ON HOPF ALGEBRAS OVER QUANTUM SUBGROUPS

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ABSTRACT. Using the standard filtration associated with a generalized lifting method, we determine all finite-dimensional Hopf algebras over an algebraically closed field of characteristic zero whose coradical generates a Hopf subalgebra isomorphic to the smallest non-pointed non-cosemisimple Hopf algebra $\mathcal{K}$ and the corresponding infinitesimal module is an indecomposable object in ${}^{\mathcal{K}}_{\mathcal{K}}\mathcal{YD}$ (we assume that the diagrams are Nichols algebras). As a byproduct, we obtain new Nichols algebras of dimension 8 and new Hopf algebras of dimension 64.

INTRODUCTION

Let $\Bbbk$ be an algebraically closed field of characteristic zero. The problem of classifying all Hopf algebras over $\Bbbk$ of a given dimension was posed by Kaplansky in 1975 [8]. Some progress has been made but, in general, it is a difficult question. One of the few general techniques is the so-called *Lifting Method* [3], under the assumption that the coradical is a subalgebra, *i.e.*, the Hopf algebra has the Chevalley Property. More recently, Andruskiewitsch and Cuadra [1] proposed to extend this technique by considering the subalgebra generated by the coradical and the related wedge filtration. It turns out that this filtration is a Hopf algebra filtration, provided that the antipode is injective, what is true in the finite-dimensional context.

We describe the lifting method briefly. Let H be a Hopf algebra over $\Bbbk$. Recall that the coradical filtration $\{H_n\}_{n\geq 0}$ of H is defined recursively by

- the coradical H_0, which is the sum of all simple subcoalgebras, and
- $H_n = \bigwedge^{n+1} H_0 = \{h \in H : \Delta(h) \in H \otimes H_0 + H_{n-1} \otimes H\}$.

This filtration corresponds to the filtration of H^* given by the powers of the Jacobson radical. It is always a coalgebra filtration and if H_0 is a Hopf subalgebra, then it is indeed a Hopf algebra filtration; in particular, its associated graded object $\operatorname{gr} H = \bigoplus_{n\geq 0} H_n/H_{n-1}$ is a graded Hopf algebra, where $H_{-1} = 0$. Let $\pi : \operatorname{gr} H \to H_0$ be the homogeneous projection. It turns out that $\operatorname{gr} H \simeq R\# H_0$ as Hopf algebras, where $R = (\operatorname{gr} H)^{\operatorname{co}\pi} = \{h \in H : (\operatorname{id}\otimes\pi)\Delta(h) = h\otimes 1\}$ is the algebra of coinvariants and $\#$ stands for the Radford-Majid biproduct or *bosonization* of R with H_0. The algebra R is not a usual Hopf algebra, but a graded connected Hopf algebra in the category ${}^{H_0}_{H_0}\mathcal{YD}$ of left Yetter-Drinfeld modules over H_0. The subalgebra generated by the elements of degree one is the *Nichols algebra* $\mathfrak{B}(V)$ of $V = R(1)$; here V is a braided vector space called the *infinitesimal braiding*.

Let us fix a finite-dimensional cosemisimple Hopf algebra A. The lifting method then consists of the description of all finite-dimensional Nichols algebras $\mathfrak{B}(V) \in {}^{A}_{A}\mathcal{YD}$, the determination of all possible deformations of the bosonization $\mathfrak{B}(V)\#A$, and the proof that all Hopf algebras H with $H_0 = A$ satisfy that $\operatorname{gr} H \simeq \mathfrak{B}(V)\#A$.

The main idea in [1] is to replace the coradical filtration by a more general but adequate filtration: the **standard filtration** $\{H_{[n]}\}_{n\geq 0}$, which is defined recursively by

- the subalgebra $H_{[0]}$ of H generated by H_0, called the *Hopf coradical*, and
- $H_{[n]} = \bigwedge^{n+1} H_{[0]}$.

If the coradical H_0 is a Hopf subalgebra, then $H_{[0]} = H_0$ and the coradical filtration coincides with the standard one.

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Let A be a Hopf algebra generated by its coradical. We will say that H is a *Hopf algebra over* A if $H_{[0]} \simeq A$ as Hopf algebras.

Assume that the antipode $\mathcal{S}$ of H is injective. Then by [1, Lemma 1.1], it holds that $H_{[0]}$ is a Hopf subalgebra of H, $H_n \subseteq H_{[n]}$ and $\{H_{[n]}\}_{n\geq 0}$ is a Hopf algebra filtration of H. In particular, the graded algebra $\operatorname{gr} H = \bigoplus_{n\geq 0} H_{[n]}/H_{[n-1]}$ with $H_{[-1]} = 0$ is a Hopf algebra associated with the standard filtration. Write $\pi : \operatorname{gr} H \to H_{[0]}$ for the homogeneous projection. Then, as before, it splits the inclusion of $H_{[0]}$ in $\operatorname{gr} H$, the *diagram* $R = (\operatorname{gr} H)^{\operatorname{co}\pi}$ is a Hopf algebra in the category ${}^{H_{[0]}}_{H_{[0]}}\mathcal{YD}$ of Yetter-Drinfeld modules over $H_{[0]}$ and $\operatorname{gr} H \simeq R\# H_{[0]}$ as Hopf algebras. It turns out that $R = \bigoplus_{n\geq 0} R(n)$ is also graded and connected. We call again the linear space $R(1)$ consisting of elements of degree one, the *infinitesimal braiding*.

The procedure to describe explicitly any Hopf algebra as above defines a proposal for the classification of general finite-dimensional Hopf algebras over a fixed Hopf subalgebra A which is generated by a cosemisimple coalgebra. The main steps are the following:

(a) determine all Yetter-Drinfeld modules V in ${}^A_A\mathcal{YD}$ such that the Nichols algebra $\mathfrak{B}(V)$ is finite-dimensional,

(b) for such V, compute all Hopf algebras L such that $\operatorname{gr} L \simeq \mathfrak{B}(V)\# A$. We call L a *lifting* of $\mathfrak{B}(V)$ over A.

(c) Prove that any finite-dimensional Hopf algebra over A is generated by the first term of the standard filtration.

In this paper, we study these questions (a) and (b) in the case that $A = \mathcal{K}$ is the smallest Hopf algebra whose coradical is not a subalgebra. It is an 8-dimensional Hopf algebra whose dual is a pointed Hopf algebra. The dual Hopf algebra A^* was first introduced by Radford [11], who addressed the problem of finding a Hopf algebra whose Jacobson radical is not a Hopf ideal.

Let ξ be a primitive 4-th root of 1. As an algebra, $\mathcal{K}$ is generated by the elements a, b, c, d satisfying the following relations:

$$\begin{array}{lllll} ab = \xi ba, & ac = \xi ca, & 0 = cb = bc, & cd = \xi dc, & bd = \xi db, \\ ad = da, & ad = 1, & 0 = b^2 = c^2, & a^2c = b, & a^4 = 1. \end{array} \tag{1}$$

The coalgebra structure and its antipode are determined by

$$\begin{array}{llll} \Delta(a) = a\otimes a + b\otimes c, & \Delta(b) = a\otimes b + b\otimes d, & \varepsilon(a) = 1, & \varepsilon(b) = 0, \\ \Delta(c) = c\otimes a + d\otimes c, & \Delta(d) = c\otimes b + d\otimes d, & \varepsilon(c) = 0, & \varepsilon(d) = 1, \\ \mathcal{S}(a) = d, & \mathcal{S}(b) = \xi b, & \mathcal{S}(c) = -\xi c, & \mathcal{S}(d) = a. \end{array} \tag{2}$$

See Section 2 for more details.

