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Two infinite families of critical clique–Helly graphs

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A graph is *clique–Helly* if every family of pairwise intersecting (maximal) cliques has non-empty total intersection. Dourado, Protti and Szwarcfiter conjectured that every clique–Helly graph contains a vertex whose removal maintains it as a clique–Helly graph. We present here two infinite families of counterexamples to this conjecture.

1. Introduction

A set family $\mathcal{F}$ satisfies the *Helly property* if the intersection of all the members of every pairwise intersecting subfamily of $\mathcal{F}$ is non-empty. This property, originated in the famous work of Eduard Helly on convex sets in the Euclidean space, has been widely studied in diverse areas of theoretical and applied mathematics such as extremal hypergraph theory, logic, optimization, theoretical computer science, computational biology, databases, image processing and graph theory. A few surveys have been written on the Helly property, see for instance [2,4–6,8].

From the computational and algorithmic point of view, the relevance of the Helly property has been highlighted in the survey [5]. In the section *Proposed Problems* of that work, the authors posed the following open question:

Conjecture 1.1 (*Dourado, Protti and Szwarcfiter [5]*)**.** *Every clique–Helly graph contains a vertex whose removal maintains it as a clique–Helly graph.*

In this work, we prove the conjecture is false: we will exhibit two infinite families of clique–Helly graphs G such that $G - x$ (the graph obtained from G by removing vertex x) is not clique–Helly for every vertex x of G. Moreover, the family in Section 3 contains only *self-clique* graphs and the family in Section 4 contains only 2-*self-clique* graphs. It is a classic result that any clique–Helly graph without dominated vertices is either self-clique or 2-self-clique (Escalante 1973, [7]), and any counterexample to the conjecture cannot contain dominated vertices (since the removal of a dominated vertex from a clique–Helly graph, preserves clique–Hellyness).

A preliminary version of this work appeared in [1] where we showed that one counterexample to the conjecture exists.

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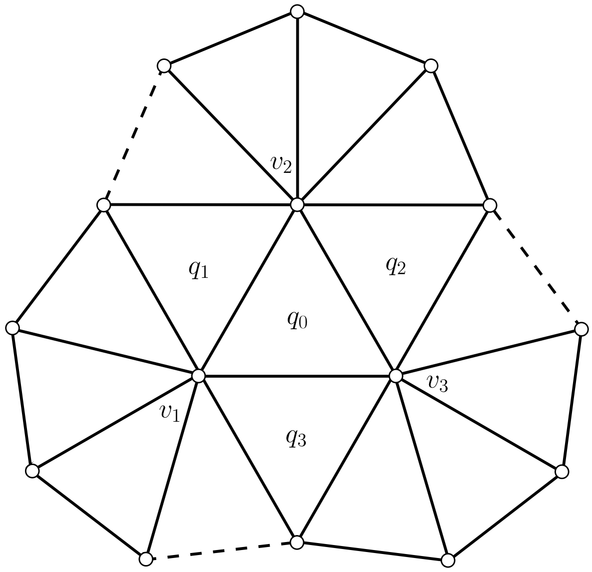

Fig. 1. A partial drawing of a locally C_d graph.

2. Preliminaries

Our graphs are finite and simple. We identify induced subgraphs with their vertex set, in particular we usually write $x \in G$ instead of $x \in V(G)$. Also, for a vertex $x \in G$, we write $G - x$ instead of $G - \{x\}$. The *open* and the *closed neighborhood* of a vertex $x \in G$ are denoted by $N(x)$ and $N[x]$ respectively. The *degree* of x is the cardinality of $N(x)$. We write $x \simeq y$ when x is *adjacent-or-equal* to y.

The *complete graph* on n vertices is denoted by K_n. A *clique* is a maximal complete subgraph. Let $\mathcal{C}(G)$ be the family of all cliques of G. When $\mathcal{C}(G)$ satisfies the Helly property, we say that G is a *clique–Helly graph*.

