

# Classical and quantum field-theoretical approach to the non-linear $q$ -Klein-Gordon equation

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**Abstract** – In the wake of efforts made Nobre and Rego-Monteiro in *EPL*, **97** (2012) 41001 and *J. Math. Phys.*, **54** (2913) 103302, we extend them here by developing the conventional Lagrangian treatment of a classical field theory (FT) to the  $q$ -Klein-Gordon equation advanced in *Phys. Rev. Lett.*, **106** (2011) 140601 and *J. Math. Phys.*, **54** (2913) 103302 by the same authors, and the quantum theory corresponding to  $q = \frac{3}{2}$ . This makes it possible to generate a putative conjecture regarding black matter. Our theory reduces to the usual FT for  $q \rightarrow 1$ .

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**Introduction.** – Motivated by the need for understanding a number of physical phenomena related to complex systems, interesting proposals for localized solutions have been proposed in the last five years, based on modifications of the linear Klein-Gordon and Schrödinger equations. This is done by turning them into non-linear equations (NLKG and NLSE, respectively) [1,2]. In the wake of efforts made in [1,3], we extend them here by developing a *conventional* classical field theory (FT) corresponding to the  $q$ -Klein-Gordon equation of [2] (*the FT of [3] is not the customary one, but the higher-order FT of [4,5] is, see below*). We also advance the concomitant quantum theory for  $q = \frac{3}{2}$ .

The NLSE may be employed for describing components of dark matter. The structure of the action variational principle leading to the NLSE implies that it might describe particles that do not interact with the electromagnetic field [1]. Note also that the NLSE exhibits a remarkable similarity with the Schrödinger equation associated to a particle with a time-position-dependent effective mass [6–9], involving quantum particles in non-local potentials (*e.g.*, the energy density functional treatment of the quantum many-body problem [10]).

We first develop the conventional classical field theory (CFT) associated to the  $q$ -Klein-Gordon equation proposed in [2] and deduced in [11] from the hypergeometric differential equation (HDE). We define the corresponding physical fields via an analogy with treatments in string theory [12] for defining physical states of the bosonic

string. Our ensuing theory reduces to the conventional Klein-Gordon (KG) field theory for  $q \rightarrow 1$ . As a second step we develop the quantum theory for  $q = \frac{3}{2}$ .

Recently, Rego-Monteiro and Nobre [3] advanced an interesting classical field theory for the generalized  $q$ -Klein-Gordon equation of [2] through the use of Lagrangian procedures for higher-order equations. This valuable effort deserves an extension, that will be tackled here. More to the point:

- Rego-Monteiro and Nobre [3] use the higher-order Lagrangian procedures of Bollini and Giambiagi [4,5], while ours is the usual Lagrangian treatment.
- They are unable, in their procedure, to throw away total divergences in the Lagrangian, since, if they do that, they do not obtain the correct expression for the four-momentum of the field, while we do it here.
- They do not obtain the physical fields, that is, the admissible fields for which probability is conserved.

*Most importantly: we add the quantum field theory for  $q = \frac{3}{2}$ , while the approach of [3] is purely classical.*

**A non-linear  $q$ -Klein-Gordon equation.** –

*Classical approach.* Consider then the  $q$ -Klein-Gordon equation, advanced in [2] and HDE-deduced in [11]:

$$\square\phi(x_\mu) + qm^2[\phi(x_\mu)]^{(2q-1)} = 0. \quad (1)$$

A possible solution to this equation is

$$\phi(x_\mu) = [1 + i(1 - q)(\vec{k} \cdot \vec{x} - \omega t)]^{\frac{1}{1-q}}. \quad (2)$$

We wish to formulate the CFT associated to (1). We start with the classical action

$$\mathcal{S} = \frac{1}{(6q-2)V} \int_M \left\{ \partial_\mu \phi(x_\mu) \partial^\mu \psi(x_\mu) + \partial_\mu \phi^+(x_\mu) \partial^\mu \psi^+(x_\mu) - qm^2 \left[ \phi^{(2q-1)}(x_\mu) \psi(x_\mu) + \phi^{+(2q-1)}(x_\mu) \psi^+(x_\mu) \right] \right\} d^n x. \quad (3)$$

