SYMMETRIC AFFINE SURFACES WITH TORSION

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ABSTRACT. We study symmetric affine surfaces which have non-vanishing torsion tensor. We give a complete classification of the local geometries possible if the torsion is assumed parallel. This generalizes a previous result of Opozda in the torsion free setting; these geometries are all locally homogeneous. If the torsion is not parallel, we assume the underlying surface is locally homogeneous and provide a complete classification in this setting as well.

1. INTRODUCTION AND STATEMENT OF RESULTS

In differential geometry, a connection ∇ on the tangent bundle of a smooth manifold M gives rise to the notion of parallelism. The pair $\mathcal{M} = (M, \nabla)$ is called an *affine manifold*; if $\dim\{M\} = 2$, $\mathcal{M}$ is called an *affine surface*. We emphasize that in contrast to the usage employed by some authors, we permit the torsion tensor $T(X,Y) := \frac{1}{2}(\nabla_X Y - \nabla_Y X - [X,Y])$ to be non-zero.

The study of various properties of affine manifolds is relevant in *non-metric extensions* of General Relativity, i.e. geometries where the connection ∇ does not arise as the Levi-Civita connection of some underlying pseudo-Riemannian metric. The standard formulation of General Relativity regards the metric as a canonical field which determines the affine structure by means of the Levi-Civita connection. Open questions in our current understanding of gravitation have led physicists to study generalizations of this scenario. In non-metric extensions of General Relativity [7, 22], the affine connection provides an independent degree of freedom; in particular, Einstein-Cartan theory regards the torsion tensor as a new canonical field.

Spacetimes with torsion give different dynamics for matter fields [20] (see [6] for an account of experiments aimed at measuring the existence of torsion). As a dynamical field, torsion also plays an important role in alternative models of the early universe [19, 21]. For recent articles on Einstein-Cartan gravity, we refer to [9, 12]. Note also that two-dimensional theories of gravity constitute an area of interest on its own; for studies of torsion in this context see [5, 10, 16] and the references therein. Finally, non-metric connections can be used to study defects in condensed matter; in this setting, the torsion describes dislocations in solids [8, 11, 15]. Thus, apart from their purely mathematical relevance, the affine properties of manifolds are of interest in physical contexts.

1.1. **Notational conventions.** A diffeomorphism of the underlying manifold M is said to be an *affine diffeomorphism* if it preserves the connection; the geometry $\mathcal{M}$ is said to be *affine homogeneous* if the group of affine diffeomorphisms acts transitively. There is a corresponding local theory. Let $R(X,Y) := \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$ be the *curvature*

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operator of an affine geometry $\mathcal{M} = (M, \nabla)$. We contract indices to define the *Ricci tensor* $\rho(X,Y) := \operatorname{Trace}(Z \to R(Z,X)Y)$. We say that $\mathcal{M}$ has a *symmetric Ricci tensor* if $\rho(X,Y) = \rho(Y,X)$ for all X and Y; this is always the case in the metrizable setting but need not hold in general. We say that $\mathcal{M}$ is a *symmetric affine surface* if $\mathcal{M}$ is an affine surface satisfying $\nabla R = 0$ or, equivalently as we are in the 2-dimensional setting, if the Ricci tensor is parallel, i.e. $\nabla\rho = 0$. We will show presently in Lemma 9 that any symmetric affine surface has a symmetric Ricci tensor.

If (x^1, x^2) is a system of local coordinates on an affine surface, we expand $\nabla_{\partial_{x^i}}\partial_{x^j} = \Gamma_{ij}{}^k\partial_{x^k}$; the *Christoffel symbols* $\Gamma_{ij}{}^k$ determine the connection and we shall specify geometries by giving their (possibly) non-zero Christoffel symbols. We say that an affine surface $\mathcal{M}_1 = (M_1, \nabla_1)$ is *modeled* on an affine surface $\mathcal{M} = (M, \nabla)$ if $\mathcal{M}$ is homogeneous and if there is a cover of M_1 by open sets which are affine isomorphic to open subsets of M. This implies that $\mathcal{M}_1$ is locally homogeneous.

1.2. **Symmetric affine surfaces with vanishing torsion.** We say $\mathcal{M}$ is *torsion free* if $T = 0$. The torsion free symmetric affine surfaces have been classified by Opozda [17]. The Ricci tensor is symmetric and there are 6 possible signatures. If $\operatorname{Rank}\{\rho\} = 0$, then $\rho = 0$; if $\operatorname{Rank}\{\rho\} = 1$, then ρ is either positive semi-definite ($\rho \geq 0$) or negative semi-definite ($\rho \leq 0$); if $\operatorname{Rank}\{\rho\} = 2$, then ρ is either positive definite ($\rho > 0$), negative definite ($\rho < 0$), or indefinite. The symmetric affine surfaces without torsion are all locally homogeneous and modeled on one of six non-isomorphic geometries which are distinguished by the signature of the Ricci tensor. The first four of the geometries, given in Assertions (1–4) below, are metrizable, i.e. the connection is the associated Levi-Civita connection. The remaining two geometries, given in Assertions (5,6) below, are not metrizable.

Theorem 1 (Opozda)**.** *Let $\mathcal{M}$ be a symmetric affine surface without torsion. Then $\mathcal{M}$ is locally homogeneous and modeled on one of the following geometries:*

(1) *The flat plane $\mathbb{R}^2$ with $ds^2 = (dx^1)^2 + (dx^2)^2$; $\rho = 0$.*
(2) *The hyperbolic plane $\mathbb{R}^+ \times \mathbb{R}$ with $ds^2 = \frac{(dx^1)^2 + (dx^2)^2}{(x^1)^2}$; $\rho < 0$.*
(3) *The Lorentzian plane $\mathbb{R}^+ \times \mathbb{R}$ with $ds^2 = \frac{(dx^1)^2 - (dx^2)^2}{(x^1)^2}$; ρ is indefinite.*
(4) *The round sphere S^2; $\rho > 0$.*
(5) *The non-metrizable geometry with $\Gamma_{11}{}^1 = 1$ and $\Gamma_{22}{}^1 = +1$; $\rho \geq 0$.*
(6) *The non-metrizable geometry with $\Gamma_{11}{}^1 = 1$ and $\Gamma_{22}{}^1 = -1$; $\rho \leq 0$.*

1.3. **Symmetric affine surfaces with non-vanishing and parallel torsion.** We will prove the following result in Section 2 extending Theorem 1. Were we to take $u = 0$ in Theorem 2 (2), then the torsion would vanish and we would obtain the geometries described in Theorem 1 (5, 6).

Theorem 2. *Let $\mathcal{M}$ be a symmetric affine surface with non-vanishing and parallel torsion. Then $\mathcal{M}$ is modeled on one of the following structures.*

(1) $\Gamma_{11}{}^1 = 1$, $\Gamma_{12}{}^1 = 2$, $\rho = 0$.
(2) $\Gamma_{11}{}^1 = 1$, $\Gamma_{12}{}^1 = 2u$ *for* $u > 0$, $\Gamma_{22}{}^1 = \pm 1$, $\rho = \pm\, dx^2 \otimes dx^2$.

These geometries are all inequivalent affine structures and homogeneous.

1.4. **Locally homogeneous affine surfaces.** Opozda [18] classified the locally homogeneous affine surfaces without torsion. Subsequently, Arias-Marco and Kowalski [1] extended this classification to the more general setting; a different proof of this result has been given recently by Brozos-Vázquez et al. [2]. Previous studies of locally homogeneous surfaces in the torsion free setting include [13, 14]. For a different approach in higher dimensions we refer to [4].

Theorem 3. *Let $\mathcal{M}$ be a locally homogeneous affine surface, possibly with torsion. At least one of the following possibilities holds:*

(1) *Type $\mathcal{A}$: There is a coordinate atlas for $\mathcal{M}$ so ${\Gamma_{ij}}^k \in \mathbb{R}$.*
(2) *Type $\mathcal{B}$: There is a coordinate atlas for $\mathcal{M}$ so $x^1{\Gamma_{ij}}^k \in \mathbb{R}$ and $x^1 > 0$.*
(3) *Type $\mathcal{C}$: The geometry is locally isomorphic to the geometry of the round sphere S^2 with the associated Levi-Civita connection.*

The possibilities of Theorem 3 are not exclusive; there are geometries which can be realized both as Type $\mathcal{A}$ and Type $\mathcal{B}$ structures. However, no Type $\mathcal{A}$ or Type $\mathcal{B}$ structure is also Type $\mathcal{C}$. We refer to Calviño-Louzao et al. [3] for additional information in this regard.