Definition 2.1. A graph G is *critical clique–Helly* if G is clique–Helly and $G - x$ is not clique–Helly for every $x \in G$.

Notice that in terms of the previous definition the conjecture of Dourado, Protti and Szwarcfiter postulates that there are no critical clique–Helly graphs.

The *clique graph* $K(G)$ of G is the intersection graph of $\mathcal{C}(G)$: the vertices of $K(G)$ are the cliques of G and two different cliques of G are adjacent in $K(G)$ if and only if they have non-empty intersection. The second clique graph of G is $K^2(G) = K(K(G))$. Then the vertices of $K^2(G)$ are the cliques of $K(G)$ which are said to be cliques of cliques of G. Given a vertex v of a graph G, the *star* of v is the set of all the cliques of G which contain v, i.e. $v^* = \{q \in V(K(G)) : v \in q\}$. Stars of G are not always vertices of $K^2(G)$: They are always complete subgraphs of $K(G)$, but not always maximal. Any clique of cliques of G which is not a star will be said to be a *necktie*. An example of a necktie is $Q = \{q_0, q_1, q_2, q_3\}$, where q_i is the clique formed by the vertices of the corresponding triangles in Fig. 1. We say that G is *self-clique* if $K(G) \cong G$, and that it is 2-*self-clique* whenever $K^2(G) \cong G \ncong K(G)$.

A *cycle* in G is a sequence of at least three distinct vertices $v_1, v_2, \ldots, v_d$ of G such that two of them are adjacent in G if and only if they are consecutive in the sequence or they are v_1 and v_d. The positive integer d is the *length* of the cycle. The cycle of length d is denoted by C_d. A graph G is *locally cyclic* if each open neighborhood in G induces a cycle, and G is a *locally C_d graph* if $N(v)$ induces a C_d for every $v \in G$. The *girth* $g(G)$ of G is the length of a shortest cycle in G (if G has no cycles, then $g(G) = \infty$). The *local girth of G at a vertex* $v \in G$, $lg_v(G)$, is the girth of the subgraph induced by the open neighborhood of v in G, i.e. $lg_v(G) = g(N(v))$. The minimum of these local girths is denoted by $lg(G)$ and is called the *local girth of G*, i.e.

$$lg(G) = \min\{lg_v(G) : v \in G\}.$$

Theorem 2.2 ([10]). *If the local girth of the graph G is greater than 6 (i.e.* $lg(G) \geq 7$*) then* $K(G)$ *is clique–Helly.*

For $d \geq 7$ and G a locally C_d graph, a detailed analysis of the cliques and cliques of cliques of G was done in [10]. We transcribe here the most relevant properties for our purposes (which can all be verified straightforwardly):

Remark 2.3 (*[10], Section 3.1*). If $d \geq 7$, and G is a locally C_d graph, then:

1. All the cliques of G are triangles.
2. For every vertex $v \in G$, v^* is a clique of cliques of G.
3. For every triangle $T = \{v_1, v_2, v_3\}$ of G, there is a necktie $Q_T = \{q \in K(G) : |q \cap T| \geq 2\}$, which is a clique of cliques of G and it is always of the form $Q_T = \{q_0, q_1, q_2, q_3\}$ (see Fig. 1).
4. Every clique of cliques of G is either a star v^* or a necktie Q_T.
5. In $K^2(G)$, $v_1^* \simeq v_2^*$ if and only if $v_1 \simeq v_2$ in G.
6. In $K^2(G)$, $Q_T \simeq Q_{T'}$ if and only if either T and T' share an edge, or they share a vertex and there is an edge joining a vertex of $T \setminus T'$ with a vertex of $T' \setminus T$.
7. In $K^2(G)$, $v^* \simeq Q_T$ if and only if $v \in \cup Q_T = \cup_{i=0}^{3} q_i$ where $Q_T = \{q_0, q_1, q_2, q_3\}$.

