Application of Affine Estimators to Single Tone Frequency Estimation

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Abstract. Affine estimation has emerged as a promising technique to reduce the mean squared error (MSE) between the estimated parameters and the true value of these parameters. The aim of this paper is to obtain an affine estimator for the frequency of a complex sinusoid corrupted by white gaussian noise. Additionally, an adaptive technique is presented. The simulation results clearly show that affine estimators have better performance than unbiased estimators such as the maximum likelihood estimator (MLE) and the Fu-Kam approximation.

Keywords: Single Tone Estimation, Affine Estimators, Mean Squared Error, Adaptive Algorithm

1 Introduction

The estimation of the parameters (amplitude, frequency and phase) of a complex sinusoid in additive white gaussian noise has been a central field of study in communications and signal processing for the last four decades. Moreover, it is still a very active research area because of its wide applicability.

Rife and Boorstyn [1] published in 1974 the first method for obtaining the maximum likelihood estimators (MLE) of the parameters from discrete-time observations. This method consists in using the periodogram for obtaining all the relevant information for constructing the estimators. They also presented the Cramér-Rao Lower Bound (CRLB) for all the parameters.

In 1985, Tretter [2] proposed a simplified noise model and developed a linear regression estimator that has the advantage of being computationally simpler than the periodogram. This estimator converges to the MLE for large values of signal-to-noise ratio (SNR). In 1989, Kay [3] extended this noise model for obtaining a faster estimator.

Several other techniques were developed since then, including the use of FFT methods with different windows [4], Markov-based estimation [5], adaptive filtering [6], Kalman filtering [7], nonlinear techniques [8], among others.

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In particular, H. Fu and P. Y. Kam [9] published in 2007 a more accurate noise model and presented an approximation of the MLE for high SNR that admits a recursive calculation of the estimator, as more samples become available.

It is well known that there exists biased estimators that outperform unbiased ones in terms of **MSE** and that these biased estimators can be obtained from transformations of the unbiased ones [10,11,12]. One particular idea for these transformations, developed by Y. C. Eldar [13,14,15], consists in obtaining a biased estimator through an affine transformation of an unbiased one,

$$h_1(\mathfrak{X}) = a\,h(\mathfrak{X}) + b \tag{1}$$

where $\mathfrak{X}$ is the sample, a, b are constants, and $h(\mathfrak{X})$ is an unbiased estimator, $\mathbb{E}[h(\mathfrak{X})] = \theta$. Estimators obtained in this way will be called **affine estimators** or **estimators with affine bias**. An exhaustive analysis of these type of estimators was done in [16] and a complete characterization and a further generalization was carried out in [17].

The main objective of this paper is to obtain affine estimators for the frequency of a complex sinusoid corrupted by white gaussian noise by application of appropriate affine transformations [17] to the maximum likelihood estimator [1] and the Fu-Kam [9] approximation. Moreover, a recursive algorithm for improving the estimation is also presented.

In section 2 the model used will be discussed, in section 3 the Cramér-Rao Lower Bound is obtained, and in sections 4 and 5 the maximum likelihood estimator and the Fu-Kam approximation to the MLE are developed. The affine estimators are constructed in section 6, results of some simulations can be found in section 7 and, finally, conclusions are presented in section 8.

For the rest of this work, there will be some considerations: boldface letters will be used to denote vectors and capital letters to denote random variables; $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}_{\mathbf{X}}, \boldsymbol{\Sigma}_{\mathbf{X}})$ means that $\mathbf{X}$ is a gaussian random vector with mean $\boldsymbol{\mu}_{\mathbf{X}}$ and covariance matrix $\boldsymbol{\Sigma}_{\mathbf{X}}$; $\mathbf{x}^*$ is the conjugate of the elements of $\mathbf{x}$, $\mathbf{x}^T$ is the transpose of $\mathbf{x}$, $\mathbf{x}^H$ is the hermitian (conjugate of elements and transpose) of $\mathbf{x}$; $\boldsymbol{I}_n$ is used to denote the identity matrix of dimension $n \times n$.

2 Circularly-Symmetric Gaussian Random Vectors

There are some interesting issues about complex random variables, as explained in [18], and that will be summarized here. The use of complex random variables in estimating the frequency of a complex sinusoid is essential because the noise will be modelled as a complex random variable in order to make the development more tractable. In this paper, the attention will be restricted to gaussian random variables; for the more general case, see [19,20,21].