We now examine affine symmetric surfaces with non-parallel torsion; to obtain a useful classification, we shall not consider the most general surfaces but restrict to locally homogeneous geometries. Theorem 4 (resp. Theorem 6), to be proved in Section 3 (resp. Section 4) deals with surfaces of Type $\mathcal{A}$ (resp. Type $\mathcal{B}$).

1.5. **Type $\mathcal{A}$ affine symmetric surfaces.** The general linear group $\mathrm{GL}(2,\mathbb{R})$ acts on the set of Type $\mathcal{A}$ geometries by change of basis. We will establish the following result in Section 3 which classifies the Type $\mathcal{A}$ symmetric affine surfaces with non-parallel torsion.

Theorem 4. *Let $\mathcal{M}$ be a Type $\mathcal{A}$ symmetric affine surface with non-parallel torsion tensor. Then $\mathcal{M}$ is flat (i.e. $\rho = 0$) and $\mathcal{M}$ is equivalent under the action of the gauge group $\mathrm{GL}(2,\mathbb{R})$ to one of the following 6 geometries for $\alpha, \eta \in \mathbb{R}$, $\beta \in \mathbb{R} - \{0, 2\}$, $\gamma \in \mathbb{R} - \{0\}$, $\varepsilon = \pm 1$, $\eta \geq 0$, and $T = (dx^1 \wedge dx^2) \otimes \partial_{x^2}$, where no two different surfaces are linearly equivalent:*

(1) $${\Gamma_{11}}^1 = \gamma,\quad {\Gamma_{11}}^2 = \gamma - 1,\quad {\Gamma_{12}}^1 = 0,\quad {\Gamma_{12}}^2 = 1,\quad {\Gamma_{21}}^1 = 0,\quad {\Gamma_{21}}^2 = -1,\quad {\Gamma_{22}}^1 = 0,\quad {\Gamma_{22}}^2 = 1,\qquad \nabla T = \begin{pmatrix} 0 & -\gamma \\ 0 & 0 \end{pmatrix}.$$

(2) $${\Gamma_{11}}^1 = 0,\quad {\Gamma_{11}}^2 = \alpha,\quad {\Gamma_{12}}^1 = 1,\quad {\Gamma_{12}}^2 = 2,\quad {\Gamma_{21}}^1 = 1,\quad {\Gamma_{21}}^2 = 0,\quad {\Gamma_{22}}^1 = 0,\quad {\Gamma_{22}}^2 = 1,\qquad \nabla T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(3) $${\Gamma_{11}}^1 = \gamma,\quad {\Gamma_{11}}^2 = 0,\quad {\Gamma_{12}}^1 = 0,\quad {\Gamma_{12}}^2 = \gamma,\quad {\Gamma_{21}}^1 = 0,\quad {\Gamma_{21}}^2 = \gamma - 2,\quad {\Gamma_{22}}^1 = 0,\quad {\Gamma_{22}}^2 = 0,\qquad \nabla T = \begin{pmatrix} 0 & -\gamma \\ 0 & 0 \end{pmatrix}.$$

(4) $${\Gamma_{11}}^1 = 2,\quad {\Gamma_{11}}^2 = 1,\quad {\Gamma_{12}}^1 = 0,\quad {\Gamma_{12}}^2 = 2,\quad {\Gamma_{21}}^1 = 0,\quad {\Gamma_{21}}^2 = 0,\quad {\Gamma_{22}}^1 = 0,\quad {\Gamma_{22}}^2 = 0,\qquad \nabla T = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}.$$

(5) $${\Gamma_{11}}^1 = \beta,\quad {\Gamma_{11}}^2 = 0,\quad {\Gamma_{12}}^1 = 0,\quad {\Gamma_{12}}^2 = 2,\quad {\Gamma_{21}}^1 = 0,\quad {\Gamma_{21}}^2 = 0,\quad {\Gamma_{22}}^1 = 0,\quad {\Gamma_{22}}^2 = 0,\qquad \nabla T = \begin{pmatrix} 0 & -\beta \\ 0 & 0 \end{pmatrix}.$$

(6) $${\Gamma_{11}}^1 = \omega,\quad {\Gamma_{11}}^2 = 0,\quad {\Gamma_{12}}^1 = 0,\quad {\Gamma_{12}}^2 = \omega,\quad {\Gamma_{21}}^1 = 0,\quad {\Gamma_{21}}^2 = \omega - 2,\quad {\Gamma_{22}}^1 = \varepsilon,\quad {\Gamma_{22}}^2 = \eta,\qquad \nabla T = \begin{pmatrix} 0 & -\omega \\ \varepsilon & 0 \end{pmatrix}.$$

Remark 5. We choose $\eta \geq 0$ because any surface of family (6) given by (ω, η) is equivalent to $(\omega, -\eta)$ thru $x_2 \to -x_2$. The constraint $\beta \neq 2$ in the surfaces of family (5) ensures non-equivalence with the surface (3) with $\gamma = 2$.

1.6. **Type $\mathcal{B}$ affine symmetric surfaces.** If $\tilde{\Gamma}_{ij}{}^k \in \mathbb{R}$, we construct a Type $\mathcal{B}$ geometry by setting $\Gamma_{ij}{}^k = \frac{1}{x^1}\tilde{\Gamma}_{ij}{}^k$. To simplify denominators, we evaluate at $x^1 = 1$ to define $\tilde{\rho}$, $\tilde{T}$, and $\widetilde{\nabla T}$. We then have $\rho = (x^1)^{-2}\tilde{\rho}$, $T = (x^1)^{-1}\tilde{T}$, and $\nabla T = (x^1)^{-2}\widetilde{\nabla T}$. The $ax+b$ group acts on the set of Type $\mathcal{B}$ geometries by the linear change of basis $(x^1, x^2) \to (x^1, ax^2 + bx^1)$. In Section 4, we complete our classification of the locally homogeneous symmetric affine surfaces by establishing the following result.

Theorem 6. *Let $\mathcal{M}$ be a Type $\mathcal{B}$ symmetric affine surface with non-parallel torsion tensor. Then $\mathcal{M}$ is equivalent under the action of the $ax+b$ gauge group to one of the following 9 structures with associated parameters ξ, η, α, β, γ, $\delta \in \mathbb{R}$ for $\alpha \geq 0$; $\eta + 1 \neq 2\delta$; $\gamma \neq -\frac{1}{2}$. The torsion is given by $T = (x^1)^{-1}(dx^1 \wedge dx^2) \otimes (T^1\partial_{x^1} + T^2\partial_{x^2})$. No two different surfaces in this classification are linearly equivalent.*

(1) $\tilde{\Gamma}_{11}{}^1 = -2$, $\tilde{\Gamma}_{11}{}^2 = \xi$, $\tilde{\Gamma}_{12}{}^1 = 0$, $\tilde{\Gamma}_{12}{}^2 = 0$, $\tilde{\Gamma}_{21}{}^1 = -1$, $\tilde{\Gamma}_{21}{}^2 = \xi$, $\tilde{\Gamma}_{22}{}^1 = 0$, $\tilde{\Gamma}_{22}{}^2 = 1$, $T^1 = \frac{1}{2}$, $T^2 = -\frac{\xi}{2}$,
$$\tilde{\rho} = \begin{pmatrix} \xi & 1 \\ 1 & 0 \end{pmatrix}, \quad \widetilde{\nabla T} = -\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