Brown and Connelly proved in [3] that there exists at least one finite locally C_d graph for each $d \geq 3$. Larrión, Neumann–Lara and Pizaña obtained the next theorem extending the result of Brown and Connelly for $d \geq 7$.

Theorem 2.4 (*[10]*). *Let d be any integer greater than or equal to 7. Then there are infinitely many non-isomorphic locally C_d graphs.*

In the *strong product* of graphs, $G \boxtimes H$, two vertices $(g_1, h_1), (g_2, h_2) \in V(G \boxtimes H) = V(G) \times V(H)$ are adjacent-or-equal whenever $g_1 \simeq g_2$ in G and $h_1 \simeq h_2$ in H. We refer the reader to [9] for the known results on the strong product. A classic result on clique graphs (Neumann–Lara, 1978) states that the clique operator distributes over the strong product of graphs:

Theorem 2.5 (*[11]*). *$K(G \boxtimes H) \cong K(G) \boxtimes K(H)$.*

3. The self-clique family

Let $r, s, t \geq 4$. Take $G(r, s, t) = C_r \boxtimes C_s \boxtimes C_t$. We claim that these graphs are the sought self-clique counterexamples:

Theorem 3.1. *All the graphs $G(r, s, t)$ are self-clique critical clique–Helly graphs.*

Proof. Certainly they are all self-clique since, by Theorem 2.5, we have $K(G(r, s, t)) = K(C_r \boxtimes C_s \boxtimes C_t) \cong K(C_r) \boxtimes K(C_s) \boxtimes K(C_t) \cong C_r \boxtimes C_s \boxtimes C_t = G(r, s, t)$.

Evidently, every C_d is clique–Helly. Let us see that the strong product of clique–Helly graphs is again clique–Helly:

Suppose X and Y are clique–Helly graphs. The cliques of $X \boxtimes Y$ are of the form $q = q_1 \times q_2$ with $q_1 \in \mathcal{C}(X)$ and $q_2 \in \mathcal{C}(Y)$. Now assume you have a pairwise intersecting family of cliques of $X \boxtimes Y$, namely: $q^1 = q_1^1 \times q_2^1, q^2 = q_1^2 \times q_2^2, \ldots, q^m = q_1^m \times q_2^m$. It follows that $q_1^1, q_1^2, \ldots, q_1^m$ are pairwise intersecting cliques in X and $q_2^1, q_2^2, \ldots, q_2^m$ are pairwise intersecting cliques in Y. Since X and Y are clique–Helly, there are some vertices $x \in \cap_{i=1}^m q_1^i \subseteq X$ and $y \in \cap_{i=1}^m q_2^i \subseteq Y$. Clearly, (x, y) belongs to the total intersection of $q^1, q^2, \ldots, q^m$ and therefore $X \boxtimes Y$ is clique–Helly.

It follows that $G(r, s, t) = C_r \boxtimes C_s \boxtimes C_t$ is clique–Helly.

Now we will show that $G(r, s, t) - x$ is not clique–Helly for every vertex x. Since $G(r, s, t)$ is clearly vertex-transitive, it is sufficient for us to prove it for any particular vertex x. Assume the vertices of each cycle C_d are numbered as $\{1, 2, \ldots, d\}$ for $d \in \{r, s, t\}$ and take $x = (2, 2, 2) \in G(r, s, t)$. Define $H = G(r, s, t) - x$. Now in $G(r, s, t)$ take the cliques $q_1 = \{2, 3\} \times \{1, 2\} \times \{1, 2\}$, $q_2 = \{1, 2\} \times \{2, 3\} \times \{1, 2\}$, $q_3 = \{1, 2\} \times \{1, 2\} \times \{2, 3\}$. The corresponding cliques in H, $\bar{q}_i = q_i \cap H = q_i - (2, 2, 2)$ for $i = 1, 2, 3$, are pairwise intersecting (each $q_i \cap q_j$ contains one of the following vertices: $(1, 2, 2)$, $(2, 1, 2)$ or $(2, 2, 1)$) but have no common total intersection: the only vertex in $q_1 \cap q_2 \cap q_3$ in $G(r, s, t)$ is $(2, 2, 2)$ which is not present in H. It follows that H is not clique–Helly. □