The most striking aspect of the work by Picinbono [18] is that if two jointly gaussian random vectors, $\mathbf{X}$ and $\mathbf{Y}$, are to be described in a complex form, say $\mathbf{Z} = \mathbf{X} + j\mathbf{Y}$, then it is not sufficient with the knowledge of the probability density function (pdf) of $\mathbf{Z}$, but it is the joint distribution of $[\mathbf{Z}^T\ \mathbf{Z}^H]^T$ what is actually needed. This implies that the vector space of $[\mathbf{X}^T\ \mathbf{Y}^T]^T \in \mathbb{R}^{2n}$ cannot be simply reduced to the complex space of $\mathbb{C}^n$.

Let $\mathbf{X} \sim \mathcal{N}(0, \boldsymbol{\Sigma}_\mathbf{X}) \in \mathbb{R}^n$ and $\mathbf{Y} \sim \mathcal{N}(0, \boldsymbol{\Sigma}_\mathbf{Y}) \in \mathbb{R}^n$, the joint pdf is

$$p_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y}) = \frac{1}{(2\pi)^n (\det(\boldsymbol{\Sigma}_{\boldsymbol{R}}))^{\frac{1}{2}}} \exp\left(-\frac{1}{2} \left[\mathbf{x}^T\ \mathbf{y}^T\right] \begin{bmatrix} \boldsymbol{\Sigma}_\mathbf{X} & \boldsymbol{\Sigma}_\mathbf{XY} \\ \boldsymbol{\Sigma}_\mathbf{YX} & \boldsymbol{\Sigma}_\mathbf{Y} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \right) \quad (2)$$

where $\boldsymbol{\Sigma}_\mathbf{XY} = \mathbb{E}[\mathbf{X}\mathbf{Y}^T]$, $\boldsymbol{\Sigma}_\mathbf{YX} = \mathbb{E}[\mathbf{Y}\mathbf{X}^T]$ and

$$\boldsymbol{\Sigma}_{\boldsymbol{R}} = \begin{bmatrix} \boldsymbol{\Sigma}_\mathbf{X} & \boldsymbol{\Sigma}_\mathbf{XY} \\ \boldsymbol{\Sigma}_\mathbf{YX} & \boldsymbol{\Sigma}_\mathbf{Y} \end{bmatrix} \quad (3)$$

Through the invertible linear transformation $\left[\mathbf{Z}^T\ \mathbf{Z}^H\right]^T = \boldsymbol{M}\left[\mathbf{X}^T\ \mathbf{Y}^T\right]^T$, with

$$\boldsymbol{M} = \frac{1}{2} \begin{bmatrix} \boldsymbol{I}_n & \boldsymbol{I}_n \\ -j\boldsymbol{I}_n & j\boldsymbol{I}_n \end{bmatrix} \quad (4)$$

and where $\boldsymbol{I}_n$ is used to denote the identity matrix of dimension $n \times n$, the pdf of $\left[\mathbf{Z}^T\ \mathbf{Z}^H\right]^T$ can be obtained,

$$p_{\mathbf{Z},\mathbf{Z}^*}(\mathbf{z},\mathbf{z}^*) = \frac{1}{\pi^n (\det(\boldsymbol{\Sigma}_{\boldsymbol{C}}))^{\frac{1}{2}}} \exp\left(-\frac{1}{2} \left[\mathbf{z}^T\ \mathbf{z}^H\right] \begin{bmatrix} \boldsymbol{\Sigma}_\mathbf{Z} & \boldsymbol{C}_\mathbf{Z} \\ \boldsymbol{C}_\mathbf{Z}^H & \boldsymbol{\Sigma}_\mathbf{Z}^* \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{z} \\ \mathbf{z}^* \end{bmatrix} \right) \quad (5)$$

where $\boldsymbol{\Sigma}_\mathbf{Z} = \mathbb{E}[\mathbf{Z}\mathbf{Z}^H]$ is the covariance matrix and $\boldsymbol{C}_\mathbf{Z} = \mathbb{E}[\mathbf{Z}\mathbf{Z}^T]$ is another second order statistic, called the *relation matrix*. It was shown, then, that the joint distribution of $\left[\mathbf{Z}^T\ \mathbf{Z}^H\right]^T$ is needed in order to completely describe the vector $\left[\mathbf{X}^T\ \mathbf{Y}^T\right]^T$.

