On the putative essential discreteness of q-generalized entropies

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Abstract

It has been argued in [EPL **90** (2010) 50004], entitled *Essential discreteness in generalized thermostatistics with non-logarithmic entropy*, that "continuous Hamiltonian systems with long-range interactions and the so-called q-Gaussian momentum distributions are seen to be outside the scope of non-extensive statistical mechanics". The arguments are clever and appealing. We show here that, however, some mathematical subtleties render them unconvincing
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1 Introduction

During more than 25 years, an important topic in statistical mechanics theory revolved around the notion of generalized q-statistics, pioneered by Tsallis [1]. It has been amply demonstrated that, in many occasions, the celebrated Boltzmann-Gibbs logarithmic entropy does not yield a correct description of the system under scrutiny [2]. Other entropic forms, called q-entropies, produce a much better performance [2]. One may cite a large number of such instances. For example, non-ergodic systems exhibiting a complex dynamics [2].
The non-extensive statistical mechanics of Tsallis' has been employed to fruitfully discuss phenomena in variegated fields. One may mention, for instance, high-energy physics [3]-[4], spin-glasses [5], cold atoms in optical lattices [6], trapped ions [7], anomalous diffusion [8], [9], dusty plasmas [10], low-dimensional dissipative and conservative maps in dynamical systems [11], [12], [13], turbulent flows [14], Levy flights [15], the QCD-based Nambu, Jona, Lasinio model of a many-body field theory [16], etc. Notions related to q-statistical mechanics have been found useful not only in physics but also in chemistry, biology, mathematics, economics, and informatics [17], [18], [19].

In this work we revisit results presented in [20]. First, we note that [20] has been criticized, in a manner unrelated to ours here, in a Comment [21]. There is also a reply by Abe to that Comment [22]. It is stated in [20] that one encounters an essential discreteness in generalized thermostatistics with non-logarithmic entropy. Thus, "continuous Hamiltonian systems with long-range interactions and the so-called q-Gaussian momentum distributions are seen to be outside the scope of non-extensive statistical mechanics" [20]. The pertinent arguments are clever and appealing. However, as we will show here that, some mathematical subtleties render them unconvincing. The main reason is the Functional Analysis is the branch of mathematics operative in this context, not simple Calculus. Functional variational procedures are described, for instance, in Ref. [23].

2 Continuous variational Tsallis' case

The functional MaxEnt treatment is given in [24]. Let P stand for the pertinent probability distribution (PD). One evaluates mean values here in the customary fashion, linear in P, i.e.,

$< R >= \int_M RP\, d\mu$. It is well known that the MaxEnt variational Tsallis functional is [2]

$$F_S(P) = -\int\limits_M P^q \ln_q(P)\, d\mu + \alpha\left(\int\limits_M PH\, d\mu - < U >\right) + \gamma\left(\int\limits_M P\, d\mu - 1\right). \tag{2.1}$$

For the variational increment h we have [23]

$$F_S(P+h) - F_S(P) = -\int\limits_M (P+h)^q \ln_q(P+h)\, d\mu + \alpha\left[\int\limits_M (P+h)H\, d\mu - < U >\right] +$$

$$\gamma\left[\int\limits_M (P+h)\, d\mu - 1\right] + \int\limits_M P^q \ln_q(P)\, d\mu - \alpha\left(\int\limits_M PH\, d\mu - < U >\right) -$$

,

$$\gamma\left(\int\limits_M P\, d\mu - 1\right). \tag{2.2}$$

Eq. (2.2) can be recast as

$$F_S(P+h) - F_S(P) = \int\limits_M \left[\left(\frac{q}{1-q}\right) P^{q-1} + \alpha H + \gamma\right] h\, d\mu -$$

$$\int\limits_M q P^{q-2} \frac{h^2}{2}\, d\mu + O(h^3). \tag{2.3}$$

Eq. (2.3) leads now to the following equations

$$\left(\frac{q}{1-q}\right) P^{q-1} + \alpha H + \gamma = 0, \tag{2.4}$$

$$-\int\limits_M q P^{q-2} h^2\, d\mu \leq C||h||^2. \tag{2.5}$$

Eq. (2.4) is the Euler-Lagrange one while (2.5) gives bounds originating from the second variation [23]. Starting with (2.4) we use the procedure given in

[2]. One first gives the Lagrange multipliers α and β a prescribed *form* in terms of a (thus far unknown) quantity Z:

$$\alpha = \beta q Z^{1-q}, \tag{2.6}$$

$$\gamma = \frac{q}{q-1} Z^{1-q}, \tag{2.7}$$

and then Z is determined by appeal to normalization. Accordingly, one has

$$P = \frac{[1+\beta(1-q)H]^{\frac{1}{q-1}}}{Z} = e_{2-q}(-\beta H)/Z, \tag{2.8}$$

and, on account of normalization,

$$Z = \int\limits_M [1+\beta(1-q)H]^{\frac{1}{q-1}}\ d\mu. \tag{2.9}$$

For Eq. (2.9) we have,

$$W = -\int\limits_M qP^{q-2}h^2\ d\mu = -\int\limits_M qZ^{2-q}[1+\beta(1-q)H]^{\frac{q-2}{q-1}}h^2\ d\mu. \tag{2.10}$$

How to obtain a bound is discussed in [24].

