2\pi)^{2k}} \int \prod_{i=1}^{k} d^2x_i d^2y_i \\ &\times \prod_{i=1}^{k} (C_0(x_i) + C_1(x_i))(C_0(y_i) - C_1(y_i)) \\ &\times \exp\{2g \sum_{i=1}^{k} (\Phi(x_i) - \Phi(y_i))\} \\ &\times \frac{\prod_{i>j}^{k} ((c\rho)^2 \mid x_i - x_j \mid \ \mid y_i - y_j \mid)^2}{\prod_{i,j}^{k} (c\rho \mid x_i - y_j \mid)^2} \end{aligned} \tag{3.11}$$

where $c\rho$ is a normal ordering parameter.

In order to obtain a bosonic description of the present problem we shall now propose the following bosonic Lagrangian density:

$$\mathcal{L}_B = \frac{1}{2}(\partial_\mu \varphi)^2 + \frac{\alpha_0}{2\beta^2}(m_+ e^{i\beta\varphi} + m_- e^{-i\beta\varphi}) \tag{3.12}$$

with β, $m_+(x)$ and $m_-(x)$ to be determined. The quantity α_0 is just a constant that we include to facilitate comparison of our procedure with previous works on local bosonization [10] [9]. Please notice that for $m_+ = m_- = 1$ this model coincides with the well known sine-Gordon model that can be used to describe a neutral Coulomb gas. In this context $\frac{\alpha_0}{\beta^2}$ is nothing but the corresponding fugacity [15]. We shall now consider the partition function

$$Z_B = \int \mathcal{D}\varphi \ \exp - \int d^2x \ \mathcal{L}_{\ B} \tag{3.13}$$

and perform a formal expansion taking the fugacity as perturbative parameter. It is quite straightforward to extend the analysis of each term, already performed for $m_+ = m_- = 1$, to the present case in which these objects are neither equal nor necessarily constants. The result is

$$\begin{aligned} Z_B &= \sum_{l=1}^{\infty} \frac{1}{(l!)^2} (\frac{\alpha}{2\beta^2})^{2l} \int (\prod_{i=1}^{l} d^2x_i \ d^2y_i)(\prod_{i=1}^{l} m_+(x_i) \ m_-(y_i)) \\ & \frac{\prod_{i>j}^{l}((c\rho)^2 \mid x_i - x_j \mid \ \mid y_i - y_j \mid)^{\frac{1}{2\pi}\beta^2}}{\prod_{i,j}^{l}(c\rho \mid x_i - y_j \mid)^{\frac{1}{2\pi}\beta^2}}. \end{aligned} \tag{3.14}$$

where we have defined a renormalized quantity α (remember that there are infrared and ultraviolet singularities involved in the correlation functions that contribute to every order).

Comparing this result with equation (3.11), we see that both series coincide if the following identities hold:

$$\begin{aligned} \beta &= \pm 2\sqrt{\pi} \\ \alpha &= g' \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} m_+(x_i) &= (C_0(x_i) + C_1(x_i)) \ e^{2g\Phi(x_i)} \\ m_-(y_i) &= (C_0(y_i) - C_1(y_i)) \ e^{-2g\Phi(y_i)} \end{aligned} \tag{3.16}$$

This is our second non-trivial result. We have found a bosonic representation for the fermionic determinant (3.11). This is given by (3.13) together with the identities (3.15) and (3.16). Let us emphasize that equations (3.15) are completely analogous to the bosonization formulae first obtained by Coleman [10] whereas equations (3.16) constitute a new result, specially connected to the present problem.

4 Final result and next steps

In this paper we have presented an extension of the recently proposed NLT, which can be used to describe a system of 1d strongly correlated particles when not only forward but also backward and umklapp scattering is considered. We were able to write the vacuum to vacuum functional of this model in terms of a non-trivial fermionic determinant (See eq.(3.11)). Our main achievement was to obtain a bosonic representation for this determinant (See equations (3.13),(3.15) and (3.16)). Using this result in (3.6) we have expressed the partition function of this system in terms of five scalars: Φ, η, C_0, C_1 and φ. Based on our previous experience with the fs model [4] we know that the physical fields (the ones that describe the collective excitations of the system) are Φ and η. The others are just auxiliary variables that we have to integrate in order to analyze the physics of the normal modes. The integrals in C_0 and C_1, being quadratic can be easily performed yielding:

$$Z = N \int \mathcal{D}\Phi \; \mathcal{D}\eta \; \mathcal{D}\varphi \; e^{-S_{eff}[\Phi,\eta,\varphi]} \tag{4.1}$$

with

$$\begin{aligned} S_{eff}[\Phi,\eta,\varphi] &= \frac{1}{2}\int d^2x d^2y (\partial_\mu \eta + \epsilon_{\mu\nu}\partial_\nu \Phi)(x) b_{(\mu)}(x,y)(\partial_\mu \eta + \epsilon_{\mu\nu}\partial_\nu \Phi)(y) \\ &+ \int d^2x [\frac{g^2}{2\pi}(\partial_\mu \Phi)^2 + \frac{1}{2}(\partial_\mu \varphi)^2] \\ &- \frac{1}{2}(\frac{g'}{2\pi})^2 \int d^2x d^2y [f_\mu(x) \; U_{(\mu)}(x,y) \; f_\mu(y)] \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} f_0(x) &= \cosh(2g\Phi + i\sqrt{4\pi}\varphi)(x) \\ f_1(x) &= \sinh(2g\Phi + i\sqrt{4\pi}\varphi)(x) \end{aligned} \tag{4.3}$$

The analysis of the action $S_{eff}[\Phi, \eta, \varphi]$ is beyond the scope of the present article. We are currently studying this problem by means of a saddle point computation. We hope to report our results in the close future.

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