In order to determine finite-dimensional Hopf algebras over $\mathcal{K}$, we first compute the Drinfeld double $D := D(\mathcal{K}^{\operatorname{cop}})$ of $\mathcal{K}^{\operatorname{cop}}$ and describe the simple left D-modules, their projective covers and some indecomposable left D-modules. In fact, we prove in Theorem 2.9 that there are sixteen simple left D-modules pairwise non-isomorphic: four 1-dimensional ones and twelve 2-dimensional ones. The former correspond to characters on $\mathbb{Z}_4$ and the latter are parametrized by the set $\Lambda = \{(i,j) \in \mathbb{Z}_4 \times \mathbb{Z}_4 |\ 2i \neq j\}$. We compute the separation diagram of D and show that D is of tame representation type.

Using that the braided monoidal categories ${}_D\mathcal{M}$ and ${}^{\mathcal{K}}_{\mathcal{K}}\mathcal{YD}$ are equivalent, we then translate the description above to simple and indecomposable modules in ${}^{\mathcal{K}}_{\mathcal{K}}\mathcal{YD}$. Then, using the description of the braiding in ${}^{\mathcal{K}}_{\mathcal{K}}\mathcal{YD}$, we obtain our first main result, see Section 3 for definitions.

Theorem A. *Let $\mathfrak{B}(V)$ be a finite-dimensional Nichols algebra over an indecomposable object V in ${}^{\mathcal{K}}_{\mathcal{K}}\mathcal{YD}$. Then V is simple and isomorphic either to $\Bbbk_{\chi}$, $\Bbbk_{\chi^3}$, $V_{2,1}$, $V_{2,3}$, $V_{3,1}$ or $V_{3,3}$.*

It turns out that $\mathfrak{B}(\Bbbk_{\chi^\ell}) \simeq \bigwedge \Bbbk_{\chi^\ell}$ is an exterior algebra for $\ell = 1, 3$ with $\dim \mathfrak{B}(\Bbbk_{\chi^\ell}) = 2$ and $\mathfrak{B}(V)$ is an 8-dimensional algebra for $V = V_{2,1}$, $V_{2,3}$, $V_{3,1}$ and $V_{3,3}$. It is possible to check that these braidings are triangular [14]. These 8-dimensional examples are new examples of finite-dimensional Nichols algebras. They are isomorphic to quantum linear spaces as algebras, but not as coalgebras since the braiding differs; in our case, the braiding is not of diagonal type, see the Appendix of the first `arXiv` version of this paper.

As the study of Nichols algebras over semisimple modules is a hard problem that demands different techniques to be applied, we focus on the description of Hopf algebras over $\mathcal{K}$ such that their infinitesimal braiding is indecomposable, *i.e.*, the liftings of the Nichols algebras in Theorem A. Thus we define two Hopf algebras $\mathfrak{A}_{3,1}(\mu)$ and $\mathfrak{A}_{3,3}(\mu)$ depending on a parameter $\mu \in \Bbbk$ and prove our second main result, see Section 5 for definitions.

Theorem B. *Let H be a finite-dimensional Hopf algebra over $\mathcal{K}$ such that its infinitesimal braiding is an indecomposable module V in ${}^{\mathcal{K}}_{\mathcal{K}}\mathcal{YD}$. Assume that the diagram R is a Nichols algebra. Then V is simple and H is isomorphic either to*

(i) $(\bigwedge \Bbbk_{\chi^\ell})\#\mathcal{K}$ *with* $\ell = 1, 3$;
(ii) $\mathfrak{B}(V_{2,1})\#\mathcal{K}$;
(iii) $\mathfrak{B}(V_{2,3})\#\mathcal{K}$;
(iv) $\mathfrak{A}_{3,1}(\mu)$ *for some* $\mu \in \Bbbk$;
(v) $\mathfrak{A}_{3,3}(\mu)$ *for some* $\mu \in \Bbbk$.

The Hopf algebras $(\bigwedge \Bbbk_{\chi^\ell})\#\mathcal{K}$ with $\ell = 1, 3$ have dimension 16 and are duals of pointed Hopf algebras. They have already appeared in [4]. The Hopf algebras $\mathfrak{B}(V_{2,1})\#\mathcal{K}$ and $\mathfrak{B}(V_{2,3})\#\mathcal{K}$ are dual of pointed Hopf algebras of dimension 64. The Hopf algebras $\mathfrak{A}_{3,1}(\mu)$ and $\mathfrak{A}_{3,3}(\mu)$ are non-pointed with non-pointed duals. To the best of the authors knowledge, they constitute new examples of Hopf algebras of dimension 64.

The paper is organized as follows. In Section 1 we recall some invariants associated with a Hopf algebra, define Yetter-Drinfeld modules, Nichols algebras and the Drinfeld double of a finite-dimensional Hopf algebra. We also recall the relation between Hopf algebras with a projection and bosonizations. In Section 2 we describe the structure of $\mathcal{K}$ and give the presentation of the double $D = D(\mathcal{K}^{\mathrm{cop}})$ by generators and relations. We also determine the simple left D-modules, their projective covers and some indecomposable left D-modules. We compute the Ext-Quiver of D and show that D is of tame representation type.

Then, using the equivalence ${}_D\mathcal{M} \simeq {}^{\mathcal{K}}_{\mathcal{K}}\mathcal{YD}$, we determine in Section 3 the corresponding objects of the latter and describe their braidings. In Section 4 we show that if $\mathfrak{B}(V)$ is a finite-dimensional Nichols algebra in ${}^{\mathcal{K}}_{\mathcal{K}}\mathcal{YD}$, then V is necessarily a semisimple object and prove Theorem A by describing first the Nichols algebra of the simple modules. Finally, in Section 5 we prove Theorem B.

1. Preliminaries

1.1. **Conventions.** We work over an algebraically closed field $\Bbbk$ of characteristic zero and with Hopf algebras which have bijective antipode. Our references for Hopf algebra theory are [10] and [12].

For a Hopf algebra H over $\Bbbk$, the comultiplication, counit and antipode are denoted by Δ, ε and $\mathcal{S}$, respectively. Comultiplication and coactions are written using the Sweedler notation with summation sign suppressed, *e.g.*, $\Delta(h) = h_{(1)} \otimes h_{(2)}$ for $h \in H$. A Hopf algebra in a braided monoidal category is called a *braided* Hopf algebra. We denote by ${}_H\mathcal{M}$ the category of finite-dimensional left H-modules.

The set $G(H) = \{h \in H \setminus \{0\} : \Delta(h) = h \otimes h\}$ denotes the group of *group-like elements*. The *coradical* H_0 of H is the sum of all simple subcoalgebras of H; in particular, $\Bbbk G(H) \subseteq H_0$. The subalgebra $H_{[0]}$ generated by H_0 is a Hopf subalgebra which is called the *Hopf coradical*. For $h, g \in G(H)$, the linear space of (h, g)-*primitive elements* is

$$\mathcal{P}_{h,g}(H) := \{x \in H \mid \Delta(x) = x \otimes h + g \otimes x\}.$$

In case $g = 1 = h$, the linear space $\mathcal{P}(H) = \mathcal{P}_{1,1}(H)$ is called the set of *primitive elements*.