(2) $\tilde{\Gamma}_{11}{}^1 = \eta$, $\tilde{\Gamma}_{11}{}^2 = 0$, $\tilde{\Gamma}_{12}{}^1 = 0$, $\tilde{\Gamma}_{12}{}^2 = \eta + 1$, $\tilde{\Gamma}_{21}{}^1 = 0$, $\tilde{\Gamma}_{21}{}^2 = \eta + 1 - 2\delta$, $\tilde{\Gamma}_{22}{}^1 = 0$, $\tilde{\Gamma}_{22}{}^2 = 0$, $T^1 = 0$, $T^2 = \delta$,
$$\rho = 0, \quad \widetilde{\nabla T} = \begin{pmatrix} 0 & -\delta(1+\eta) \\ 0 & 0 \end{pmatrix}.$$

(3) $\tilde{\Gamma}_{11}{}^1 = 2\beta - 2$, $\tilde{\Gamma}_{11}{}^2 = 1$, $\tilde{\Gamma}_{12}{}^1 = 0$, $\tilde{\Gamma}_{12}{}^2 = 2\beta - 1$, $\tilde{\Gamma}_{21}{}^1 = 0$, $\tilde{\Gamma}_{21}{}^2 = -1$, $\tilde{\Gamma}_{22}{}^1 = 0$, $\tilde{\Gamma}_{22}{}^2 = 0$, $T^1 = 0$, $T^2 = \beta$,
$$\rho = 0, \quad \widetilde{\nabla T} = \begin{pmatrix} 0 & -\beta(2\beta - 1) \\ 0 & 0 \end{pmatrix}.$$

(4) $\tilde{\Gamma}_{11}{}^1 = 0$, $\tilde{\Gamma}_{11}{}^2 = \xi$, $\tilde{\Gamma}_{12}{}^1 = 1$, $\tilde{\Gamma}_{12}{}^2 = 2\beta$, $\tilde{\Gamma}_{21}{}^1 = 0$, $\tilde{\Gamma}_{21}{}^2 = 0$, $\tilde{\Gamma}_{22}{}^1 = 0$, $\tilde{\Gamma}_{22}{}^2 = 0$, $T^1 = \frac{1}{2}$, $T^2 = \beta$,
$$\rho = 0, \quad \widetilde{\nabla T} = \frac{1}{2}\begin{pmatrix} -1 & \xi - 2\beta \\ 0 & 0 \end{pmatrix}.$$

(5) $\tilde{\Gamma}_{11}{}^1 = \xi$, $\tilde{\Gamma}_{11}{}^2 = 0$, $\tilde{\Gamma}_{12}{}^1 = 0$, $\tilde{\Gamma}_{12}{}^2 = 2\beta$, $\tilde{\Gamma}_{21}{}^1 = 0$, $\tilde{\Gamma}_{21}{}^2 = 0$, $\tilde{\Gamma}_{22}{}^1 = 0$, $\tilde{\Gamma}_{22}{}^2 = 0$, $T^1 = 0$, $T^2 = \beta$,
$$\rho = 0, \quad \widetilde{\nabla T} = \begin{pmatrix} 0 & -\beta(1+\xi) \\ 0 & 0 \end{pmatrix}.$$

(6) $\tilde{\Gamma}_{11}{}^1 = 2\beta$, $\tilde{\Gamma}_{11}{}^2 = 1$, $\tilde{\Gamma}_{12}{}^1 = 0$, $\tilde{\Gamma}_{12}{}^2 = 2\beta$, $\tilde{\Gamma}_{21}{}^1 = 0$, $\tilde{\Gamma}_{21}{}^2 = 0$, $\tilde{\Gamma}_{22}{}^1 = 0$, $\tilde{\Gamma}_{22}{}^2 = 0$, $T^1 = 0$, $T^2 = \beta$,
$$\rho = 0, \quad \widetilde{\nabla T} = \begin{pmatrix} 0 & -\beta(1+2\beta) \\ 0 & 0 \end{pmatrix}.$$

(7) $\tilde\Gamma_{11}{}^1=\xi,\quad \tilde\Gamma_{11}{}^2=0,\quad \tilde\Gamma_{12}{}^1=2\alpha,\quad \tilde\Gamma_{12}{}^2=-1,\quad \tilde\Gamma_{21}{}^1=0,$
$\tilde\Gamma_{21}{}^2=0,\quad \tilde\Gamma_{22}{}^1=\varepsilon,\quad \tilde\Gamma_{22}{}^2=0,\quad T^1=\alpha,\quad T^2=-\frac{1}{2},$

$$\tilde\rho=\left(\begin{array}{cc}0&0\\0&\varepsilon\xi\end{array}\right),\qquad \widetilde{\nabla T}=-\tfrac{1}{2}\left(\begin{array}{cc}2\alpha&-\xi-1\\ \varepsilon&0\end{array}\right).$$

(8) $\tilde\Gamma_{11}{}^1=2\gamma+1,\quad \tilde\Gamma_{11}{}^2=0,\quad \tilde\Gamma_{12}{}^1=2\alpha,\quad \tilde\Gamma_{12}{}^2=2\gamma,\quad \tilde\Gamma_{21}{}^1=0,$
$\tilde\Gamma_{21}{}^2=0,\quad \tilde\Gamma_{22}{}^1=\varepsilon,\quad \tilde\Gamma_{22}{}^2=0,\quad T^1=\alpha,\quad T^2=\gamma,$

$$\rho=0,\qquad \widetilde{\nabla T}=\left(\begin{array}{cc}-\alpha&-2\gamma(\gamma+1)\\ \varepsilon\gamma&0\end{array}\right).$$

(9) $\tilde\Gamma_{11}{}^1=-1,\quad \tilde\Gamma_{11}{}^2=-2\varepsilon\alpha(2\gamma+1),\quad \tilde\Gamma_{12}{}^1=2\alpha,\quad \tilde\Gamma_{12}{}^2=-1,$
$\tilde\Gamma_{21}{}^1=0,\quad \tilde\Gamma_{21}{}^2=-2\gamma-1,\quad \tilde\Gamma_{22}{}^1=\varepsilon,\quad \tilde\Gamma_{22}{}^2=0,$
$T^1=\alpha,\quad T^2=\gamma,$

$$\tilde\rho=\left(\begin{array}{cc}-2\gamma-1&0\\0&-\varepsilon\end{array}\right),\qquad \widetilde{\nabla T}=\left(\begin{array}{cc}2\alpha\gamma&-2\varepsilon\alpha^2(2\gamma+1)\\ \varepsilon\gamma&-\alpha(2\gamma+1)\end{array}\right).$$

Remark 7. We choose $\alpha\geq 0$ because any surface of family (7) given by (ξ,α) is equivalent to $(\xi,-\alpha)$ thru $x_2\to -x_2$. The same observation holds for families (8) and (9). The remaining constraints on η,γ,δ ensure non-equivalence between different families. Note that $\gamma=-1/2$ in (8) gives (7) for $\xi=0$, and that $\gamma=-1/2$ in (9) gives (7) for $\xi=-1$.

1.7. **Symmetric affine surfaces which are not locally homogeneous.** As for the most part, we shall be concerned with locally homogeneous geometries, we conclude the introduction by presenting two examples of symmetric affine surfaces which are not locally homogeneous.

Example 8.

(1) Let the non-zero symbols be $\Gamma_{12}{}^2=\frac{1}{2}\tanh(x^1)$ and $\Gamma_{21}{}^2=-\frac{1}{2}\tanh(x^1)$. Then $\rho=dx^1\otimes dx^1$, $\nabla\rho=0$, and $\nabla T=(\cosh x_1)^{-2}\,dx^1\wedge dx^2\otimes\partial_{x^2}$. The space of affine Killing vector fields is given by $\operatorname{Span}\{1,x^1,x^2\}\partial_{x^2}$. Thus this is a symmetric affine surface of cohomogeneity 1.
(2) Let $\{X,Y\}$ be a frame for the tangent bundle of M. There is a unique connection with torsion so $\nabla X=0$ and $\nabla Y=0$. Let $\{X^*,Y^*\}$ be the corresponding dual frame for the cotangent bundle and let $[X,Y]$ denote the Lie bracket of X and Y. We have $T=\frac{1}{2}(X^*\wedge Y^*)\otimes[X,Y]$ and the geometry is flat.