In all works related with complex sinusoids, the noise is always modelled as a complex circularly-symmetric gaussian random vector. This means that the pdfs of $\mathbf{Z}$ and $e^{j\alpha}\mathbf{Z}$ are the same for all $\alpha \in \mathbb{R}$. In terms of the recent development, circular symmetry is equivalent to the relation matrix $\boldsymbol{C}_\mathbf{Z}$ being zero. This last statement implies that, if $\mathbf{Z}$ is circularly symmetric, then it completely describes the random vector $\left[\mathbf{X}^T\ \mathbf{Y}^T\right]^T$ turning the pdf of $\mathbf{Z}$ into

$$p_\mathbf{Z}(\mathbf{z}) = \frac{1}{\pi^n \det(\boldsymbol{\Sigma}_\mathbf{Z})} \exp\left(-\mathbf{z}^H \boldsymbol{\Sigma}_\mathbf{z}^{-1} \mathbf{z}\right) \quad (6)$$

Moreover, if $\boldsymbol{\Sigma}_\mathbf{Z}$ is real, then $\mathbf{X}$ and $\mathbf{Y}$ have to be independent and identically distributed [19],

$$\boldsymbol{\Sigma}_\mathbf{Z} \in \mathbb{R}^{n\times n} \quad \Rightarrow \quad \boldsymbol{\Sigma}_\mathbf{X} = \boldsymbol{\Sigma}_\mathbf{Y} = \boldsymbol{\Sigma}\,,\ \boldsymbol{\Sigma}_\mathbf{Z} = 2\boldsymbol{\Sigma}\,,\ \boldsymbol{\Sigma}_\mathbf{XY} = \boldsymbol{\Sigma}_\mathbf{YX} = 0 \quad (7)$$

Finally, the model for the data will be

$$R(k) = ae^{j(\omega k+\theta)} + N(k) \quad (8)$$

where a and θ are known, ω is deterministic and unknown, $\{N(k)\}$ is a discrete-time, circularly-symmetric, zero-mean, complex additive white gaussian noise

(AWGN), with pdf given by (6) and $\mathbb{E}[N(k)N^*(k+m)] = \sigma^2\delta(m)$, with σ^2 known. The pdf of the sample $\{R(k)\}_{k=0}^{n-1}$ will be given by

$$p_{\{R(k)\}}(\{r(k)\}|\omega) = \frac{1}{(\pi\sigma^2)^n} \exp\left(-\frac{1}{\sigma^2}\sum_{k=0}^{n-1}|r(k) - ae^{j(\omega k+\theta)}|\right) \tag{9}$$

where $\{r(k)\} = \{r(k)\}_{k=0}^{n-1}$ is a realization of the sample. The SNR is defined as $\mathrm{SNR} = a^2/\sigma^2$.

3 The Cramér-Rao Lower Bound

The Cramér-Rao Lower Bound [22,23] is a useful benchmark to which estimators can be compared. It sets the minimum possible variance an unbiased estimator can have. In this case, it is observed that the pdf (9) can be written as

$$\ln\left(p_{\{R(k)\}}(\{r(k)\}|\omega)\right) = \tag{10}$$

$$n\ln\left(\frac{1}{\pi\sigma^2}\right) - \frac{1}{\sigma^2}\sum_{k=0}^{n-1}\left((r_{\mathfrak{Re}}(k) - a\cos(\omega k+\theta))^2 + (r_{\mathfrak{Im}}(k) - a\sin(\omega k+\theta))^2\right)$$

where $r(k) = r_{\mathfrak{Re}}(k) + jr_{\mathfrak{Im}}(k)$. Differentiating twice with respect to ω and taking the opposite, yields

$$u(\{r(k)\},\omega) = \frac{2a}{\sigma^2}\sum_{k=0}^{n-1}k^2\left(r_{\mathfrak{Re}}(k)\cos(\omega k+\theta) + r_{\mathfrak{Im}}(k)\sin(\omega k+\theta)\right) \tag{11}$$

