On the variational homotopy perturbation method for nonlinear oscillators

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In this paper, I discuss a recent application of a variational homotopy perturbation method to rather simple nonlinear oscillators. It is shown that the main equations are inconsistent and for that reason the results may be of scarce utility. © *2012 American Institute of Physics*. [doi:10.1063/1.3681790]

I. INTRODUCTION

There has recently been great interest in developing simple solutions to textbook models of nonlinear oscillators (Refs. 1–5, and references therein). Some of those results are of questionable utility as argued, for example, by Sanchez:[6] "A perturbation technique valid for large parameters was presented by He.[1] It is not an appropriate procedure and it leads to a wrong conclusion" or Rajendran *et al.*[7] who concluded that He's calculations of the limit cycle of the van der Pol oscillator[2] "contain several errors which once rectified make the method inapplicable to it." I have disclosed several inconsistencies in a paper by Ren and He[3] and even proposed how to tidy up and improve their calculations.[8]

Here I discuss a recent application of a variational homotopy perturbation method to rather simple nonlinear oscillators.[5] In Sec. II, I analyse their results and in Sec. III draw conclusions.

II. VARIATIONAL HOMOTOPY PERTURBATION METHOD FOR NONLINEAR OSCILLATORS

Akbarzade and Langari[5] were interested in equations of the form

$$A(u) - f(r) = L(u) + N(u) - f(r) = 0, \tag{1}$$

where L and N are the linear and nonlinear parts of the operator A, and u is the solution. They proposed the "homotopy perturbation structure,"

$$H(v, p) = (1 - p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \tag{2}$$

where p is an embedding parameter (dummy perturbation parameter in the language of the well-known perturbation theory) and u_0 is the first approximation that satisfies the boundary conditions.

They expanded the solution in p-power series $v = v_0 + v_1 p + v_2 p^2 + \ldots$ and obtained the solution to Eq. (1) as $u = v_0 + v_1 + v_2 + \ldots$ provided that the series converges for $p = 1$.

In particular, the authors concentrated in nonlinear oscillators of the form

$$u'' + \omega_0^2 u + \epsilon f(u) = 0, \tag{3}$$

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where f is a nonlinear function of u'', u', and u, and considered the "variational functional" (Ref. 5 and references therein)

$$J(u) = \int_0^t \left[-\frac{1}{2}u'^2 + \frac{1}{2}\omega_0^2 u^2 + \epsilon F(u) \right] dt, \tag{4}$$

where $dF/du = f$. Note that I have corrected a misprint in the authors' Eq. (9). Obviously, $J(u)$ is minus the well-known action integral[9] for a particular time interval.

In order to introduce the basic idea, the authors first modified the well-known Duffing equation[10]

$$u'' + u + \epsilon u^3 = 0, \quad u(0) = A, \quad u'(0) = 0 \tag{5}$$

as

$$u'' + \omega^2 u + p \left[\epsilon u^3 + (1 - \omega^2) u \right] = 0 \tag{6}$$

and derived the perturbation equations of order zero,

$$u_0'' + \omega^2 u_0 = 0, \tag{7}$$

and first order

$$u_1'' + \omega^2 u_1 + \epsilon u_0^3 + (1 - \omega^2) u_0 = 0, \tag{8}$$

where

$$u_0(t) = A \cos(\omega t) \tag{9}$$

already satisfies the boundary conditions, and

$$u_1(0) = u_1'(0) = 0. \tag{10}$$

Note that in this way the approximate solution $u_{app}(t) = u_0(t) + u_1(t)$ satisfies the correct boundary conditions at $t = 0$.

According to the authors, "ω will be identified from the variational formulation for u_1, which reads"

$$J(u_1) = \int_0^T \left[-\frac{1}{2}u_1'^2 + \frac{1}{2}\omega^2 u_1^2 + (1 - \omega^2) u_0 u_1 + \epsilon u_0^3 u_1 \right] dt, \quad T = \frac{2\pi}{\omega}. \tag{11}$$

They argued that the simplest trial function is[5]

$$u_1 = B \left[\cos(\omega t) - \frac{1}{3} \cos(5\omega t) \right]. \tag{12}$$

Surprisingly, this function satisfies one of the boundary conditions $u_1'(0) = 0$ but not the other one because $u_1(0) = 2B/3 = 0$ leads to the unwanted trivial solution.

From the variational conditions $\partial J/\partial B = 0$ and $\partial J/\partial \omega = 0$, the authors obtained

$$\omega = \sqrt{1 + \frac{3}{4}\epsilon A^2}, \quad B = 0. \tag{13}$$

Although the estimated value of ω is reasonable, the result $B = 0$ leads to the trivial solution $u_1 = 0$ that restricts considerably the practical utility of the approach because the approximate solution $u_{app}(t) = u_0(t)$ is rather poor.