2. Proof of Theorem 2

2.1. **The Ricci tensor of a symmetric affine surface.** The fact that the Ricci tensor of a symmetric affine surface is a symmetric 2-tensor is due to Opozda [17] in the torsion free setting; it is not known if a similar statement holds in higher dimensions. We can extend this result to the setting of affine surfaces with torsion.

Lemma 9. *If $\mathcal{M}$ is a connected symmetric affine surface, then the Ricci tensor of $\mathcal{M}$ is a symmetric 2-tensor which has constant rank.*

Proof. Extend the action of the curvature operator to tensors of all types. The *alternating Ricci tensor* is defined by setting $\rho_a := (\rho_{12} - \rho_{21}) dx^1 \wedge dx^2$. As the commutator of covariant differentiation is given by curvature, one has:

$$\begin{aligned}\rho_{a;21} - \rho_{a;12} &= (\rho_{12} - \rho_{21}) R_{12}(dx^1 \wedge dx^2) \\ &= (\rho_{12} - \rho_{21})(-R_{121}{}^1 - R_{122}{}^2)(dx^1 \wedge dx^2) \\ &= (\rho_{12} - \rho_{21})(-\rho_{21} + \rho_{12})(dx^1 \wedge dx^2)\,.\end{aligned}$$

If $\nabla\rho = 0$, then $\nabla\rho_a = 0$ and thus $(\rho_{12} - \rho_{21})^2 = 0$ and $\rho_a = 0$. This shows that the Ricci tensor of $\mathcal{M}$ is symmetric.

We have assumed that M is connected. Given points P and Q, let $\sigma(t)$ be a curve from P to Q. Let $\{e_1(t), e_2(t)\}$ be a parallel frame for the tangent bundle along $\sigma(t)$. Since $\nabla\rho = 0$, we compute

$$\begin{aligned}&\partial_t\{\rho(e_i(t), e_j(t))\} \\ = \;&\{\nabla_{\partial_t}\rho\}(e_i(t), e_j(t)) - \rho(\nabla_{\partial_t} e_i(t), e_j(t)) - \rho(e_i(t), \nabla_{\partial_t} e_j(t)) = 0\,.\end{aligned}$$

Thus the matrix of ρ is constant and $\mathrm{Rank}(\rho)$ is constant. □

2.2. **Abstract torsion tensors.** Let (M, ∇) be an affine surface. Let (x^1, x^2) be a system of local coordinates on M. In terms of the Christoffel symbols the torsion tensor takes the form

$$T := \tfrac{1}{2}(dx^1 \wedge dx^2) \otimes \{(\Gamma_{12}{}^1 - \Gamma_{21}{}^1)\partial_{x^1} + (\Gamma_{12}{}^2 - \Gamma_{21}{}^2)\partial_{x^2}\}\,.$$

Let $\mathfrak{T}(M)$ be the vector space of 2-form valued tangent vector fields; $S \in \mathfrak{T}(M)$ if there are smooth functions S^1 and S^2 so that $S = (dx^1 \wedge dx^2) \otimes (S^1\partial_{x^1} + S^2\partial_{x^2})$. This is the space of *abstract torsion tensors*. Let $\mathfrak{P}(\mathcal{M}) := \{S \in \mathfrak{T}(\mathcal{M}) : \nabla S = 0\}$ be the subspace of *parallel abstract torsion tensors*.

Lemma 10. *If $\mathcal{M}$ is an affine surface, then* $\mathrm{Rank}\{\rho\} + \dim\{\mathfrak{P}\} \le 2$.

Proof. Since M is connected, a parallel tensor is determined by its value at any point of M. Thus $\dim\{\mathfrak{P}\} \le 2$. Suppose that $0 \ne S \in \mathfrak{P}(\mathcal{M})$ is a parallel abstract torsion tensor. We compute:

$$\begin{aligned}0 \;&=\; S_{;12} - S_{;21} = R_{12}(dx^1 \wedge dx^2) \otimes S^k\partial_{x^k} + (dx^1 \wedge dx^2) \otimes R_{12\ell}{}^k S^\ell \partial_{x^k} \\ &=\; \{(-R_{121}{}^1 - R_{122}{}^2)S^1 + (R_{121}{}^1 S^1 + R_{122}{}^1 S^2)\}(dx^1 \wedge dx^2) \otimes \partial_{x^1} \\ &\quad\; + \{(-R_{121}{}^1 - R_{122}{}^2)S^2 + (R_{121}{}^2 S^1 + R_{122}{}^2 S^2)\}(dx^1 \wedge dx^2) \otimes \partial_{x^2} \\ &=\; (dx^1 \wedge dx^2) \otimes \{(\rho_{12}S^1 + \rho_{22}S^2)\partial_{x^1} - (\rho_{11}S^1 + \rho_{21}S^2)\partial_{x^2}\}\,.\end{aligned}$$

Consequently

$$\begin{pmatrix} \rho_{11} & \rho_{21} \\ \rho_{12} & \rho_{22} \end{pmatrix} \begin{pmatrix} S^1 \\ S^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{1}$$

Thus if $\mathfrak{P}$ is non-trivial, $\mathrm{Rank}\{\rho\} \le 1$. Fix $P \in M$ and let (x^1, x^2) be a system of local coordinates centered at P. Suppose $\dim\{\mathfrak{P}\} = 2$. Choose $S_i \in \mathfrak{P}(\mathcal{M})$ so $S_1(P) = \partial_{x^1}$ and $S_2(P) = \partial_{x^2}$. Equation (1) then implies $\rho = 0$. □

2.3. **The associated torsion free surface.** If $\mathcal{M}$ is an affine surface without torsion and if $S \in \mathfrak{T}(M)$, then we can perturb the Christoffel symbols of $\mathcal{M}$ to create a new affine manifold $\mathcal{M}(S) = (M, {}^S\nabla)$ with S as the associated torsion tensor by setting

$$\begin{array}{lll} {}^S\Gamma_{11}{}^1 = \Gamma_{11}{}^1, & {}^S\Gamma_{11}{}^2 = \Gamma_{11}{}^2, & \\ {}^S\Gamma_{12}{}^1 = \Gamma_{12}{}^1 + S^1, & {}^S\Gamma_{21}{}^1 = \Gamma_{12}{}^1 - S^1, & {}^S\Gamma_{22}{}^1 = \Gamma_{22}{}^1, \\ {}^S\Gamma_{12}{}^2 = \Gamma_{12}{}^2 + S^2, & {}^S\Gamma_{21}{}^2 = \Gamma_{12}{}^2 - S^2, & {}^S\Gamma_{22}{}^2 = \Gamma_{22}{}^2. \end{array} \tag{2}$$

Thus every abstract torsion tensor can be realized geometrically. Conversely, if $\mathcal{M}$ is an affine manifold with torsion, set ${}^0\nabla_X Y = \nabla_X Y - T(X,Y)$ and obtain an associated surface ${}^0\mathcal{M} = (M, {}^0\nabla)$ such that ${}^0\mathcal{M}(T) = \mathcal{M}$. We then have

$${}^0\Gamma_{ij}{}^k = \tfrac{1}{2}\{\Gamma_{ij}{}^k + \Gamma_{ji}{}^k\}\,.$$

Let $\mathcal{M}_{u,v}$ be the geometry with (possibly) non-zero Christoffel symbols $\Gamma_{11}{}^1 = 1$, $\Gamma_{12}{}^1 = 2u$, and $\Gamma_{22}{}^1 = v$ for $(u,v) \in \mathbb{R}^2$. The associated torsion free geometry ${}^0\mathcal{M}_{u,v}$ has Christoffel symbols ${}^0\Gamma_{11}{}^1 = 1$, ${}^0\Gamma_{12}{}^1 = {}^0\Gamma_{21}{}^1 = u$, and ${}^0\Gamma_{22}{}^1 = v$. The torsion tensor of $\mathcal{M}_{u,v}$ is given by $T = (dx^1 \wedge dx^2) \otimes (u\partial_{x^1})$. We make a direct computation to see

$$\begin{array}{lll} \rho_{\mathcal{M}_{u,v}} = v\,dx^2 \otimes dx^2, & \nabla\rho_{\mathcal{M}_{u,v}} = 0, & \nabla T = 0, \\ \rho_{{}^0\mathcal{M}_{u,v}} = (v-u^2)\,dx^2 \otimes dx^2, & {}^0\nabla(\rho_{{}^0\mathcal{M}_{u,v}}) = 0, & {}^0\nabla T = 0. \end{array}$$

Thus both $\mathcal{M}_{u,v}$ and ${}^0\mathcal{M}_{u,v}$ are symmetric affine surfaces; the torsion tensor T of $\mathcal{M}_{u,v}$ is parallel both with respect to ∇ and with respect to ${}^0\nabla$.