In  $\mathcal{S}$  we detect the appearance of the Klein-Gordon field  $\phi$  the auxiliary field  $\psi$ . The second arises because on the non-linearity of the  $q$ -Klein-Gordon. We recast the action (3) as

$$\mathcal{S} = \int_M \mathcal{L}(\phi, \psi, \phi^+, \psi^+, \partial_\mu \phi, \partial_\mu \psi, \partial_\mu \phi^+, \partial_\mu \psi^+) d^n x, \quad (4)$$

where  $M$  stands for Minkowski's space and  $\mathcal{L}$  is the pertinent Lagrangian. From the minimum action principle one gets the motion equations for the two fields

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} = 0; \quad \frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi)} = 0. \quad (5)$$

The first equation coincides with (1). The auxiliary field equations is

$$\square \psi(x_\mu) + q(2q-1)m^2 [\phi(x_\mu)]^{(2q-2)} \psi(x_\mu) = 0. \quad (6)$$

The solution associated to (2) is

$$\psi(x_\mu) = [1 + i(1 - q)(\vec{k} \cdot \vec{x} - \omega t)]^{\frac{2q-1}{q-1}}. \quad (7)$$

For  $q \rightarrow 1$ ,  $\psi$  becomes the conjugated of  $\phi$ .

We wish to ascertain that the relations between energy and momentum in (2) remain intact in our formulation. For this we need to evaluate these two field-quantities. The field's energy-momentum is

$$\begin{aligned} \mathcal{T}_\mu^\nu &= \frac{\partial \mathcal{L}}{\partial (\partial_\nu \phi)} \partial_\mu \phi + \frac{\partial \mathcal{L}}{\partial (\partial_\nu \psi)} \partial_\mu \psi + \frac{\partial \mathcal{L}}{\partial (\partial_\nu \phi^+)} \partial_\mu \phi^+ \\ &+ \frac{\partial \mathcal{L}}{\partial (\partial_\nu \psi^+)} \partial_\mu \psi^+ - \delta_\mu^\nu \mathcal{L}. \end{aligned} \quad (8)$$

Its expression in terms of the two fields becomes

$$\begin{aligned} \mathcal{T}_\mu^\nu &= \frac{1}{(6q-2)V} [\partial^\nu \psi \partial_\mu \phi + \partial^\nu \phi \partial_\mu \psi + \partial^\nu \psi^+ \partial_\mu \phi^+ \\ &+ \partial^\nu \phi^+ \partial_\mu \psi^+] - \delta_\mu^\nu \mathcal{L}. \end{aligned} \quad (9)$$

The four-momentum is

$$\mathcal{P}_\mu = \int_V \mathcal{T}_\mu^0 d^{n-1} x, \quad (10)$$

where  $V$  is the Euclidian volume. The time component of the four-momentum is the field energy (up to spatial divergences)

$$\begin{aligned} \mathcal{P}_0 &= \frac{1}{(6q-2)V} \int_V (\partial_0 \psi \partial_0 \phi + \partial_0 \psi^+ \partial_0 \phi^+ \\ &- \psi \partial_0^2 \phi - \psi^+ \partial_0^2 \phi^+) d^{n-1} x. \end{aligned} \quad (11)$$

Using the solutions (2) and (7) we find for the energy

$$\mathcal{P}_0 = \frac{1}{(6q-2)V} \int_V (6q-2)\omega^2 d^{n-1} x, \quad (12)$$

or

$$\mathcal{P}^0 = \mathcal{P}_0 = \omega^2. \quad (13)$$

Up to spatial divergences, the field-momentum is

$$\begin{aligned} \mathcal{P}_j &= \frac{1}{(6q-2)V} \int_V (\partial_0 \psi \partial_j \phi + \partial_0 \psi^+ \partial_j \phi^+ \\ &- \psi \partial_0 \partial_j \phi - \psi^+ \partial_0 \partial_j \phi^+) d^{n-1} x. \end{aligned} \quad (14)$$

Specializing this for the solutions (2) and (7) one has

$$\mathcal{P}_j = -\frac{1}{(6q-2)V} \int_V (6q-2)\omega k_j d^{n-1} x \quad (15)$$

or

$$\mathcal{P}^j = -\mathcal{P}_j = \omega k_j. \quad (16)$$

We see that eqs. (13)–(16) are proportional to the energy and momentum of the  $q$ -exponential wave (1), while the proportionality constant is the wave energy  $\omega$ . This happens because we did not use a  $q$ -exponential divided by  $\sqrt{2\omega}$  as is the case with the usual Klein-Gordon field when one appeals to waves  $e^{i(\vec{k} \cdot \vec{x} - \omega t)}$  instead of the more common waves  $\frac{e^{i(\vec{k} \cdot \vec{x} - \omega t)}}{\sqrt{2\omega}}$ .