4. The 2-self-clique family

Theorem 4.1. *Let G be a locally C_d graph with $d \geq 7$. Then $K(G)$ is a critical clique–Helly graph.*

Proof. Since G is locally C_d graph, the local girth of G equals $d \geq 7$, therefore, by Theorem 2.2, $K(G)$ is clique–Helly.

Let q_0 be any vertex of $K(G)$ (a clique of G). We will prove that $K(G) - q_0$ is not clique–Helly. By Remark 2.3(1) every clique in G is a triangle. Without loss of generality assume $q_0 = \{v_1, v_2, v_3\}$ as in Fig. 1.

Consider the following cliques of $K(G) - q_0$: $v_1^* - q_0$, $v_2^* - q_0$ and $v_3^* - q_0$. We claim these three cliques are pairwise intersecting but the intersection of all three of them is empty: indeed, the vertices of $K(G) - q_0$ corresponding to the cliques q_1, q_2 and q_3 of G (as in Fig. 1) belong to $v_1^* \cap v_2^*$, $v_2^* \cap v_3^*$ and $v_3^* \cap v_0^*$, respectively. Finally, assume in order to obtain a contradiction that a vertex q of $K(G) - q_0$ belongs to $v_1^* \cap v_2^* \cap v_3^*$, then, by definition of these sets, q is a clique of G such that $v_i \in q$ for $i \in \{1, 2, 3\}$. Thus, $q = \{v_1, v_2, v_3\} = q_0$ which contradicts our assumption that q is a vertex of $K(G) - q_0$. Hence $K(G) - q_0$ is not clique–Helly. □

Theorem 4.2. *Let G_1 and G_2 be two non-isomorphic locally C_d graphs. Then $K(G_1)$ and $K(G_2)$ are also non-isomorphic.*

Proof. Assume, by way of contradiction, that $K(G_1) \cong K(G_2)$. Then we also have $K^2(G_1) \cong K^2(G_2)$. Now, by Remark 2.3(4), the vertices of $K^2(G_1)$ (and those of $K^2(G_2)$) are either stars or neckties. By Remark 2.3(5–7), stars have degree $3d$ (d of the neighbors are stars and $2d$ are neckties) and neckties have degree 15 (6 stars and 9 neckties), see Fig. 1. Hence, any isomorphism $\phi : K^2(G_1) \to K^2(G_2)$ must map stars onto stars bijectively. Let S_1 and S_2 be the subgraphs of $K^2(G_1)$ and $K^2(G_2)$ induced by the stars of G_1 and G_2 (respectively). Hence the restriction of ϕ, $\phi' : S_1 \to S_2$, is still an isomorphism. But, by Remark 2.3(5), $G_1 \cong S_1$ and $G_2 \cong S_2$. It follows that $G_1 \cong S_1 \cong S_2 \cong G_2$, contrary to our hypothesis. □

Theorem 4.3. *There are infinitely many 2-self-clique critical clique–Helly graphs.*

Proof. Let G be a locally C_d graph and $H = K(G)$. It follows by Theorem 4.1, that H is a critical clique–Helly graph. By Theorem 2.4, there are infinitely many such examples, all of them non-isomorphic to each other by Theorem 4.2.

The fact that these examples are not self-clique follows from vertex degree comparisons: As in the proof of Theorem 4.2, the vertices of $K(H) = K^2(G)$ have a degree which is either $3d$ or 15, but the degree of all vertices of $H = K(G)$ is $3d-6$ (see Fig. 1 and Remark 2.3(1)). It follows that $K(H) \ncong H$ and hence H is not self-clique. By the result of Escalante mentioned in the introduction, H is 2-self-clique. □

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