Evaluating (11) in $\{R(k)\}$ (sample given by random variables described in equation 8) and taking the expectation gives,

$$\mathbb{E}[u(\{R(k)\},\omega)] = \frac{2a^2}{\sigma^2}\sum_{k=0}^{n-1}k^2 \tag{12}$$

due to the independence property given in 7. Finally, the CRLB for any unbiased estimator $\hat{\omega}_{\{R(k)\}}$ turns to be

$$\mathbf{Var}(\hat{\omega}_{\{R(k)\}}) \geq \frac{\sigma^2}{a^2}\frac{3}{n(n-1)(2n-1)} \tag{13}$$

and is found to depend inversely on the SNR and to be highly sensitive (n^3) to the number of samples.

4 The Maximum Likelihood Estimator

The maximum likelihood estimator, $\hat{\omega}_{ML\{R(k)\}}$, is the value of ω that maximizes the pdf of the sample (9) for a given set of values. Rife and Boorstyn developed the MLE for the case of the complex sinusoid in [1]. It is observed from equation (10) that maximization with respect to ω will be equivalent to maximizing

$$L_0 = -\frac{1}{n}\sum_{k=0}^{n-1}\Big((r_{\mathfrak{Re}}(k) - a\cos(\omega k+\theta))^2 + (r_{\mathfrak{Im}}(k) - a\sin(\omega k+\theta))^2\Big) \tag{14}$$

which can be turned into

$$L = 2a\mathfrak{Re}\left\{e^{-j\theta}\left(\frac{1}{n}\sum_{k=0}^{n-1} r(k)e^{-j\omega k}\right)\right\} \tag{15}$$

Finally, the MLE becomes

$$\hat{\omega}_{ML\{R(k)\}} = \max_{\omega}\ \mathfrak{Re}\left\{e^{-j\theta}\left(\frac{1}{n}\sum_{k=0}^{n-1} R(k)e^{-j\omega k}\right)\right\} \tag{16}$$

Rife and Boorstyn proposed a search routine, divided in two parts, first a *coarse* search and then a *fine search* around the value found in the first part. They found that the estimator becomes biased for low values of SNR due to outliers misleading the *coarse* search to incorrect values of ω. Finally, it is necessary to observe that the variance of the estimator will be equal to the CRLB as long as the estimator is unbiased.

5 The Fu-Kam Approximation

In their paper published in 2007, Fu and Kam [9] proposed an estimator that would be computationally easier and that would admit a recursive calculation. The approximation presented is valid for high values of SNR as shown in the derivation. The main difference from the MLE is that the polar decomposition of the samples is used, $r(k) = |r(k)|e^{j\measuredangle r(k)}$. Then, the pdf of the samples can be expressed as,

$$p_{\{R(k)\}}(\{r(k)\}|\omega) = c\exp\left(\frac{2A}{\sigma^2}\sum_{k=0}^{n-1}|r(k)|\cos(\measuredangle r(k) - (\omega k+\theta))\right) \tag{17}$$

where c is a constant, independent of ω. Taking logarithm, differentiating with respect to ω and setting the result equal to zero, yields

$$\sum_{k=0}^{n-1} k|r(k)|\sin(\measuredangle r(k) - (\omega k+\theta)) = 0 \tag{18}$$

If high SNR is assumed, it is reasonable to suppose that $\measuredangle r(k)$ is close to $(\omega k+\theta)$ because the noise power will not be high enough to cause large deviations of the measured angle from its true value, then the approximation $\sin(x) \approx x$ becomes valid, turning this last equation into

$$\sum_{k=0}^{n-1} k|r(k)|(\measuredangle r(k) - (\omega k + \theta)) = 0 \tag{19}$$

where the estimator can be derived,

$$\hat{\omega}_{FK\{R(k)\}} = \frac{\sum_{k=0}^{n-1} k|R(k)|(\measuredangle R(k) - \theta)}{\sum_{k=0}^{n-1} k^2|R(k)|} \tag{20}$$

It is readily observed that the numerator and the denominator of this estimator can be calculated recursively.