Let M be a left H-comodule via $\delta(m) = m_{(-1)} \otimes m_{(0)} \in H \otimes M$ for all $m \in M$. The space of *left coinvariants* is given by ${}^{\mathrm{co}\,\delta}M = \{x \in M \mid \delta(x) = 1 \otimes x\}$. In particular, for a Hopf algebra map $\pi : H \to L$, it follows that H is a left L-comodule via $(\pi \otimes \mathrm{id})\Delta$ and

$$ {}^{\mathrm{co}\,\pi}H := {}^{\mathrm{co}\ (\pi\otimes\mathrm{id})\Delta}H = \{h \in H \mid (\pi \otimes \mathrm{id})\Delta(h) = 1 \otimes h\}. $$

Right coinvariants, written $H^{\mathrm{co}\,\pi}$, are defined analogously.

1.2. **Yetter-Drinfeld modules and Nichols algebras.** Let H be a Hopf algebra. A *left Yetter-Drinfeld module* M over H is a left H-module $(M, \cdot)$ and a left H-comodule (M, δ) satisfying

$$ \delta(h \cdot m) = h_{(1)} m_{(-1)} \mathcal{S}(h_{(3)}) \otimes h_{(2)} \cdot m_{(0)} \qquad \forall\, m \in M, h \in H. $$

We denote by ${}^H_H\mathcal{YD}$ the category of left Yetter-Drinfeld modules over H. It is a braided monoidal category: for $M, N \in {}^H_H\mathcal{YD}$, the braiding $c_{M,N} : M \otimes N \to N \otimes M$ is given by

$$ c_{M,N}(m \otimes n) = m_{(-1)} \cdot n \otimes m_{(0)} \qquad \forall\, m \in M, n \in N. \tag{3} $$

Definition 1.1. [3, Definition 2.1] *Let H be a Hopf algebra and $V \in {}^H_H\mathcal{YD}$. A braided $\mathbb{N}$-graded Hopf algebra $R = \bigoplus_{n\geq 0} R(n) \in {}^H_H\mathcal{YD}$ is called the Nichols algebra of V if*

(*i*) *$R(0) \simeq \Bbbk$, $R(1) \simeq V$;*
(*ii*) *$R(1) = \mathcal{P}(R)$;*
(*iii*) *R is generated as an algebra by $R(1)$.*

In this case, R is denoted by $\mathfrak{B}(V) = \bigoplus_{n\geq 0} \mathfrak{B}^n(V)$.

For any $V \in {}^H_H\mathcal{YD}$ there is a unique up to isomorphism Nichols algebra $\mathfrak{B}(V)$ associated with it. It is the quotient of the tensor algebra $T(V)$ by the largest homogeneous two-sided ideal I satisfying:

- I is generated by homogeneous elements of degree ≥ 2, and
- $\Delta(I) \subseteq I \otimes T(V) + T(V) \otimes I$, *i.e.*, it is also a coideal.

See [3, Section 2.1] for details.

Remark 1.2. *Let c be the braiding associated to $V \in {}^H_H\mathcal{YD}$ and assume that there is $W \subseteq V$ a subspace such that $c(W \otimes W) \subseteq W \otimes W$. Then, one may identify $\mathfrak{B}(W)$ with a subalgebra of $\mathfrak{B}(V)$; perhaps belonging to different braided monoidal categories. In particular, $\mathfrak{B}(V)$ is infinite-dimensional whenever $\mathfrak{B}(W)$ is infinite-dimensional. This occurs for example, when V contains a non-zero element v such that $c(v \otimes v) = v \otimes v$.*

1.3. **Bosonization and Hopf algebras with a projection.** Let H be a Hopf algebra and B a braided Hopf algebra in ${}^H_H\mathcal{YD}$. The procedure to obtain a usual Hopf algebra from B and H is called the Majid-Radford biproduct or *bosonization*, and it is usually denoted by $B\#H$. As a vector space, $B\#H = B \otimes H$, and the multiplication and comultiplication are given by the smash-product and smash-coproduct, respectively. Explicitly, for all $b, c \in B$ and $g, h \in H$, we have

$$ (b\#g)(c\#h) = b(g_{(1)} \cdot c)\#g_{(2)}h, $$
$$ \Delta(b\#g) = b^{(1)}\#(b^{(2)})_{(-1)}g_{(1)} \otimes (b^{(2)})_{(0)}\#g_{(2)}, $$

where $\Delta_B(b) = b^{(1)} \otimes b^{(2)}$ denotes the comultiplication in $B \in {}^H_H\mathcal{YD}$. We identify $b = b\#1$ and $h = 1\#h$; in particular we have $bh = b\#h$ and $hb = h_{(1)} \cdot b\#h_{(2)}$. Clearly, the map $\iota : H \to B\#H$ given by $\iota(h) = 1\#h$ is an injective Hopf algebra map, and the map $\pi : B\#H \to H$ given by $\pi(b\#h) = \varepsilon_B(b)h$ is a surjective Hopf algebra map such that $\pi \circ \iota = \mathrm{id}_H$. Moreover, it holds that $B = (B\#H)^{\mathrm{co}\,\pi}$.

Conversely, let A be a Hopf algebra with bijective antipode. Suppose that there are Hopf algebra morphisms $\pi : A \to H$ and $\iota : H \to A$ such that $\pi \circ \iota = \mathrm{id}_H$. Then $B = A^{\mathrm{co}\,\pi}$ is a braided Hopf algebra in ${}^H_H\mathcal{YD}$ and $A \simeq B\#H$ as Hopf algebras.

1.4. **The Drinfeld double.** We briefly describe the structure of the Drinfeld double of a finite-dimensional Hopf algebra.

Let H be a finite-dimensional Hopf algebra. Consider H acting on H^* and H^* acting on H via respectively

$$h \twoheadrightarrow f = \langle f_{(3)}\mathcal{S}^{-1}(f_{(1)}), h\rangle f_{(2)} \text{ and } h \twoheadleftarrow f = \langle f, \mathcal{S}^{-1}(h_{(3)})h_{(1)}\rangle h_{(2)},\ \forall\, h \in H,\ f \in H^*.$$

The Drinfeld double of H is the Hopf algebra $D(H)$, where $D(H) = H^* \otimes H$, as vector spaces. The product and the unit are given by

$$(f \bowtie h)(g \bowtie k) = f(h_{(1)} \twoheadrightarrow g_{(2)}) \bowtie (h_{(2)} \twoheadleftarrow g_{(1)})k, \quad \text{and} \quad 1_{D(H)} = \varepsilon \bowtie 1.$$

The coproduct, counit and antipode do not play an important role in this paper. They can be found for instance in [10, Definition 10.3.5].

The following result will be central in Section 3, since we will study the simple and indecomposable left $D(\mathcal{K}^{\mathrm{cop}})$-modules first and then translate the information to ${}^{\mathcal{K}}_{\mathcal{K}}\mathcal{YD}$.