2.4. **The proof of Theorem 2 (1).** Let $\mathcal{M}$ be an affine surface which is flat with parallel non-vanishing torsion. Fix a point P of M. Since $R = 0$, we can choose a frame $\{X, Y\}$ for the tangent bundle so that $\nabla X = 0$ and $\nabla Y = 0$. Let $\{X^*, Y^*\}$ be the corresponding dual frame for the cotangent bundle; we then have dually that $\nabla X^* = \nabla Y^* = 0$. Expand $[X,Y] = aX + bY$. Then

$$T = -\tfrac{1}{2}(X^* \wedge Y^*) \otimes [X,Y] = -\tfrac{1}{2}(X^* \wedge Y^*) \otimes (a(x^1,x^2)X + b(x^1,x^2)Y)\,.$$

Since X, Y, X^*, and Y^* are parallel, the assumption that $\nabla T = 0$ implies a and b are constant. Since $T \neq 0$, we can make a linear change of frame to assume $a = 0$ and $b = -1$ and hence $[X,Y] = -Y$. Choose local coordinates (s,t) near P so $Y = \partial_t$. Expand $X = u(s,t)\partial_s + v(s,t)\partial_t$. The bracket relation $[X,Y] = -Y$ shows $\partial_t u = 0$ and $\partial_t v = 1$. Consequently, $X = u(s)\partial_s + (v(s) + t)\partial_t$. Perform a shear and set $\tilde{s} = s$ and $\tilde{t} = t + \varepsilon(s)$ where ε remains to be determined. Then

$$\begin{array}{ll} d\tilde{s} = ds, \quad d\tilde{t} = dt + \varepsilon'(s)ds, \quad \partial_{\tilde{s}} = \partial_s - \varepsilon'(s)\partial_t, & \partial_{\tilde{t}} = \partial_t, \\ X = u(\tilde{s})\partial_{\tilde{s}} + \{v(\tilde{s}) + \tilde{t} - \varepsilon(\tilde{s}) + u(\tilde{s})\varepsilon'(\tilde{s})\}\partial_{\tilde{t}}, & Y = \partial_{\tilde{t}}\,. \end{array}$$

Solve the ODE $v(\tilde{s}) - \varepsilon(\tilde{s}) + u(\tilde{s})\varepsilon'(\tilde{s}) = 0$ to express $X = u(\tilde{s})\partial_{\tilde{s}} + \tilde{t}\partial_{\tilde{t}}$. Set $x^2 = \tilde{t}$ and choose $x^1 = x^1(\tilde{s})$ so $x^1\partial_{x^1} = u(\tilde{s})\partial_{\tilde{s}}$. This expresses $X = x^1\partial_{x^1} + x^2\partial_{x^2}$. Since $\nabla\partial_{x^2} = 0$, we have $\Gamma_{12}{}^1 = \Gamma_{12}{}^2 = \Gamma_{22}{}^1 = \Gamma_{22}{}^2 = 0$. We compute:

$$\begin{array}{rcl} 0 &=& \nabla_{\partial_1}(x^1\partial_{x^1} + x^2\partial_{x^2}) = (1 + x^1\Gamma_{11}{}^1)\partial_{x^1} + x^1\Gamma_{11}{}^2\partial_{x^2}, \\ 0 &=& \nabla_{\partial_2}(x^1\partial_{x^1} + x^2\partial_{x^2}) = x^1\Gamma_{21}{}^1\partial_{x^1} + (x^1\Gamma_{21}{}^2 + 1)\partial_{x^2}. \end{array}$$

This defines a Type $\mathcal{B}$ structure where the only non-zero Christoffel symbols are $\Gamma_{11}{}^1 = \Gamma_{21}{}^2 = -(x^1)^{-1}$. On the other hand, the structure $\mathcal{M}_{1,0}$ has non-zero parallel torsion with vanishing Ricci tensor. Consequently this structure is isomorphic to $\mathcal{M}_{1,0}$. This establishes Theorem 2 (1). $\square$

2.5. **The proof of Theorem 2 (2).** Let $\mathcal{M}$ be a symmetric affine surface with parallel non-zero torsion which is not flat. By Lemma 9 and Lemma 10, the Ricci tensor ρ of $\mathcal{M}$ is symmetric and has rank 1. Define a smooth 1-dimensional distribution by setting $\ker(\rho) := \{\xi : \rho(\xi,\eta) = 0 \;\forall\; \eta\}$. Suppose $\xi \in \ker(\rho)$. Let η be an arbitrary tangent vector field. Since $\nabla\rho = 0$, we compute

$$0 = (\nabla\rho)(\xi,\eta) = d\rho(\xi,\eta) - \rho(\nabla\xi,\eta) - \rho(\xi,\nabla\eta) = 0 - \rho(\nabla\xi,\eta) - 0\,.$$

Consequently, the distribution $\ker(\rho)$ is invariant under ∇. Let $0 \neq \xi \in \ker(\rho)$. Choose local coordinates so $\xi = \partial_{x^1}$. We then have $\rho = \rho_{22}dx^2 \otimes dx^2$. Since $\ker(\rho)$ is invariant under ∇, we may expand $\nabla_{\partial_{x^1}}\partial_{x^1} = \omega_1\partial_{x^1}$ and $\nabla_{\partial_{x^2}}\partial_{x^1} = \omega_2\partial_{x^1}$. The commutator of covariant differentiation is given by curvature so

$$(\nabla_{\partial_{x^1}}\nabla_{\partial_{x^2}} - \nabla_{\partial_{x^2}}\nabla_{\partial_{x^1}})\partial_{x^1} = R_{121}{}^1\partial_{x^1} + R_{121}{}^2\partial_{x^2} = \rho_{21}\partial_{x^1} - \rho_{11}\partial_{x^2} = 0\,.$$

We may also compute directly

$$\begin{aligned} &(\nabla_{\partial_{x^1}}\nabla_{\partial_{x^2}} - \nabla_{\partial_{x^2}}\nabla_{\partial_{x^1}})\partial_{x^1} = \nabla_{\partial_{x^1}}\{\omega_2\partial_{x^1}\} - \nabla_{\partial_{x^2}}\{\omega_1\partial_{x^1}\} \\ = \;& (\omega_1\omega_2 + \partial_{x^1}\omega_2 - \omega_2\omega_1 - \partial_{x^2}\omega_1)\partial_{x^1}\,. \end{aligned}$$

This implies $\partial_{x^1}\omega_2 - \partial_{x^2}\omega_1 = 0$. Consequently, there exists a smooth function f so that $\omega_1 = \partial_{x^1}f$ and $\omega_2 = \partial_{x^2}f$. Let $\tilde\xi = e^{-f}\xi$. We then have $\nabla\tilde\xi = 0$ so $\tilde\xi$ is a parallel vector field on $\mathcal{M}$. We replace ξ by $\tilde\xi$ and obtain

$$\rho = \rho_{22}dx^2 \otimes dx^2, \quad \Gamma_{11}{}^1 = \Gamma_{11}{}^2 = \Gamma_{21}{}^1 = \Gamma_{21}{}^2 = 0\,.$$

Let $A_{ij}{}^k$ be the Christoffel symbols of ${}^0\mathcal{M}$. We adopt the notation of Equation (2) and obtain

$$\begin{array}{llll} A_{11}{}^1 = 0, & A_{11}{}^2 = 0, & A_{12}{}^1 = T^1, & A_{12}{}^2 = T^2, \\ A_{22}{}^1 = \Gamma_{22}{}^1, & A_{22}{}^2 = \Gamma_{22}{}^2\,. \end{array}$$