The remedy is to choose the constant appearing in the field action as  $\frac{1}{(6q-2)V\omega}$  instead of  $\frac{1}{(6q-2)V}$ . In such a case the four-momentum becomes

$$\mathcal{P}_\mu \rightarrow \frac{\mathcal{P}_\mu}{\omega}, \quad (17)$$

and one finds, as expected,

$$\mathcal{P}^\mu = (\omega, \vec{k}), \quad (18)$$

in complete agreement with the conventional field formulation. Note that, from (3), our theory is not gauge invariant, save for  $q \rightarrow 1$ . This entails that our fields cannot interact with light. In other words, for  $q \neq 1$ , we can have free massive particles of a non-linear character, that seem to be incapable to couple with light. This might suggest a mechanism able to describe the presence of dark matter [1].

As for probability conservation, we define the four-current as

$$\mathcal{J}_\mu = \frac{i}{4mV} [\psi \partial_\mu \phi - \phi \partial_\mu \psi + \phi^+ \partial_\mu \psi^+ - \psi^+ \partial_\mu \phi^+]. \quad (19)$$

Thus, the four-divergence of the four-current does not vanish. It is now

$$\partial_\mu \mathcal{J}^\mu = \mathcal{K}, \quad (20)$$

where  $\mathcal{K}$  is

$$\mathcal{K} = \frac{i}{4mV} q(2q-2) \left[ \psi \phi^{(2q-1)} - \psi^+ \phi^{+(2q-1)} \right]. \quad (21)$$

Note that  $\mathcal{K}$  vanishes for  $q \rightarrow 1$ .

We appeal then to bosonic string's theory [12] and define (in a similar way to that for the definition of physical states) the *physical* fields as those that make  $\mathcal{K}$  vanish. The waves (2) and (7) make  $\mathcal{K}$  vanish. Also,

$$\mathcal{J}^\mu = (\rho, \vec{j}), \quad (22)$$

where  $\rho$  is

$$\rho = \frac{i}{4mV} [\psi \partial_t \phi - \phi \partial_t \psi + \phi^+ \partial_t \psi^+ - \psi^+ \partial_t \phi^+]. \quad (23)$$

Note that unlike the usual instance,  $\rho$  is not positive-definite and that  $\vec{j}$  is

$$\vec{j} = -\frac{i}{4mV} [\psi \nabla \phi - \phi \nabla \psi + \phi^+ \nabla \psi^+ - \psi^+ \nabla \phi^+]. \quad (24)$$

All quantities defined in this section become identical to those of the usual CFT for  $q \rightarrow 1$ .

**Quantum approach for  $q = \frac{3}{2}$ .** – For  $q = \frac{3}{2}$  and small  $m$ , field quantization can be performed perturbatively. We write the corresponding action as

$$\mathcal{S} = \int_M \left\{ \partial_\mu \phi(x_\mu) \partial^\mu \psi(x_\mu) + \partial_\mu \phi^+(x_\mu) \partial^\mu \psi^+(x_\mu) - \frac{3}{2} m^2 [\phi^2(x_\mu) \psi(x_\mu) + \phi^{+2}(x_\mu) \psi^+(x_\mu)] \right\} d^4 x. \quad (25)$$

Now we define i) the free action  $\mathcal{S}_0$  and ii) that corresponding to the interaction  $\mathcal{S}_I$  as

$$\mathcal{S}_0 = \int_M [\partial_\mu \phi(x_\mu) \partial^\mu \psi(x_\mu) + \partial_\mu \phi^+(x_\mu) \partial^\mu \psi^+(x_\mu)] d^4 x, \quad (26)$$

$$\mathcal{S}_I = -\frac{3}{2} m^2 \int_M [\phi^2(x_\mu) \psi(x_\mu) + \phi^{+2}(x_\mu) \psi^+(x_\mu)] d^4 x. \quad (27)$$