Proposition 1.3. [10, Proposition 10.6.16] *Let H be a finite-dimensional Hopf algebra. The category ${}^{H}_{H}\mathcal{YD}$ of left Yetter-Drinfeld modules over H can be identified with the category ${}_{D(H^{\mathrm{cop}})}\mathcal{M}$ of left modules over the Drinfeld double $D(H^{\mathrm{cop}})$.* □

2. The Hopf algebra $\mathcal{K}$ and its Drinfeld double $D(\mathcal{K}^{\mathrm{cop}})$

All pointed nonsemisimple Hopf algebras of dimension 8 were determined by Ştefan [13]. Except for one case (up to isomorphism), these pointed Hopf algebras have pointed duals. The exception is given by

$$\mathcal{A}_4'' := \Bbbk\langle g, x \mid g^4 - 1 = x^2 - g^2 + 1 = gx + xg = 0\rangle,$$

with $\Delta(g) = g \otimes g$ and $\Delta(x) = x \otimes g + 1 \otimes x$. Moreover, it holds that $\mathcal{K}$, presented by (1) and (2), is isomorphic to $(\mathcal{A}_4'')^*$, see [5]. Up to isomorphism, $\mathcal{K}$ is the only Hopf algebra of dimension 8 which is neither semisimple nor pointed nor has the Chevalley property. The next proposition gives us a presentation of $\mathcal{K}$ and some useful relations that will be used in the sequel. The proof follows from [5, Lemma 3.3].

Throughout the paper, we fix a primitive 4-th root of unity ξ.

Proposition 2.1.

(*i*) *$\mathcal{K}$ is generated as an algebra by the elements a, b, c, d satisfying* (1)*.*

(*ii*) *The set $\{1, a, b, c, d, a^2, ab, ac\}$ is a linear basis of $\mathcal{K}$.*

(*iii*) *The coalgebra structure and the antipode are determined by* (2)*. In particular,*

$$\Delta(ab) = ab \otimes 1 + a^2 \otimes ab, \qquad \Delta(ac) = ac \otimes a^2 + 1 \otimes ac, \qquad \Delta(a^2) = a^2 \otimes a^2, \tag{4}$$
$$\mathcal{S}(a^2) = a^2, \qquad \mathcal{S}(ab) = -ac, \qquad \mathcal{S}(ac) = ab.$$

(*iv*) *The multiplication table of $\mathcal{K}$ is*

	1	a	b	c	d	a^2	ab	ac
1	1	a	b	c	d	a^2	ab	ac
a	a	a^2	ab	ac	1	d	c	b
b	b	$-\xi ab$	0	0	ξac	$-c$	0	0
c	c	$-\xi ac$	0	0	ξab	$-b$	0	0
d	d	1	ac	ab	a^2	a	b	c
a^2	a^2	d	c	b	a	1	ac	ab
ab	ab	$-\xi c$	0	0	ξb	$-ac$	0	0
ac	ac	$-\xi b$	0	0	ξc	$-ab$	0	0

(*v*) *$\mathcal{K} \simeq H_4 \oplus \mathcal{M}^*(2, \Bbbk)$ as coalgebras, where H_4 is the Sweedler's Hopf algebra and $\mathcal{M}^*(2, \Bbbk)$ is a comatrix coalgebra of dimension 4.*

□

Remarks 2.2.
(a) *Denote by* $\{1^*, a^*, b^*, c^*, d^*, (a^2)^*, (ab)^*, (ac)^*\}$ *the basis of* $\mathcal{K}^*$ *dual to* $\{1, a, b, c, d, a^2, ab, ac\}$. *Using the multiplication table in Proposition 2.1 (iv), it follows that*

$$\begin{aligned}
\Delta(1^*) &= 1^* \otimes 1^* + a^* \otimes d^* + d^* \otimes a^* + (a^2)^* \otimes (a^2)^*,\\
\Delta(a^*) &= 1^* \otimes a^* + a^* \otimes 1^* + (a^2)^* \otimes d^* + d^* \otimes (a^2)^*,\\
\Delta(d^*) &= 1^* \otimes d^* + d^* \otimes 1^* + (a^2)^* \otimes a^* + a^* \otimes (a^2)^*,\\
\Delta((a^2)^*) &= 1^* \otimes (a^2)^* + (a^2)^* \otimes 1^* + a^* \otimes a^* + d^* \otimes d^*,\\
\Delta(b^*) &= 1^* \otimes b^* + b^* \otimes 1^* + a^* \otimes (ac)^* - \xi(ac)^* \otimes a^* +\\
&\quad + (a^2)^* \otimes c^* - c^* \otimes (a^2)^* + \xi(ab)^* \otimes d^* + d^* \otimes (ab)^*,\\
\Delta(c^*) &= 1^* \otimes c^* + c^* \otimes 1^* - \xi(ab)^* \otimes a^* + a^* \otimes (ab)^* +\\
&\quad + (a^2)^* \otimes b^* - b^* \otimes (a^2)^* + \xi(ac)^* \otimes d^* + d^* \otimes (ac)^*,\\
\Delta((ab)^*) &= 1^* \otimes (ab)^* + (ab)^* \otimes 1^* - \xi b^* \otimes a^* + a^* \otimes b^* +\\
&\quad + d^* \otimes c^* + \xi c^* \otimes d^* - (ac)^* \otimes (a^2)^* + (a^2)^* \otimes (ac)^*,\\
\Delta((ac)^*) &= 1^* \otimes (ac)^* + (ac)^* \otimes 1^* - \xi c^* \otimes a^* + a^* \otimes c^* +\\
&\quad + d^* \otimes b^* + \xi b^* \otimes d^* - (ab)^* \otimes (a^2)^* + (a^2)^* \otimes (ab)^*.
\end{aligned}$$

(b) *Let* $\alpha \in G(\mathcal{K}^*) = \mathrm{Alg}(\mathcal{K}, \Bbbk)$. *The relations* (1) *implies that* $\alpha(a)$ *is a* 4*-th root of unity,* $\alpha(b) = \alpha(c) = 0$ *and* $\alpha(d) = \alpha(a)^{-1}$. *Thus* $G(\mathcal{K}^*)$ *consists of the elements*

$$\alpha_j = 1^* + \xi^{-j} a^* + \xi^j d^* + (-1)^j (a^2)^*, \qquad j = 0, 1, 2, 3.$$

Note that $\alpha_0 = \varepsilon$ *and* $\alpha_1^j = \alpha_j$. *In particular,* $G(\mathcal{K}^*) \simeq \mathbb{Z}/4\mathbb{Z}$ *and* α_1, α_3 *are generators.*
(c) *The multiplication table of* $\mathcal{K}^*$ *is*

	1^*	a^*	b^*	c^*	d^*	$(a^2)^*$	$(ab)^*$	$(ac)^*$
1^*	1^*	0	0	0	0	0	0	$(ac)^*$
a^*	0	a^*	b^*	0	0	0	0	0
b^*	0	0	0	a^*	b^*	0	0	0
c^*	0	c^*	d^*	0	0	0	0	0
d^*	0	0	0	c^*	d^*	0	0	0
$(a^2)^*$	0	0	0	0	0	$(a^2)^*$	$(ab)^*$	0
$(ab)^*$	$(ab)^*$	0	0	0	0	0	0	0
$(ac)^*$	0	0	0	0	0	$(ac)^*$	0	0

In order to compute the Drinfeld double $D(\mathcal{K}^{\mathrm{cop}})$, we need to describe the isomorphism $\mathcal{K}^* \simeq \mathcal{A}_4''$ explicitly.