A direct computation shows

$$\begin{array}{l} \rho_{11} = 0, \quad \rho_{21} = 0, \quad \rho_{12} = 2\partial_{x^2}A_{12}{}^2 - \partial_{x^1}A_{22}{}^2, \\ \rho_{22} = -2A_{12}{}^2A_{22}{}^1 + 2A_{12}{}^1A_{22}{}^2 - 2\partial_{x^2}A_{12}{}^1 + \partial_{x^1}A_{22}{}^1\,. \end{array} \tag{3}$$

We express the equation $\nabla T = \begin{pmatrix} T_{1;1} & T_{1;2} \\ T_{2;1} & T_{2;2} \end{pmatrix}$ in terms of the A variables:

$$0 = \nabla T = \begin{pmatrix} \partial_{x^1}(A_{12}{}^1) & A_{12}{}^2A_{22}{}^1 - A_{12}{}^1A_{22}{}^2 + \partial_{x^2}(A_{12}{}^1) \\ \partial_{x^1}(A_{12}{}^2) & \partial_{x^2}(A_{12}{}^2) \end{pmatrix}.$$

Consequently, $\partial_{x^1}(A_{12}{}^1) = 0$, $\partial_{x^1}(A_{12}{}^2) = 0$, and $\partial_{x^2}(A_{12}{}^2) = 0$. By Lemma 9, $\rho_{12} = \rho_{21} = 0$. Since $\partial_{x^2}(A_{12}{}^2) = 0$, Equation (3) implies $\partial_{x^1}A_{22}{}^2 = 0$. Thus

$$\begin{array}{l} A_{12}{}^1(x^1,x^2) = a_{12}{}^1(x^2), \quad A_{12}{}^2(x^1,x^2) = c_{12}{}^2 \in \mathbb{R}, \\ A_{22}{}^2(x^1,x^2) = a_{22}{}^2(x^2)\,. \end{array}$$

Since $\mathrm{Rank}\{\rho\} = 1$, $0 \neq \rho_{22}$. We compute $\rho_{22} + 2T_{1;2} = \partial_{x^1}(A_{22}{}^1)$. Since $T_{1;2} = 0$, we may conclude $\partial_{x^1}(A_{22}{}^1) \neq 0$. We have

$$0 = T_{1;2} = c_{12}{}^2A_{22}{}^1(x^1,x^2) - a_{12}{}^1(x^2)a_{22}{}^2(x^2) + \partial_{x^2}(a_{12}{}^1(x^2))\,.$$

Since $A_{22}{}^1$ exhibits non-trivial dependence on x^1, we have that $c_{12}{}^2 = 0$. Thus

$$A_{12}{}^1(x^1,x^2) = a_{12}{}^1(x^2), \quad A_{12}{}^2(x^1,x^2) = 0, \quad A_{22}{}^2(x^1,x^2) = a_{22}{}^2(x^2)\,.$$

We may then compute $0 = T_{1;2} = -a_{12}{}^1 a_{22}{}^2 + (a_{12}{}^1)'$. Let $u = a_{12}{}^1(0)$ and let $a(x^2)$ be a smooth function so $a(0) = 0$ and $a'(x^2) = a_{22}{}^2(x^2)$. We can then solve the ODE $0 = -a_{12}{}^1 a_{22}{}^2 + (a_{12}{}^1)'$ to see:

$$A_{12}{}^1(x^1, x^2) = ue^{a(x^2)}, \quad A_{12}{}^2(x^1, x^2) = 0, \quad A_{22}{}^2(x^1, x^2) = a'(x^2)\,.$$

There are only two non-trivial equations remaining to ensure $\nabla\rho = 0$:

$$\begin{aligned} 0 &= (\partial_{x^1})^2 A_{22}{}^1(x^1, x^2), \\ 0 &= -2a'(x^2)\partial_{x^1} A_{22}{}^1(x^1, x^2) + (\partial_{x^1}\partial_{x^2}) A_{22}{}^1(x^1, x^2)\,. \end{aligned}$$

This implies $A_{22}{}^1(x^1, x^2) = b(x^2) + x^1 ve^{2a(x^2)}$ for some constant $v \in \mathbb{R}$. We then compute $\rho = ve^{2a(x^2)} dx^2 \otimes dx^2$. Since $\mathcal{M}$ is not flat, $v \neq 0$. We can renormalize x^2 so $\rho_{22} = v dx^2 \otimes dx^2$ for $v \neq 0$. The non-zero Christoffel symbols are then (renaming $e^{-2a(x^2)} b(x^2) \to b(x^2)$)

$$\Gamma_{12}{}^1 = 2u \text{ and } \Gamma_{22}{}^1 = b(x^2) + vx^1\,.$$

We perform a shear and set $y^1 = x^1 + \alpha(x^2)$ and $y^2 = x^2$. We then have $\partial_{y^1} = \partial_{x^1}$ and $\partial_{y^2} = \partial_{x^2} - \alpha'(x^2)\partial_{x^1}$. Consequently

$$\begin{aligned} &\nabla_{\partial_{y^1}} \partial_{y^1} = \nabla_{\partial_{x^1}} \partial_{x^1} = 0, \\ &\nabla_{\partial_{y^2}} \partial_{y^1} = \nabla_{\partial_{x^2}} \partial_{x^1} - \alpha' \nabla_{\partial_{x^1}} \partial_{x^1} = 0, \\ &\nabla_{\partial_{y^1}} \partial_{y^2} = \nabla_{\partial_{x^1}} \partial_{x^2} - \alpha' \nabla_{\partial_{x^1}} \partial_{x^1} = 2u\partial_{x^1} = 2u\partial_{y^1}, \\ &\nabla_{\partial_{y^2}} \partial_{y^2} = \nabla_{(\partial_{x^2} - \alpha'\partial_{x^1})}(\partial_{x^2} - \alpha'\partial_{x^1}) \\ &\quad = (b(x^2) + vx^1)\partial_{x^1} - 2u\alpha'\partial_{x^1} - \alpha''\partial_{x^1}\,. \end{aligned}$$

Choose κ so that $(x^1 + \kappa) > 0$ in a neighborhood of the point in question. We solve the ODE $b(x^2) - 2u\alpha'(x^2) - \alpha''(x^2) = v\kappa$ to ensure the only non-zero Christoffel symbols are $\Gamma_{12}{}^1 = 2u$ and $\Gamma_{22}{}^1 = v(x^1 + \kappa)$. We make the change of variables $\partial_{z^1} = (x^1 + \kappa)\partial_{y^1}$ and $\partial_{z^2} = \partial_{y^2}$. We compute

$$\begin{aligned} &\nabla_{\partial_{z^1}} \partial_{z^1} = (x^1 + \kappa)\nabla_{\partial_{y^1}}((x^1 + \kappa)\partial_{y^1}) = (x^1 + \kappa)\partial_{y^1} = \partial_{z^1}, \\ &\nabla_{\partial_{z^2}} \partial_{z^1} = \nabla_{\partial_{y^2}}((x^1 + \kappa)\partial_{y^1}) = 0, \\ &\nabla_{\partial_{z^1}} \partial_{z^2} = (x^1 + \kappa)\nabla_{\partial_{y^1}} \partial_{y^2} = 2u(x^1 + \kappa)\partial_{y^1} = 2u\partial_{z^1}, \\ &\nabla_{\partial_{z^2}} \partial_{z^2} = \nabla_{\partial_{y^2}} \partial_{y^2} = v(x^1 + \kappa)\partial_{y^1} = v\partial_{z^1}\,. \end{aligned}$$

The non-zero Christoffel symbols now take the form

$$\Gamma_{11}{}^1 = 1, \quad \Gamma_{12}{}^1 = 2u, \quad \Gamma_{22}{}^1 = v\,.$$

We can rescale x^2 to assume $v = \pm 1$. We must have $u \neq 0$ to ensure the torsion is non-zero. Replacing x^2 by $-x^2$ replaces u by $-u$. We may therefore assume $u > 0$ and obtain the structures which are given in Theorem 2 (2). □