The fields in the interaction representation satisfy the equations of motion for free fields, corresponding to the action  $\mathcal{S}_0$ . This is to satisfy the usual massless Klein-Gordon equation. As a consequence, we can cast the fields  $\phi$  and  $\psi$  in the form

$$\phi(x_\mu) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{-\infty}^{\infty} \left[ \frac{a(\vec{k})}{\sqrt{2\omega}} e^{ik_\mu x_\mu} + \frac{b^+(\vec{k})}{\sqrt{2\omega}} e^{-ik_\mu x_\mu} \right] d^3 k, \quad (28)$$

$$\psi(x_\mu) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{-\infty}^{\infty} \left[ \frac{c(\vec{k})}{\sqrt{2\omega}} e^{ik_\mu x_\mu} + \frac{d^+(\vec{k})}{\sqrt{2\omega}} e^{-ik_\mu x_\mu} \right] d^3 k, \quad (29)$$

where  $k_0 = \omega = |\vec{k}|$ . The quantification of these two fields is immediate and the usual one, given by

$$\begin{aligned} [a(\vec{k}), a^+(k')] &= [b(\vec{k}), b^+(k')] = [c(\vec{k}), c^+(k')] = \\ [d(\vec{k}), d^+(k')] &= \delta(\vec{k} - \vec{k}'). \end{aligned} \quad (30)$$

The naked propagator corresponding to both fields is the customary one, and it is just the Feynman propagator for massless fields:

$$\Delta_0(k_\mu) = \frac{i}{k^2 + i0}, \quad (31)$$

where  $k^2 = k_0^2 - \vec{k}^2$ . The dressed propagator, which takes into account the interaction, is given by

$$\Delta(k_\mu) = \frac{i}{k^2 + i0 - i\Sigma(k_\mu)}, \quad (32)$$

where  $\Sigma(k_\mu)$  is the self-energy.

Let us calculate the self-energy for the field  $\phi$  at second order in perturbation theory. To this order, the self-energy is composed of two Feynman diagrams, of which one is null (this is easily demonstrated using the regularization of Guelfand for integrals containing powers of  $x$  [13]). Therefore, we have for self-energy the expression

$$\Sigma(k_\mu) = \frac{9m^4}{4} \frac{i}{k^2 + i0} * \frac{i}{k^2 + i0}. \quad (33)$$

The convolution of the two Feynman's propagators of zero mass is calculated directly using the theory of convolution of ultradistributions [14–17]. Its result is simply

$$\frac{i}{k^2 + i0} * \frac{i}{k^2 + i0} = i\pi^2 \ln(k^2 + i0). \quad (34)$$

The self-energy is then

$$\Sigma(k_\mu) = \frac{9\pi^2 m^4 i}{4} \ln(k^2 + i0). \quad (35)$$

As a consequence, the dressed propagator, up to second order, is given by

$$\Delta(k_\mu) = \frac{4i}{4k^2 + 9\pi^2 m^4 \ln(k^2 + i0) + i0}. \quad (36)$$

For both fields  $\phi$  and  $\psi$  the self-energy and the dressed propagator coincide up to second order.

Note that the current of probability is given by

$$\mathcal{J}_\mu = \frac{i}{4m} [\psi \partial_\mu \phi - \phi \partial_\mu \psi + \phi^+ \partial_\mu \psi^+ - \psi^+ \partial_\mu \phi^+] \quad (37)$$

and it is verified that

$$\partial_\mu \mathcal{J}^\mu = 0. \quad (38)$$

This implies that the fields defined in the representation of interaction are physical fields.

**Conclusions.** – We have here developed further weapons for the formidable arsenal being erected in the wake of the pioneer work of ref. [2], in order to be better able to face the complex physics associated to non-linear quantum equations.

First, we developed the classical field theory corresponding to the non-linear  $q$ -Klein-Gordon equation, improving upon the work of Rego-Monteiro and Nobre [3].

1) Rego-Monteiro and Nobre [3] use the higher-order Lagrangian procedures of Bollini and Giambiagi [4,5], while we have used the conventional Lagrangian treatment.

2) They were unable, in their procedure, to throw away total divergences in the Lagrangian, since if they were to do that, they would not obtain the correct expression for the four-momentum of the field. In our procedure we have removed the total divergences.

3) We have obtained the physical fields, this is, the admissible fields for which probability is conserved.

4) *Most importantly: we have added the quantum field theory for  $q = \frac{3}{2}$ .*

We hope that our next stage will be extending things to a quantum field theory for  $q$  a real number such that  $1 \leq q < 2$  is a difficult task indeed.

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