For obtaining the bias and the mean squared error of the estimator, an approximated model for the noise is developed [9] which is an improvement of Tretter's model [2]. Using this model, which is valid for high SNR, the estimator is shown to be unbiased, and the variance turns out to be

$$\mathbf{Var}(\hat{\omega}_{FK\{R(k)\}}) = \frac{1}{2a^2/\sigma^2 + 1} \frac{5}{3n^2 - 3n - 1} \tag{21}$$

6 The Affine Estimators

The technique of affine estimation consists in introducing a well-known and controlled bias to some unbiased estimator in order to reduce the mean squared error (**MSE**), at least in a region of interest, typically exploiting some previous information about the unknown parameter.

In particular, the affine estimators developed in [16,17,24] assume that the parameter lies in some known region $\mathcal{R} = \{\omega \in \mathbb{R} : \omega_A < \omega < \omega_B\}$. This is a perfectly valid assumption for the frequency of a single tone estimation, making these estimators ideal for lowering the **MSE** of any unbiased estimator, especially for low SNR or a small sample size.

The basic idea for obtaining the affine estimator, $\hat{\omega}_{\varepsilon\{R(k)\}}$, is to apply an appropriate affine transformation to an unbiased estimator $\hat{\omega}_{\{R(k)\}}$ of the parameter, this is,

$$\hat{\omega}_{\varepsilon\{R(k)\}} = a\,\hat{\omega}_{\{R(k)\}} + b \tag{22}$$

The transformation has to be such that $\mathbf{MSE}(\hat{\omega}_{\varepsilon\{R(k)\}}) < \mathbf{MSE}(\hat{\omega}_{\{R(k)\}})$ for all $\omega \in \mathcal{R}$, where

$$\mathbf{MSE}(\hat{\omega}_{\varepsilon\{R(k)\}}) = (a-1)^2\omega^2 + 2b(a-1)\omega + b^2 + a^2 V \tag{23}$$

with $V = \mathbf{Var}(\hat{\omega}_{\{R(k)\}}) =$ constant, and ω the true value of the frequency.

It was shown in [17] that the complete family of affine estimators of the form (22) is characterized by the values of a and b given by

$$a = \frac{\left(\frac{(\omega_B+\varepsilon)-(\omega_A-\varepsilon)}{2}\right)^2 - V}{\left(\frac{(\omega_B+\varepsilon)-(\omega_A-\varepsilon)}{2}\right)^2 + V} \tag{24a}$$

$$b = \frac{2V}{\left(\frac{(\omega_B+\varepsilon)-(\omega_A-\varepsilon)}{2}\right)^2 + V}\,\frac{\omega_A+\omega_B}{2} \tag{24b}$$

for any $\varepsilon > 0$, and where V is the constant variance of the unbiased estimator, $\mathbf{MSE}(\hat{\omega}_{\{R(k)\}}) = V$. This is the case for the estimators developed in sections 4 and 5 whose variance is observed not to depend on ω (for other forms of the variance, see [17]).

It can also be found in [17] that if $\varepsilon = 0$, then the estimator of equations (22,24) is equal to the RB-Affine estimator [16],

$$\hat{\omega}_{RB\{R(k)\}} = a_{RB}\,\hat{\omega}_{\{R(k)\}} + b_{RB} \tag{25a}$$

$$a_{RB} = \frac{\gamma-1}{\gamma+1} \quad ; \quad b_{RB} = \frac{2}{\gamma+1}\,\frac{\omega_A+\omega_B}{2} \tag{25b}$$

where

$$\gamma = \frac{\left(\frac{\omega_B-\omega_A}{2}\right)^2}{V} \tag{26}$$

Also, if $\varepsilon = \sqrt{V + \frac{(\omega_B-\omega_A)^2}{2}} - \left(\frac{\omega_B-\omega_A}{2}\right)$, then the estimator (22,24) becomes equal to the estimator developed in [24],

$$\hat{\omega}_{E\{R(k)\}} = a_E\hat{\omega}_{\{R(k)\}} + b_E \tag{27a}$$

$$a_E = \frac{\gamma}{\gamma+1} \quad ; \quad b_E = \frac{1}{\gamma+1}\,\frac{\omega_A+\omega_B}{2} \tag{27b}$$

Finally, the other important remark made in [17] is the fact that there is only one value of ε which minimizes $\mathbf{MSE}(\hat{\omega}_{\varepsilon\{R(k)\}})$ for a given value of ω. This observation led the way to the development of an adaptive algorithm that adjusts the value of ε recursively as more samples become available. It consists in proposing an inital ε and using it to estimate the value of ω from the unbiased estimator $\hat{\omega}_{\{R(k)\}}$. Then, this estimation is used for finding the ε that minimizes $\mathbf{MSE}(\hat{\omega}_{\varepsilon\{R(k)\}})$ as if that estimation were the true value of the parameter. Finally, this recently obtained value of ε is used for performing a new estimation, this time with another sample and a better value of $\hat{\omega}_{\{R(k)\}}$, obtaining a better approximation of ω and then correcting again the value of ε, recursively.