Lemma 2.3. *The algebra map* $\varphi : \mathcal{A}_4'' \to \mathcal{K}^*$ *given by*

$$\varphi(g) = \alpha_1 \quad \text{and} \quad \varphi(x) = \sqrt{2}\xi(b^* + c^* + (ab)^* + (ac)^*),$$

is a Hopf algebra isomorphism.

Proof. A direct computation shows that φ is a coalgebra map. Hence, the image of φ is a Hopf subalgebra of $\mathcal{K}^*$ of dimension greater than 4, because it contains the group algebra $\Bbbk G(\mathcal{K}^*)$ and the image of the skew-primitive element x. By the Nichols-Zoeller theorem it follows that φ is surjective and whence an isomorphism. □

Remark 2.4. *Consider the basis* $\{g^j, xg^j\}_{0 \le j \le 3}$ *of* $\mathcal{A}_4''$. *By Remark 2.2 (c) and Lemma 2.3, it follows that*

$$\begin{aligned}
\varphi(g^j) &= \alpha_j && \text{for all} \quad 0 \le j \le 3,\\
\varphi(xg^j) &= \sqrt{2}\xi(\xi^j b^* + \xi^{-j} c^* + (ab)^* + (-1)^j (ac)^*) && \text{for all} \quad 0 \le j \le 3.
\end{aligned}$$

2.1. **Description of $D(\mathcal{K}^{\rm cop})$.** In this subsection we describe the Drinfeld double $D(\mathcal{K}^{\rm cop})$. To make the notation lighter, from now on we write $D = D(\mathcal{K}^{\rm cop})$.

Proposition 2.5. *D is the $\Bbbk$-algebra generated by the elements a, b, c, d, x, g such that a, b, c, d satisfy the relations of $\mathcal{K}^{\rm cop}$; x, g satisfy the relations of $(\mathcal{A}_4'')^{\rm op\,cop}$; and all together they satisfy the following relations:*

$$\begin{aligned} ax + \xi xa &= \sqrt{2}\xi(b + gc), & bx - \xi xb &= \sqrt{2}\xi(a - gd),\\ ag &= ga, & bg &= -gb,\\ cg &= -gc, & dg &= gd,\\ cx + \xi xc &= \sqrt{2}\xi(d - ga), & dx - \xi xd &= \sqrt{2}\xi(c + gb). \end{aligned}$$

Proof. Since $(f \bowtie 1)(g \bowtie k) = fg \bowtie k$ and $(f \bowtie h)(1 \bowtie k) = f \bowtie hk$ for all $f, g \in (\mathcal{A}_4'')^{\rm op\,cop}$ and $h, k \in \mathcal{K}^{\rm cop}$, it is enough to describe the relations derived from products of the form $(1_{\mathcal{A}_4''} \bowtie h)(y \bowtie 1_{\mathcal{K}})$, where $h \in \mathcal{K}^{\rm cop}$ and $y \in (\mathcal{A}_4'')^{\rm op\,cop}$ are algebra generators.

Assume first that $y = g$. Since $h \twoheadrightarrow g = \langle \mathcal{S}_{\rm cop}^{-1}(g)g, h\rangle g = \langle \mathcal{S}(g)g, h\rangle g = \langle 1, h\rangle g = \varepsilon(h)g$, for all $h \in \mathcal{K}^{\rm cop}$, it follows that

$$(1_{\mathcal{A}_4''} \bowtie \begin{Bmatrix} a\\ b\\ c\\ d \end{Bmatrix})(g \bowtie 1_{\mathcal{K}}) = \begin{Bmatrix} a_{(1)_{\rm cop}} \twoheadrightarrow g_{(2)} \bowtie a_{(2)_{\rm cop}} \twoheadleftarrow g_{(1)}\\ b_{(1)_{\rm cop}} \twoheadrightarrow g_{(2)} \bowtie b_{(2)_{\rm cop}} \twoheadleftarrow g_{(1)}\\ c_{(1)_{\rm cop}} \twoheadrightarrow g_{(2)} \bowtie c_{(2)_{\rm cop}} \twoheadleftarrow g_{(1)}\\ d_{(1)_{\rm cop}} \twoheadrightarrow g_{(2)} \bowtie d_{(2)_{\rm cop}} \twoheadleftarrow g_{(1)} \end{Bmatrix}$$

$$= \begin{Bmatrix} a \twoheadrightarrow g \bowtie a \twoheadleftarrow g + c \twoheadrightarrow g \bowtie b \twoheadleftarrow g\\ b \twoheadrightarrow g \bowtie a \twoheadleftarrow g + d \twoheadrightarrow g \bowtie b \twoheadleftarrow g\\ a \twoheadrightarrow g \bowtie c \twoheadleftarrow g + c \twoheadrightarrow g \bowtie d \twoheadleftarrow g\\ b \twoheadrightarrow g \bowtie c \twoheadleftarrow g + d \twoheadrightarrow g \bowtie d \twoheadleftarrow g \end{Bmatrix} = g \bowtie \begin{Bmatrix} a \twoheadleftarrow g\\ b \twoheadleftarrow g\\ c \twoheadleftarrow g\\ d \twoheadleftarrow g \end{Bmatrix}$$

$$= g \bowtie \begin{Bmatrix} \langle g, \mathcal{S}_{\rm cop}^{-1}(a_{(3)_{\rm cop}})a_{(1)_{\rm cop}}\rangle a_{(2)_{\rm cop}}\\ \langle g, \mathcal{S}_{\rm cop}^{-1}(b_{(3)_{\rm cop}})b_{(1)_{\rm cop}}\rangle b_{(2)_{\rm cop}}\\ \langle g, \mathcal{S}_{\rm cop}^{-1}(c_{(3)_{\rm cop}})c_{(1)_{\rm cop}}\rangle c_{(2)_{\rm cop}}\\ \langle g, \mathcal{S}_{\rm cop}^{-1}(d_{(3)_{\rm cop}})d_{(1)_{\rm cop}}\rangle d_{(2)_{\rm cop}} \end{Bmatrix} = g \bowtie \begin{Bmatrix} \langle g, \mathcal{S}(a_{(1)})a_{(3)}\rangle a_{(2)}\\ \langle g, \mathcal{S}(b_{(1)})b_{(3)}\rangle b_{(2)}\\ \langle g, \mathcal{S}(c_{(1)})c_{(3)}\rangle c_{(2)}\\ \langle g, \mathcal{S}(d_{(1)})d_{(3)}\rangle d_{(2)} \end{Bmatrix}$$

$$= g \bowtie \begin{Bmatrix} \langle g, \mathcal{S}(a)a\rangle a + \langle g, \mathcal{S}(b)a\rangle c + \langle g, \mathcal{S}(a)c\rangle b + \langle g, \mathcal{S}(b)c\rangle d\\ \langle g, \mathcal{S}(a)b\rangle a + \langle g, \mathcal{S}(b)b\rangle c + \langle g, \mathcal{S}(a)d\rangle b + \langle g, \mathcal{S}(b)d\rangle d\\ \langle g, \mathcal{S}(c)a\rangle a + \langle g, \mathcal{S}(d)a\rangle c + \langle g, \mathcal{S}(c)c\rangle b + \langle g, \mathcal{S}(d)c\rangle d\\ \langle g, \mathcal{S}(c)b\rangle a + \langle g, \mathcal{S}(d)b\rangle c + \langle g, \mathcal{S}(c)d\rangle b + \langle g, \mathcal{S}(d)d\rangle d \end{Bmatrix}$$