2.6. **Distinguishing the structures.** The structures $\mathcal{M}_{u,v}$ are all Type $\mathcal{A}$ structures; they are invariant under the translation group and are thus homogeneous geometries. The signature of the Ricci tensor determines the parameter v. We suppose $v = \pm 1$ as there is only one model in Assertion (1). Let ${}^0\mathcal{M}_{u,v}$ be the associated torsion free geometry; ${}^0\Gamma_{11}{}^1 = 1$, ${}^0\Gamma_{12}{}^1 = {}^0\Gamma_{21}{}^1 = u$, and ${}^0\Gamma_{22}{}^1 = v$. We have $\rho_{{}^0\mathcal{M}_{u,v}} = v(v - u^2)\rho_{\mathcal{M}_{u,v}}$. Since v is determined by the signature of $\rho_{\mathcal{M}_{u,v}}$, u^2 is an invariant of the affine structure in this context. Since $u > 0$, u is determined and the structures are distinct affine structures. $\square$

3. The proof of Theorem 4

Let $\mathcal{M}$ be a Type $\mathcal{A}$ symmetric surface with non-parallel torsion tensor. By making a suitable change of basis, we may assume $T = (dx^1 \wedge dx^2) \otimes \partial_{x^2}$. This normalizes the linear changes of coordinates up to the action of the $ax+b$ subgroup of $\mathrm{GL}(2,\mathbb{R})$. Let $A_{ij}{}^k := \frac{1}{2}(\Gamma_{ij}{}^k + \Gamma_{ji}{}^k)$ be the Christoffel symbols of ${}^0\mathcal{M}$. The following is a useful result which follows by a direct computation.

Lemma 11. *Let $(y^1, y^2) = (x^1, a^{-1}(x^2 - bx^1))$ be a change of variables which defines a shear. Then*

$$\begin{aligned}
&dy^1 = dx^1, \quad dy^2 = a^{-1}(dx^2 - bdx^1), \quad \partial_{y^1} = \partial_{x^1} + b\partial_{x^2}, \quad \partial_{y^2} = a\partial_{x^2},\\
&{}^yA_{11}{}^1 = {}^xA_{11}{}^1 + 2b\,{}^xA_{12}{}^1 + b^2\,{}^xA_{22}{}^1,\\
&{}^yA_{11}{}^2 = \tfrac{1}{a}\{{}^xA_{11}{}^2 + b(2\,{}^xA_{12}{}^2 - {}^xA_{11}{}^1) + b^2({}^xA_{22}{}^2 - 2\,{}^xA_{12}{}^1) - b^3\,{}^xA_{22}{}^1\},\\
&{}^yA_{12}{}^1 = a({}^xA_{12}{}^1 + b\,{}^xA_{22}{}^1),\\
&{}^yA_{12}{}^2 = {}^xA_{12}{}^2 + b\,{}^xA_{22}{}^2 - b({}^xA_{12}{}^1 + b\,{}^xA_{22}{}^1),\\
&{}^yA_{22}{}^1 = a^2\,{}^xA_{22}{}^1,\\
&{}^yA_{22}{}^2 = a({}^xA_{22}{}^2 - b\,{}^xA_{22}{}^1).
\end{aligned}$$

By Lemma 11, if $\Gamma_{22}{}^1 \neq 0$, we can always fix the gauge so $\Gamma_{22}{}^1 = \pm 1$ and $\Gamma_{22}{}^2 = 0$. If $\Gamma_{22}{}^1 = 0$, we can rescale x^2 to assume $\Gamma_{22}{}^2 \in \{0, 1\}$ but the gauge is not yet fixed. This gives rise to three cases. We will use a similar gauge normalization in the Type $\mathcal{B}$ setting.

Case 1: $\Gamma_{22}{}^1 = 0$ and $\Gamma_{22}{}^2 \neq 0$. Rescale x^2 to assume $\Gamma_{22}{}^2 = 1$ and set $a = 1$ in Lemma 11. We compute that $0 = \rho_{22;2} = 2(A_{12}{}^1 - 1)A_{12}{}^1$. There are two subcases:

Case 1.1: $A_{12}{}^1 = 0$. The only remaining non-zero component of $\nabla\rho$ is given by $\rho_{11;1} = -2A_{11}{}^1[1 + A_{11}{}^2 + A_{11}{}^1(-1 + A_{12}{}^2) - (A_{12}{}^2)^2]$. There are two subpossibilities:

Case 1.1.1: $A_{11}{}^1 = 0$. We have $\nabla T = 0$ so we reject this case.

Case 1.1.2: $A_{11}{}^1 \neq 0$ and $1 + A_{11}{}^2 + A_{11}{}^1(-1 + A_{12}{}^2) - (A_{12}{}^2)^2 = 0$. This fixes $A_{11}{}^2$. Choose b in Lemma 11 to ensure $A_{12}{}^2 = 0$; this gives Assertion (1).

Case 1.2: $A_{12}{}^1 = 1$. We have $\rho_{12;2} = 2(1 - A_{12}{}^2)$. We set $A_{12}{}^2 = 1$ and obtain $\nabla\rho = 0$. Choose b in Lemma 11 to ensure $A_{11}{}^1 = 0$; this gives Assertion (2).

Case 2: $\Gamma_{22}{}^1 = 0$ and $\Gamma_{22}{}^2 = 0$. Since $\rho_{22;1} = 4(A_{12}{}^1)^2$, we have $A_{12}{}^1 = 0$. $\rho_{11;1} = -2A_{11}{}^1(A_{11}{}^1 - A_{12}{}^2 - 1)(A_{12}{}^2 - 1)$ is the only non-zero component of ∇R. Furthermore, the only non-zero component of ∇T is $T_{2;1} = -A_{11}{}^1$ so $A_{11}{}^1 \neq 0$.

Case 2.1: $A_{12}{}^2 = A_{11}{}^1 - 1$. If $A_{11}{}^1 \neq 2$, we set $a = 1$ and choose b in Lemma 11 so $A_{11}{}^2 = 0$. Rescaling x^2 then plays no role. This normalizes the gauge and we obtain Assertion (3). If on the other hand $A_{11}{}^1 = 2$ and $A_{11}{}^2 \neq 0$, then we can rescale x^2 to obtain Assertion (4) and again we have fixed the gauge as the parameter b plays no role. Finally, if $A_{11}{}^1 = 2$ and $A_{11}{}^2 = 0$, we again obtain Assertion (3).

Case 2.2: $A_{12}{}^2 \neq A_{11}{}^1 - 1$ and $A_{12}{}^2 = 1$. Thus $A_{11}{}^1 \neq 2$ and we can choose the parameter b in Lemma 11 so that $A_{11}{}^2 = 0$. We obtain Assertion (5).

Case 3: $A_{22}{}^1 \neq 0$. We use Lemma 11 to make a gauge transformation and fix the gauge so $A_{22}{}^1 = \varepsilon = \pm 1$ and $A_{12}{}^1 = 0$. We set $A_{11}{}^1 = \omega$ and $A_{22}{}^2 = \eta$ and compute $0 = \rho_{12;2} - \rho_{22;1} = -4\varepsilon(1 + A_{12}{}^2 - \omega)$. We set $A_{12}{}^2 = \omega - 1$ and compute $0 = \rho_{22;2} = 2A_{11}{}^2$. We then have $\nabla\rho = 0$ and obtain Assertion (6). $\square$

4. The proof of Theorem 6

The essential technical point in performing the analysis is to fix the gauge; otherwise the problem is combinatorially intractable. The torsion tensor plays an essential role in this regard. For Type $\mathcal{A}$ surfaces we used the action of $\mathrm{GL}(2,\mathbb{R})$ to set $T=(dx^1\wedge dx^2)\otimes\partial_{x^2}$. The remaining gauge freedom is then governed by the $ax+b$ group sending $(x^1,x^2)\to(x^2,ax^2+bx^1)$. The natural gauge group in the Type $\mathcal{B}$ setting is again the $ax+b$ group with the same action on the coordinates. We denote $\tilde{A}_{ij}{}^k$ the Christoffel symbols of ${}^0\mathcal{M}$ evaluated at $x_1=1$; $\tilde{A}_{ij}{}^k=x^1(\Gamma_{ij}{}^k+\Gamma_{ji}{}^k)/2$.