In order to improve the results, the authors proposed the correction

$$u_1 = B_1 \left[\cos(\omega t) - \frac{\cos(3\omega t)}{5} \right] + B_3 \left[\frac{\cos(3\omega t)}{5} - \frac{\cos(5\omega t)}{7} \right] \tag{14}$$

and from the variational conditions $\partial J/\partial B_1 = 0$, $\partial J/\partial B_3 = 0$, and $\partial J/\partial \omega = 0$, they obtained the frequency

$$\omega = \frac{\sqrt{31}}{124} \sqrt{\sqrt{510237\rho^2 + 1416576\rho + 984064} - 357\rho - 496}, \tag{15}$$

where I have introduced the relevant parameter of the model $\rho = \epsilon A^2$. They did not show the coefficients and I obtained

$$B_1 = \frac{A\left[357\rho - 496\left(\omega^2 - 1\right)\right]}{96\omega^2}, \quad B_3 = \frac{49A\left[3\rho - 4\left(\omega^2 - 1\right)\right]}{96\omega^2}. \tag{16}$$

Note that $u_1(t)$ does not satisfy one of the required boundary conditions

$$u_1(0) = -\frac{A\left(68\omega^2 - 49\rho - 68\right)}{16\omega^2} \tag{17}$$

and, consequently, $u_{app}(t) = u_0(t) + u_1(t)$ exhibits the wrong amplitude $u_{app}(0) = A + u_1(0)$. It, therefore, seems that by means of the variational approach the authors obtained a frequency for the amplitude A and an approximate trajectory $u_{app}(t)$ with a different amplitude.

Akbarzade and Langari[5] applied the method to other nonlinear oscillators that I briefly discuss below.

As a first example, the authors chose

$$u'' + u^3 = 0, \quad u(0) = A, \quad u'(0) = 0. \tag{18}$$

In this case, they expanded the solution to

$$u'' + \omega^2 u + p\left(u^3 - \omega^2 u\right) = 0 \tag{19}$$

in a Taylor series about $p = 0$ and chose $u_0(t) = A\cos(\omega t)$ that satisfies both boundary conditions. As a first-order trial function, they proposed

$$u_1(t) = B\left[\cos(\omega t) - \frac{1}{5}\cos(3\omega t)\right] \tag{20}$$

that satisfies one of the boundary conditions $u_1'(0) = 0$ but not the other one $u_1(0) = 4B/5$. Therefore, $u_{app}(t) = u_0(t) + u_1(t)$ does not satisfy $u_{app}(0) = A$ except for the trivial case $B = 0$ which is exactly the result of their variational method.[5]

The second example

$$u'' + u + u^{1/3} = 0, \quad u(0) = A, \quad u'(0) = 0 \tag{21}$$

is interesting because $u^{1/3}$ exhibits a branch point at $u = 0$. The authors did not explicitly indicate that they chose the real branch and they should have written this term, for example, as $u|u|^{-2/3}$ or $\mathrm{sgn}(u)|u|^{1/3}$ to avoid confusion. In this case, they resorted to the perturbation equation

$$u'' + \omega^2 u + p\left[u^{1/3} + \left(1 - \omega^2\right)u\right] = 0 \tag{22}$$

and expanded $u_0^{1/3}$ in a Fourier series in order to calculate the variational integral. This expansion is equivalent to choosing the real branch as discussed above. Again they proposed the trial function (20) that does not satisfy one of the boundary conditions at $t = 0$ and again they obtained the trivial result $B = 0$.

The third example

$$u'' + u^3 + u^{1/3} = 0, \quad u(0) = A, \quad u'(0) = 0 \tag{23}$$

was converted into the perturbation equation

$$u'' + \omega^2 u + p\left(u^{1/3} + u^3 - \omega^2 u\right) = 0. \tag{24}$$

The authors chose u_0 and u_1 as in the preceding example and, of course, they obtained the same trivial correction of first order. Thus, in all the three examples the authors' result was the rather poor harmonic approximation $u_{app}(t) = u_0(t)$.

III. CONCLUSIONS

Although the combination of the homotopy perturbation method and the variational principle proposed by Akbarzade and Langari[5] led to reasonable approximate frequencies, one cannot take the approach seriously because of its inconsistencies. First, the first-order corrections to the solutions do not satisfy one of the boundary conditions proposed by the authors and, second, in most of the cases the resulting corrections are trivial (that is to say, they vanish identically). In the case where this correction does not vanish, the approximate solution exhibits an amplitude that is different from the one appearing in the expression for the frequency.

The conclusions of the present paper are in line with those of Sanchez[6] who showed that the approach proposed by He[1] yielded an acceptable frequency for all amplitudes but the trajectory was quite poor for a sufficiently strong nonlinear oscillation.

[1] J.-H. He, J. Sound Vib. **229**, 1257 (2000).
[2] J.-H. He, Phys. Rev. Lett. **90**, 174301 (2003).
[3] Z.-F. Ren and J.-H. He, Phys. Lett. A **373**, 3749 (2009).
[4] J.-H. He, Phys. Lett. A **374**, 2312 (2010).
[5] M. Akbarzade and J. Langari, J. Math. Phys. **52**, 023518 (2011).
[6] N. E. Sanchez, J. Sound Vib. **282**, 1309 (2005).
[7] S. Rajendran, S. N. Pandey, and M. Lakshmanan, Phys. Rev. Lett. **93**, 069401 (2004).
[8] F. M. Fernández, On a simple approach to nonlinear oscillators, e-print arXiv:0910.0600v1 [math-ph].
[9] H. Goldstein, *Classical Mechanics*, 2nd ed. (Addison-Wesley, Reading, MA, 1980).
[10] A. H. Nayfeh, *Introduction to Perturbation Techniques* (Wiley, New York, 1981).