$$= g \bowtie \begin{Bmatrix} \langle g, da\rangle a + \langle g, \xi ba\rangle c + \langle g, dc\rangle b + \langle g, \xi bc\rangle d\\ \langle g, db\rangle a + \langle g, \xi bb\rangle c + \langle g, dd\rangle b + \langle g, \xi bd\rangle d\\ \langle g, -\xi ca\rangle a + \langle g, aa\rangle c + \langle g, -\xi cc\rangle b + \langle g, ac\rangle d\\ \langle g, -\xi cb\rangle a + \langle g, ab\rangle c + \langle g, -\xi cd\rangle b + \langle g, ad\rangle d \end{Bmatrix}$$

$$= g \bowtie \begin{Bmatrix} \langle g, 1\rangle a + \langle g, ab\rangle c + \langle g, ab\rangle b + \langle g, 0\rangle d\\ \langle g, ac\rangle a + \langle g, 0\rangle c + \langle g, a^2\rangle b + \langle g, -ac\rangle d\\ \langle g, -ac\rangle a + \langle g, a^2\rangle c + \langle g, 0\rangle b + \langle g, ac\rangle d\\ \langle g, 0\rangle a + \langle g, ab\rangle c + \langle g, ab\rangle b + \langle g, 1\rangle d \end{Bmatrix} = g \bowtie \begin{Bmatrix} a\\ -b\\ -c\\ d \end{Bmatrix}.$$

From this equalities it follows that $ag = ga, bg = -gb, cg = -gc$ and $dg = gd$.

Suppose now that $y = x$. Using the computations above, we have that

$$\begin{Bmatrix} a\\ b\\ c\\ d \end{Bmatrix} \twoheadleftarrow x = \begin{Bmatrix} \langle x, 1\rangle a + \langle x, ab\rangle c + \langle x, ab\rangle b + \langle x, 0\rangle d\\ \langle x, ac\rangle a + \langle x, 0\rangle c + \langle x, a^2\rangle b + \langle x, -ac\rangle d\\ \langle x, -ac\rangle a + \langle x, a^2\rangle c + \langle x, 0\rangle b + \langle x, ac\rangle d\\ \langle x, 0\rangle a + \langle x, ab\rangle c + \langle x, ab\rangle b + \langle x, 1\rangle d \end{Bmatrix} = \sqrt{2}\xi \begin{Bmatrix} b + c\\ a - d\\ d - a\\ b + c \end{Bmatrix},$$

and

$$\begin{aligned}\begin{Bmatrix} a\\ b\\ c\\ d\end{Bmatrix} \twoheadrightarrow x &= \langle \mathcal{S}(x_{(1)})x_{(3)}, \begin{Bmatrix} a\\ b\\ c\\ d\end{Bmatrix}\rangle x_{(2)}\\ &= \langle \mathcal{S}(x)g, \begin{Bmatrix} a\\ b\\ c\\ d\end{Bmatrix}\rangle g + \langle \mathcal{S}(1)g, \begin{Bmatrix} a\\ b\\ c\\ d\end{Bmatrix}\rangle x + \langle \mathcal{S}(1)x, \begin{Bmatrix} a\\ b\\ c\\ d\end{Bmatrix}\rangle 1\\ &= \langle -xg^3g, \begin{Bmatrix} a\\ b\\ c\\ d\end{Bmatrix}\rangle g + \langle g, \begin{Bmatrix} a\\ b\\ c\\ d\end{Bmatrix}\rangle x + \langle x, \begin{Bmatrix} a\\ b\\ c\\ d\end{Bmatrix}\rangle 1\\ &= \langle x, \begin{Bmatrix} a\\ b\\ c\\ d\end{Bmatrix}\rangle (1-g) + \langle g, \begin{Bmatrix} a\\ b\\ c\\ d\end{Bmatrix}\rangle x = \begin{Bmatrix} \xi x\\ \sqrt{2}\xi(1-g)\\ \sqrt{2}\xi(1-g)\\ -\xi x\end{Bmatrix}.\end{aligned}$$

Hence, it follows that

$$(1_{\mathcal{A}_4''} \bowtie \begin{Bmatrix} a\\ b\\ c\\ d\end{Bmatrix})(x \bowtie 1_{\mathcal{K}}) = \begin{Bmatrix} a_{(2)} \twoheadrightarrow g \bowtie a_{(1)} \twoheadleftarrow x + a_{(2)} \twoheadrightarrow x \bowtie a_{(1)} \twoheadleftarrow 1\\ b_{(2)} \twoheadrightarrow g \bowtie b_{(1)} \twoheadleftarrow x + b_{(2)} \twoheadrightarrow x \bowtie b_{(1)} \twoheadleftarrow 1\\ c_{(2)} \twoheadrightarrow g \bowtie c_{(1)} \twoheadleftarrow x + c_{(2)} \twoheadrightarrow x \bowtie c_{(1)} \twoheadleftarrow 1\\ d_{(2)} \twoheadrightarrow g \bowtie d_{(1)} \twoheadleftarrow x + d_{(2)} \twoheadrightarrow x \bowtie d_{(1)} \twoheadleftarrow 1\end{Bmatrix}$$

$$= \begin{Bmatrix} \varepsilon(a_{(2)})g \bowtie a_{(1)} \twoheadleftarrow x + a_{(2)} \twoheadrightarrow x \bowtie a_{(1)}\\ \varepsilon(b_{(2)})g \bowtie b_{(1)} \twoheadleftarrow x + b_{(2)} \twoheadrightarrow x \bowtie b_{(1)}\\ \varepsilon(c_{(2)})g \bowtie c_{(1)} \twoheadleftarrow x + c_{(2)} \twoheadrightarrow x \bowtie c_{(1)}\\ \varepsilon(d_{(2)})g \bowtie d_{(1)} \twoheadleftarrow x + d_{(2)} \twoheadrightarrow x \bowtie d_{(1)}\end{Bmatrix}$$

$$= \begin{Bmatrix} g \bowtie a \twoheadleftarrow x + a \twoheadrightarrow x \bowtie a + c \twoheadrightarrow x \bowtie b\\ g \bowtie b \twoheadleftarrow x + b \twoheadrightarrow x \bowtie a + d \twoheadrightarrow x \bowtie b\\ g \bowtie c \twoheadleftarrow x + a \twoheadrightarrow x \bowtie c + c \twoheadrightarrow x \bowtie d\\ g \bowtie d \twoheadleftarrow x + b \twoheadrightarrow x \bowtie c + d \twoheadrightarrow x \bowtie d\end{Bmatrix}$$

$$= \begin{Bmatrix} \sqrt{2}\xi g \bowtie (b+c) + \xi x \bowtie a + \sqrt{2}\xi(1-g) \bowtie b\\ \sqrt{2}\xi g \bowtie (a-d) + \sqrt{2}\xi(1-g) \bowtie a - \xi x \bowtie b\\ \sqrt{2}\xi g \bowtie (d-a) + \xi x \bowtie c + \sqrt{2}\xi(1-g) \bowtie d\\ \sqrt{2}\xi g \bowtie (b+c) + \sqrt{2}\xi(1-g) \bowtie c - \xi x \bowtie d\end{Bmatrix}$$

$$= \begin{Bmatrix} -\xi x \bowtie a + \sqrt{2}\xi g \bowtie c + \sqrt{2}\xi \bowtie b\\ \xi x \bowtie b - \sqrt{2}\xi g \bowtie d + \sqrt{2}\xi \bowtie a\\ -\xi x \bowtie c - \sqrt{2}\xi g \bowtie a + \sqrt{2}\xi \bowtie d\\ \xi x \bowtie d + \sqrt{2}\xi g \bowtie b + \sqrt{2}\xi \bowtie c\end{Bmatrix},$$

which gives us the other four relations of D. □

2.2. **Simple left D-modules.** We begin by describing the 1-dimensional D-modules. Given a character χ on D, we denote by $\Bbbk_\chi$ the module associated with it.