This will be very useful in combination with the recursive aspect of the Fu-Kam approximation.

7 Examples

For the first two examples, $n = 6$ samples of a complex discrete-time sinusoid in AWGN with $a = 1$ and $\theta = \frac{\pi}{4}$ are considered. The parameter is supposed to lie in the region $\mathcal{R}$ with $\omega_A = \frac{\pi}{32}$ and $\omega_B = \frac{3\pi}{32}$. The true value of the parameter is $\omega = \frac{\pi}{16}$. The simulation is carried out for different values of SNR repeating 1000 times for each value in order to obtain a better estimate of the **MSE**.

The results for the MLE are shown in figure 1.

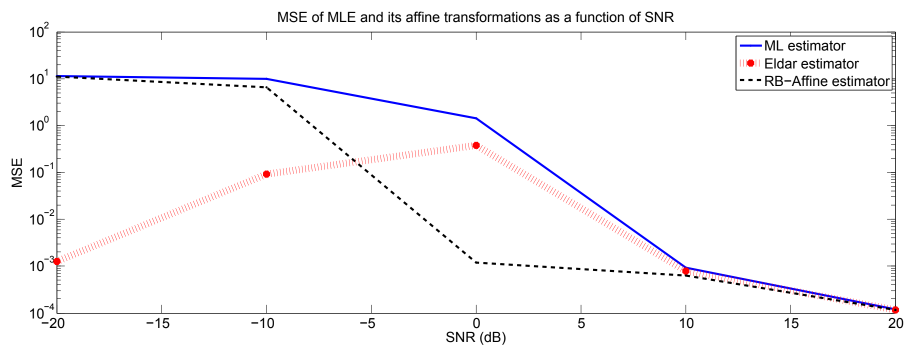

Fig. 1. MSE as a function of SNR for the MLE and the affine transformations given by equations (25) and (27).

It is easily observed that both affine estimators perform better than the MLE for low SNR.

The behaviour of the RB-Affine estimator can be easily explained: for SNR $<$ 0 dB, the value of $\mathbf{MSE}(\hat{\omega}_{ML\{R(k)\}})$ is greater than the maximum V_0 (see [16]) so $\hat{\omega}_{RB\{R(k)\}}$ is unreliable (it performs better than the MLE because the true value of ω is in the center of the region $\mathcal{R}$). Once the value of $\mathbf{MSE}(\hat{\omega}_{ML\{R(k)\}})$ is lower than V_0 (which happens at SNR > -10 dB) the estimator becomes useful and performs better than the Eldar estimator as it was expected, being the true value of the parameter in the center of the region $\mathcal{R}$ (it is shown in [17] that the optimum affine estimator for ω in the center of the region is the RB-Affine estimator).

The reason why the Eldar estimator is the best for SNR < 0 dB is that[1], for low values of γ (low values of SNR), $a_E \to 0$ and $\hat{\omega}_{E\{R(k)\}} \to b_E$ which happens to be the center of the interval. This holds down the estimator and prevents it to go to infinity as the noise rises. This is relaxed when γ becomes bigger, causing the **MSE** to worsen a little.

[1] The simulation was carried out for values of SNR $\in B$, where $B = \{-20\,\text{dB}, -10\,\text{dB}, 0\,\text{dB}, 10\,\text{dB}, 20\,\text{dB}\}$. This causes the **MSE** to be undetermined between -10 dB and 0 dB, where the transition occurs.

In figure 2 the **MSE** for the Fu-Kam estimator and its affine transformations are shown.