3 Discrete variational Tsallis' case

The concomitant Tsallis discrete functional is

$$F_S(P) = -\sum_{i=1}^{n} P_i^q \ln_q(P_i) + \lambda_1\left(\sum_{i=1}^{n} P_iU_i - <U>\right) + \lambda_2\left(\sum_{i=1}^{n} P_i - 1\right) \tag{3.1}$$

For the increment we have

$$F_S(P+h) - F_S(P) = -\sum_{i=1}^{n}(P_i+h_i)^q \ln_q(P_i+h_i) + \lambda_1\left[\sum_{i=1}^{n}(P_i+h_i)U_i - <U>\right] +$$

$$\lambda_2\left[\sum_{i=1}^{n}(P_i+h_i) - 1\right] + \sum_{i=1}^{n} P_i^q \ln_q(P_i) - \lambda_1\left(\sum_{i=1}^{n} P_iU_i - <U>\right) -$$

$$\lambda_2\left(\sum_{i=1}^{n} P_i - 1\right) \tag{3.2}$$

Eq. (3.2) can be recast as

$$F_S(P+h) - F_S(P) = \sum_{i=1}^{n}\left[\left(\frac{q}{1-q}\right)P_i^{q-1} + \lambda_1 U_i + \lambda_2\right] h_i$$

$$-\sum_{i=1}^{n} q P_i^{q-2}\frac{h_i^2}{2} + O(h^3), \tag{3.3}$$

Eq. (3.3) leads to the following equations:

$$\left(\frac{q}{1-q}\right)P_i^{q-1} + \lambda_1 U_i + \lambda_2 = 0, \tag{3.4}$$

$$-\sum_{i=1}^{n} q P_i^{q-2} h_i^2 \le C||h||^2 \tag{3.5}$$

Eq. (3.4) is the Euler-Lagrange one while (3.5) gives bounds originating from the second variation.
Starting with (3.4), we use again the procedure given in [2] and in Section 2. One first gives the Lagrange multipliers λ_1 and λ_2 a prescribed *form* in terms of a (thus far unknown) quantity Z:

$$\lambda_1 = \beta q Z^{1-q}, \tag{3.6}$$

$$\lambda_2 = \frac{q}{q-1} Z^{1-q}, \tag{3.7}$$

and then has

$$P_i = \frac{[1+\beta(1-q)U_i]^{\frac{1}{q-1}}}{Z}, \tag{3.8}$$

so that normalization demands that

$$Z = \sum_{i=1}^{n} [1+\beta(1-q)U_i]^{\frac{1}{q-1}}. \tag{3.9}$$

For Eq. (3.5) we have,

$$W = -\sum_{i=1}^{n} q P_i^{q-2} h_i^2 = -\sum_{i=1}^{n} q Z^{2-q}[1+\beta(1-q)U_i]^{\frac{q-2}{q-1}} h_i^2. \tag{3.10}$$

Just how to obtain a bound follows the lines discussed in [24], adapted to the discrete scenario.

4 Our above results vis-a-vis those of Ref. [20]

We have seen in the two previous Sections that a rigorous functional analysis, MaxEnt treatment, yields Eqs. (2.8), (2.9), (3.8), and (3.9). This entails that Eqs. (3) and (4) of [20] cannot be correct. Neither are correct Eqs. (7) and (8)of such reference. The question is the passage from the continuous to the discrete (or viceversa) scenarios. This is the main issue discussed in [20].

It is well known in Measure Theory [25] that it is not possible to pass from a discrete probability distribution (PD) to a continuous one via a simple Riemann integral, as done in [20]. Moreover, a discrete PD can be cast as an integral over a Lebesgue-Stieltjes measure, concentrated on a finite (or numerable) set of points. For instance, in the Shannon entropic case one has, in phase space (q, p), with U the energy and λ_i Lagrange multipliers:

$$S = -\sum_{i=1}^{n} P_i \ln P_i = -\int\limits_M P[U(p,q), \lambda_1 ....., \lambda_n] \ln P[U(p,q), \lambda_1 ....., \lambda_n] d\mu(p,q), \tag{4.1}$$

with

$$d\mu(p,q) = \sum_{i=1}^{n} \delta[U(p,q) - U_i] dU(p,q), \tag{4.2}$$

where

$$P_i = P[U_i, \lambda_1 ....., \lambda_n]. \tag{4.3}$$

Thus, it becomes clear that Eq. (7) of [20] is incorrect because there, the probability density is NOT concentrated on a numerable set of points. The Tsallis' entropy scenario is identical to the one above. One just must insert in (4.1) $P^q \ln_q P$ instead of $P \ln P$.

Note also that Eq. (11) of [20] is NOT the standard classical Boltzmann's entropy expression, transcribed in his tombstone at Vienna's cemetery. This, of course, does not contain Planck's constant. The derivation that follows such Eq. (11) is thus invalid.

Finally, we cite some recent works that successfully deal with the Tsallis' q-statistics with continuous probability distributions [26, 27, 28].

5 Conclusions

We have here shown, by recourse to Functional Analysis, that Tsallis's thermostatistics is fully valid in the continuous instance, notwithstanding the arguments of [20].

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