Lemma 2.6. *There are four non-isomorphic 1-dimensional left D-modules given by the characters χ^j, $0 \le j \le 3$, where*

$$\chi^j(a) = \xi^j,\ \chi^j(b) = 0,\ \chi^j(c) = 0,\ \chi^j(d) = \xi^{-j},\ \chi^j(x) = 0,\ \chi^j(g) = (-1)^j.$$

Moreover, any 1-dimensional D-module is isomorphic to $\Bbbk_{\chi^j}$ for some $0 \le j \le 3$.

Proof. Straightforward. □

We describe next the simple D-modules of dimension two. For this, consider the set

$$\Lambda = \{(i,j) \in \mathbb{Z}_4 \times \mathbb{Z}_4 |\ 2i \neq j\}.$$

Clearly, $|\Lambda| = 12$.

Lemma 2.7. *For any pair $(i,j) \in \Lambda$, there exists a simple 2-dimensional left D-module $V_{i,j}$. The action on a fixed basis is given by*

$$\rho_{i,j}(a) = \begin{pmatrix} \xi^i & 0 \\ 0 & \xi^{i+3} \end{pmatrix}, \ \rho_{i,j}(b) = \begin{pmatrix} 0 & (-1)^i \\ 0 & 0 \end{pmatrix}, \ \rho_{i,j}(c) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$\rho_{i,j}(d) = \begin{pmatrix} \xi^{-i} & 0 \\ 0 & \xi^{-i+1} \end{pmatrix}, \quad \rho_{i,j}(g) = \begin{pmatrix} \xi^j & 0 \\ 0 & \xi^{j+2} \end{pmatrix},$$

$$\rho_{i,j}(x) = \begin{pmatrix} 0 & \frac{\sqrt{2}}{2}\xi(\xi^i + \xi^{3i+j}) \\ \sqrt{2}\xi(\xi^{3i} - \xi^{i+j}) & 0 \end{pmatrix},$$

Moreover, any simple 2-dimensional D-module is isomorphic to $V_{i,j}$ for some $(i,j) \in \Lambda$, and $V_{i,j} \simeq V_{k,\ell}$ if and only if $(i,j) = (k,\ell)$.

Proof. Let $\rho : D \to \operatorname{End}(V)$ be a 2-dimensional simple representation and assume that the associated matrices of the generators of D on a fixed basis of V are given by:

$$\rho(a) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \ \rho(b) = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \ \rho(c) = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix},$$
$$\rho(d) = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}, \ \rho(x) = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, \ \rho(g) = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}.$$

As $a^4 = 1 = g^4$ and $ga = ag$, $\rho(a)$ and $\rho(g)$ are simultaneously diagonalizable and, without loss of generality, we may assume that

$$\rho(a) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad \rho(d) = \begin{pmatrix} \lambda_1^{-1} & 0 \\ 0 & \lambda_2^{-1} \end{pmatrix} \text{ and } \rho(g) = \begin{pmatrix} \lambda_3 & 0 \\ 0 & \lambda_4 \end{pmatrix},$$

where $\lambda_i^4 = 1$ for $1 \leq i \leq 4$. From the relation $ac = \xi ca$ we have that

$$\begin{pmatrix} \lambda_1 c_{11} & \lambda_1 c_{12} \\ \lambda_2 c_{21} & \lambda_2 c_{22} \end{pmatrix} = \xi \begin{pmatrix} \lambda_1 c_{11} & \lambda_2 c_{12} \\ \lambda_1 c_{21} & \lambda_2 c_{22} \end{pmatrix},$$

which implies that $c_{11} = c_{22} = 0$. Similarly, the relation $gx = -xg$ implies $x_{11} = x_{22} = 0$. Since $a^2 c = b$, we must have that

$$\rho(b) = \begin{pmatrix} 0 & \lambda_1^2 c_{12} \\ \lambda_2^2 c_{21} & 0 \end{pmatrix}.$$

Also note that from the relation $c^2 = 0$, we get that $c_{12}c_{21} = 0$. Thus, by permuting the elements of the basis, we may assume that $c_{21} = 0$. Suppose $c_{12} = 0$. That is,

$$\rho(b) = \rho(c) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \rho(x) = \begin{pmatrix} 0 & x_{12} \\ x_{21} & 0 \end{pmatrix}.$$

Clearly, these modules are simple if and only if $x_{12} \neq 0$ and $x_{21} \neq 0$. As $ax = -\xi xa + \sqrt{2}\xi(b+gc)$, it follows that $x_{12}(\lambda_1 + \xi\lambda_2) = 0$ and $x_{21}(\lambda_2 + \xi\lambda_1) = 0$. Since $x_{12}x_{21} \neq 0$, we must have that $\lambda_1 + \xi\lambda_2 = \lambda_2 + \xi\lambda_1 = 0$, which implies that $\lambda_1 = \lambda_2 = 0$, a contradiction.

Therefore, we must have that $c_{12} \neq 0$. Clearly, we may also assume that $c_{12} = 1$. From the equality $ac = \xi ca$, we get that $\lambda_2 = -\xi\lambda_1$. Moreover, seeing that $cg = -gc$, we must have $\lambda_4 = -\lambda_3$. Now, the relation $ax + \xi xa = \sqrt{2}\xi(b + gc)$ yields

$$\begin{pmatrix} 0 & \lambda_1 x_{12} \\ \lambda_2 x_{21} & 0 \end{pmatrix} = -\xi \begin{pmatrix} 0 & \lambda_2 x_{12} \\ \lambda_1 x_{21} & 0 \end{pmatrix} + \sqrt{2}\xi \begin{pmatrix} 0 & \lambda_1^2 + \lambda_3 \\ 0 & 0 \end{pmatrix},$$

which implies that $x_{12} = \frac{\sqrt{2}}{2}\xi(\lambda_1 + \lambda_3\lambda_1^{-1})$. This is the same information obtained from the relation $dx - \xi xd = \sqrt{2}\xi(c + gb)$. Analogously, $cx + \xi xc = \sqrt{2}\xi(d - ga)$ yields

$$\begin{pmatrix} x_{21} & 0 \\ 0 & \xi x_{21} \end{pmatrix} = \sqrt{2}\xi \begin{pmatrix} \lambda_1^{-1} - \lambda_1\lambda_3 & 0 \\ 0 & \xi(\lambda_1^{-1} - \lambda_1\lambda_3) \end{pmatrix},$$

which gives $x_{21} = \sqrt{2}\xi(\lambda_1^{-1} - \lambda_1\lambda_3)$. Also, the relation $bx - \xi xb = \sqrt{2}\xi(a - gd)$ yields no further condition on the coefficients. Considering $g^2 = 1 + x^2$, we must have that $x_{12}x_{21} = \lambda_3^2 - 1$. In fact,

$$x_{12}x_{21} = \frac{\sqrt{2}}{2}\xi(\lambda_1 + \lambda_3\lambda_1^{-1})\sqrt{2}\xi(\lambda_1^{-1} - \lambda_1\lambda_3) = -(1 - \lambda_1^2\lambda_3 + \lambda_3\lambda_1^{-2} - \lambda_3^2) = \lambda_3^2 - 1.$$