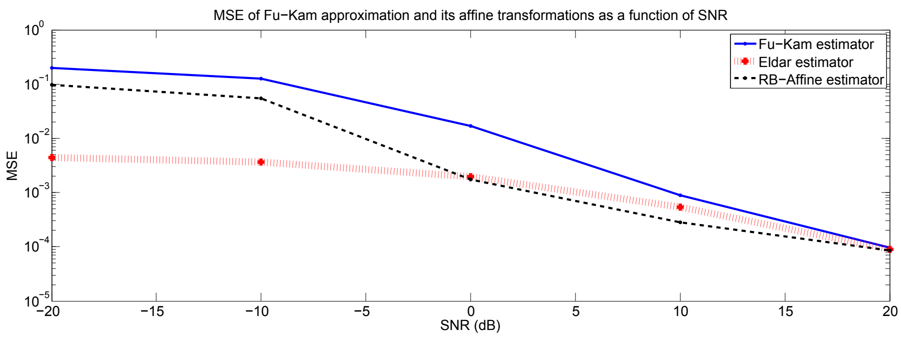

Fig. 2. MSE as a function of SNR for the Fu-Kam estimator and the affine transformations given by equations (25) and (27).

An analysis similar to that carried out for figure 1 applies for figure 2.

The estimators can be further improved if the length of the region $\mathcal{R}$ is reduced as they are highly sensitive to this length, especially for low values of SNR. On the other hand, if the region is widened they still perform better than the MLE, although much closer to it.

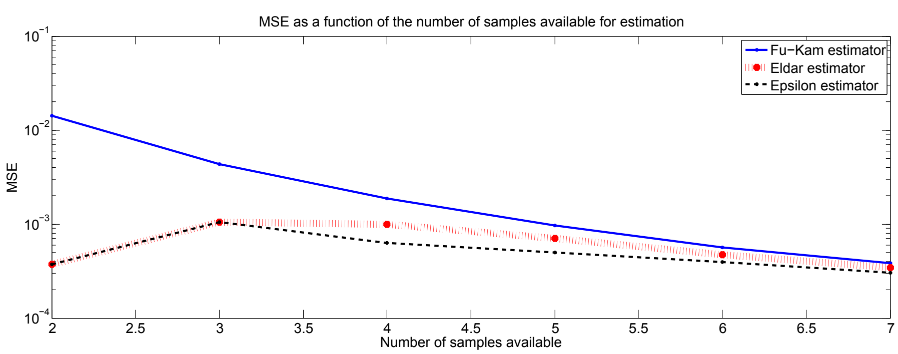

Fig. 3. MSE as a function of the number of samples available for estimation, for the Fu-Kam approximation, the Eldar estimator (27) and the $\varepsilon-$estimator (22,24).

The last example intends to show the performance of the adaptive algorithm. In this case, the situation is the same as for the previous examples (6 samples, $a = 1$, $\theta = \frac{\pi}{4}$, $\omega = \frac{\pi}{16}$, $\omega_A = \frac{\pi}{32}$, $\omega_B = \frac{3\pi}{32}$) but the SNR is held fixed at $\mathrm{SNR} = 10\,\mathrm{dB}$.

In this case, only the Fu-Kam estimator is used and it is calculated recursively as each of the 6 samples is available. The $\hat{\omega}_{\varepsilon\{R(k)\}}$ estimator is obtained (equations 22,24) using the adaptive algorithm explained in the previous section. The Eldar estimator (equation 27) is also shown for comparison.

Figure 3 show the **MSE** of the three estimators as a function of the number of samples available for estimation.

This simulation clearly shows that as more samples are available, the $\hat{\omega}_{\varepsilon\{R(k)\}}$ estimator becomes better because of the adjusting value of ε. As seen in equations (23,24), the mean squared error of the estimator $\hat{\omega}_{\varepsilon\{R(k)\}}$ depends on the value of ε and, in the algorithm illustrated in figure 3, this value is being continuously adjusted in order to decrease the **MSE**, using each estimation of ω as if it were the true value of the parameter.

8 Conclusions

In this paper, affine estimators were used to estimate the frequency of a complex sinusoid corrupted by additive white gaussian noise.

It was shown that affine estimators perform better than the MLE and the Fu-Kam approximation, especially for low values of SNR and/or small sample size, making them excellent choices when the situation is adverse (in terms of SNR).

An adaptive algorithm was also presented for improving recursively the affine estimator, using the Fu-Kam approximation as the unbiased estimator.

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