Let $\mathcal{M}$ be a symmetric affine surface of Type $\mathcal{B}$. The Ricci tensor is symmetric. This yields the relation $\tilde{A}_{12}{}^1=\tilde{T}^1-\tilde{A}_{22}{}^2$. This is analogous to using the general linear group in the Type $\mathcal{A}$ setting to fix the gauge. The $ax+b$ group now acts and we have the same 3 cases as in Lemma 11 in the Type $\mathcal{A}$ setting. Note that $\tilde{T}^2$ is still a free parameter.

Case 1: $\tilde{A}_{22}{}^1=0$ and $\tilde{A}_{22}{}^2\neq0$. We rescale x^2 to assume $\tilde{A}_{22}{}^2=1$. We then have $\tilde{\rho}_{22;2}=-4(2\tilde{T}^1-1)$ so $\tilde{T}^1=\frac12$ and $\tilde{A}_{12}{}^1=\tilde{T}^1-\tilde{A}_{22}{}^2=-\frac12$. Since $\tilde{\rho}_{12;1}=-2-\tilde{A}_{11}{}^1-\tilde{A}_{12}{}^2-\tilde{T}^2$, we obtain $\tilde{A}_{11}{}^1=-2-\tilde{A}_{12}{}^2-\tilde{T}^2$. We finally compute $\tilde{\rho}_{11;2}=4(\tilde{A}_{11}{}^2-\tilde{A}_{12}{}^2-(\tilde{A}_{12}{}^2)^2+\tilde{T}^2+(\tilde{T}^2)^2)$, which leads to

$$\tilde{A}_{11}{}^2=\tilde{A}_{12}{}^2+(\tilde{A}_{12}{}^2)^2-\tilde{T}^2-(\tilde{T}^2)^2\,.$$

We now have $\nabla\rho=0$. Since $\tilde{A}_{22}{}^2\neq0$, we can make a shear to set $\tilde{A}_{12}{}^2+\tilde{T}^2=0$. We thus obtain Assertion (1).

Case 2: $\tilde{A}_{22}{}^1=0$ and $\tilde{A}_{22}{}^2=0$. We have $\tilde{\nabla}\rho_{22;1}=-8(\tilde{T}^1)^2(\tilde{A}_{12}{}^2-\tilde{T}^2)$. This gives rise to 2 cases.

Case 2.1: $\tilde{A}_{12}{}^2\neq\tilde{T}^2$. Thus $\tilde{T}^1=0$ so $\tilde{A}_{12}{}^1=0$. The only remaining non-zero component of $\nabla\rho$ is given by $\tilde{\rho}_{11;1}=2(1+\tilde{A}_{11}{}^1)(\tilde{A}_{12}{}^2-\tilde{T}^2)(-1-\tilde{A}_{11}{}^1+\tilde{A}_{12}{}^2+\tilde{T}^2)$. If $\tilde{A}_{11}{}^1=-1$ we have $\nabla T=0$. We take $\tilde{A}_{12}{}^2=1+\tilde{A}_{11}{}^1-\tilde{T}^2$. This ensures $\nabla\rho=0$. We now fix the gauge.

Case 2.1.1: $\tilde{A}_{11}{}^2=0$. We obtain Assertion (2).

Case 2.1.2: $\tilde{A}_{11}{}^1\neq2\tilde{T}^2-2$. We have $\tilde{A}_{22}{}^2=\tilde{A}_{12}{}^1=\tilde{A}_{22}{}^1=0$. Furthermore $2\tilde{A}_{12}{}^2-\tilde{A}_{11}{}^1=2+\tilde{A}_{11}{}^1-2\tilde{T}^2\neq0$. Thus we can use Lemma 11 to make a gauge transform to ensure $\tilde{A}_{11}{}^2=0$ which reduces to Case 2.1.1.

Case 2.1.3: $\tilde{A}_{11}{}^2\neq0$ and $\tilde{A}_{11}{}^1=2\tilde{T}^2-2$. Rescale x^2 to ensure $\tilde{A}_{11}{}^2=1$. The shear parameter b in Lemma 11 plays no role. We obtain Assertion (3).

Case 2.2: $\tilde{A}_{12}{}^2=\tilde{T}^2$. We then have $\nabla\rho=0$. We fix the gauge.

Case 2.2.1: $\tilde{T}^1\neq0$. We have $\tilde{A}_{12}{}^1=\tilde{T}^1-\tilde{A}_{22}{}^2=\tilde{T}^1\neq0$. Since $\tilde{A}_{22}{}^1=0$, we can choose b in Lemma 11 to assume $\tilde{A}_{11}{}^1=0$. We can then rescale x^2 to assume $A_{12}{}^1=\frac12$ and obtain Assertion (4).

Case 2.2.2: $\tilde{T}^1=0$ and $\tilde{A}_{11}{}^2=0$. We obtain Assertion (5); the remaining gauge freedom plays no role.

Case 2.2.3: $\tilde{T}^1=0$ and $\tilde{A}_{11}{}^1\neq2\tilde{T}^2$. We have $\tilde{A}_{22}{}^1=0$, $\tilde{A}_{12}{}^1=0$, $\tilde{A}_{22}{}^2=0$, and $2\tilde{A}_{12}{}^2-\tilde{A}_{11}{}^1\neq0$. We can therefore apply Lemma 11 to choose b so $\tilde{A}_{11}{}^2=0$ and obtain Case 2.2.2.

Case 2.2.4: $\tilde{T}^1=0$, $\tilde{A}_{11}{}^1=2\tilde{T}^2$ and $\tilde{A}_{11}{}^2\neq0$. We rescale x^2 to obtain Assertion (6).

Case 3: $\tilde{A}_{22}{}^1\neq0$. We may rescale x^2 and then use Lemma 11 to assume $\tilde{A}_{22}{}^1=\varepsilon$ and $\tilde{A}_{22}{}^2=0$ for $\varepsilon=\pm1$. We have $0=\tilde{\rho}_{12;2}=2\varepsilon(\tilde{A}_{12}{}^2-\tilde{T}^2)(\tilde{T}^2+\tilde{A}_{12}{}^2-\tilde{A}_{11}{}^1)$.

Case 3.1: $\tilde{A}_{12}{}^2 = \tilde{T}^2$. We compute $\tilde{\rho}_{22;2} = 2\tilde{A}_{11}{}^2$. We set $\tilde{A}_{11}{}^2 = 0$; the only remaining equation is $\tilde{\rho}_{22;1} = -2\varepsilon(-1 + \tilde{A}_{11}{}^1 - 2\tilde{T}^2)(1 + 2\tilde{T}^2)$.

Case 3.1.1: $\tilde{T}^2 = -\frac{1}{2}$. We obtain Assertion (7)

Case 3.1.2: $\tilde{T}^2 \neq -\frac{1}{2}$ and $\tilde{A}_{11}{}^1 = 1 + 2\tilde{T}^2$. We obtain Assertion (8).

Case 3.2: $\tilde{A}_{12}{}^2 \neq \tilde{T}^2$ and $\tilde{A}_{12}{}^2 = \tilde{A}_{11}{}^1 - \tilde{T}^2$. We obtain

$$\tilde{\rho}_{22;2} = 2\varepsilon(-2\tilde{A}_{11}{}^1\tilde{T}^1 + 4\tilde{T}^1\tilde{T}^2 + \varepsilon\tilde{A}_{11}{}^2)\,.$$

This determines $\tilde{A}_{11}{}^2$. We have $\tilde{\rho}_{22;1} = 2\varepsilon(1 + \tilde{A}_{11}{}^1)$ and hence $\tilde{A}_{11}{}^1 = -1$. To ensure that $\tilde{A}_{12}{}^2 \neq \tilde{T}^2$, we require that $\tilde{T}^2 \neq -\frac{1}{2}$. We obtain Assertion (9). □

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