From the discussion above, the matrices defining the action on the simple module V are of the form

$$\rho(a) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\xi\lambda_1 \end{pmatrix}, \ \rho(b) = \begin{pmatrix} 0 & \lambda_1^2 \\ 0 & 0 \end{pmatrix}, \ \rho(c) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ \rho(g) = \begin{pmatrix} \lambda_3 & 0 \\ 0 & -\lambda_3 \end{pmatrix},$$

$$\rho(d) = \begin{pmatrix} \lambda_1^{-1} & 0 \\ 0 & \xi\lambda_1^{-1} \end{pmatrix}, \ \rho(x) = \begin{pmatrix} 0 & \frac{\sqrt{2}}{2}\xi(\lambda_1 + \lambda_3\lambda_1^{-1}) \\ \sqrt{2}\xi(\lambda_1^{-1} - \lambda_1\lambda_3) & 0 \end{pmatrix},$$

with $\lambda_1^4 = 1 = \lambda_3^4$. Moreover, a direct computation shows that V is simple if and only if $\lambda_3 \neq \lambda_1^2$. Set $\lambda_1 = \xi^i$ and $\lambda_3 = \xi^j$ for some $i, j \in \mathbb{Z}_4$. Then $2i \neq j$ and consequently $(i, j) \in \Lambda$.

Finally, we show that $V_{i,j}$ is isomorphic to $V_{k,\ell}$ if and only if $(i, j) = (k, \ell)$. Let $T : V_{i,j} \to V_{k,\ell}$ be an isomorphism of D-modules; *i.e.*, $\rho_{k,\ell}(t)T = T\rho_{i,j}(t)$ for all $t \in D$. Denote by $[T] = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}$ the matrix of T with respect to the given basis. Using the action of c, we must have that $t_{21} = 0$ and $t_{11} = t_{22}$, because

$$\begin{pmatrix} t_{21} & t_{22} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & t_{11} \\ 0 & t_{21} \end{pmatrix}.$$

Moreover, acting by a we obtain that

$$\begin{pmatrix} \xi^k t_{11} & \xi^k t_{12} \\ 0 & -\xi^{k+1} t_{11} \end{pmatrix} = [\rho_{k,\ell}(a)][T] = [T][\rho_{i,j}(a)] = \begin{pmatrix} \xi^i t_{11} & -\xi^{i+1} t_{12} \\ 0 & -\xi^{i+1} t_{11} \end{pmatrix},$$

which implies $(\xi^k - \xi^i)t_{11} = 0$ and $(\xi^k + \xi^{i+1})t_{12} = 0$. Since T is an isomorphism, this implies that $\xi^k = \xi^i$, from which follows that $t_{12} = 0$. Consequently, $[T] = t_{11}I$. Finally, acting by g yields that $\xi^\ell = \xi^j$ and the claim follows. □

Remark 2.8. *Let V be a left D-module. Since D is a Hopf algebra, V^* inherits a left D-module structure by the formula $(h \cdot f)(v) = f(\mathcal{S}(h) \cdot v)$ for all $f \in V^*$, $v \in V$ and $h \in D$. A straightforward computation yields $V_{i,j}^* \simeq V_{-i+1,-j+2}$ for all $(i, j) \in \Lambda$, where the indices in the second term are considered modulo* 4.

We end this subsection by describing all simple left D-modules up to isomorphism.

Theorem 2.9. *There are sixteen simple left D-modules pairwise non-isomorphic: four* 1*-dimensional ones, given by Lemma 2.6, and twelve* 2*-dimensional ones, given by Lemma 2.7.*

Proof. Assume that there is a simple module of dimension $d > 2$ and let n be the amount of simple d-dimensional modules pairwise non-isomorphic. Since D is non-semisimple, the projective covers of the 1-dimensional modules have dimension at least 2. Thus, by Lemmata 2.6 and 2.7, it follows that

$$4.2 + 12.2^2 + nd^2 = 56 + nd^2 \leq \dim D = 64.$$

Then $nd^2 \leq 8$, which is a contradiction. □

2.3. **Projective covers of simple left D-modules.** In this subsection we denote by $\widehat{D}$ the set of isomorphism classes of simple left D-modules and by $P(V)$ the projective cover of a simple D-module V. The left regular D-module decomposes as

$$_D D \simeq \bigoplus_{V \in \widehat{D}} P(V)^{\dim V}.$$

Lemma 2.10.

(i) $V_{i,j} \otimes \Bbbk_{\chi^\ell} \simeq V_{i+\ell,j+2\ell}$ *and* $\Bbbk_{\chi^\ell} \otimes \Bbbk_{\chi^k} \simeq \Bbbk_{\chi^{\ell+k}}$ *for all* $(i,j) \in \Lambda$ *and* $k, \ell \in \mathbb{Z}_4$.
(ii) $P(V_{i,j}) \simeq V_{i,j}$ *for all* $(i,j) \in \Lambda$.
(iii) $P(\Bbbk_{\chi^\ell}) \simeq P(\Bbbk_\varepsilon) \otimes \Bbbk_{\chi^\ell}$ *and* $\dim P(\Bbbk_{\chi^\ell}) = 4$ *for all* $\ell \in \mathbb{Z}_4$.

Proof.
(i) follows by a direct computation.
(ii) Let $(i,j) \in \Lambda$ and $\mu = \chi^\ell$ be a character on D. Since $\operatorname{Hom}(P(V_{i,j}) \otimes \Bbbk_\mu, V_{i,j} \otimes \Bbbk_\mu) = \operatorname{Hom}(P(V_{i,j}), V_{i,j} \otimes \Bbbk_\mu \otimes \Bbbk_\mu^*) = \operatorname{Hom}(P(V_{i,j}), V_{i,j}) \neq 0$, and $P(V_{i,j}) \otimes \Bbbk_\mu$ is projective, it follows that $P(V_{i,j}) \otimes \Bbbk_\mu$ contains $P(V_{i,j} \otimes \Bbbk_\mu) \simeq P(V_{i+\ell,j+2\ell})$. This inclusion induces an isomorphism when tensoring with $\Bbbk_{\chi^{-\ell}}$. Hence, $P(V_{i,j}) \otimes \Bbbk_\mu \simeq P(V_{i+\ell,j+2\ell})$.

Assume there is $(i,j) \in \Lambda$ such that $\dim P(V_{i,j}) > \dim V_{i,j}$. Since D is unimodular, by [9, page 487] the socle of $P(V_{i,j})$ is $V_{i,j}$. Thus, $\dim P(V_{i,j}) \geq 2 \dim V_{i,j}$ and consequently $\dim P(V_{i-\ell,j-2\ell}) \geq 4$ for $\ell = 0,1,2,3$. Consider the set

$$I = \{(m,n) \in \Lambda : \ (m,n) \neq (i+k, j+2k) \text{ for all } 0 \leq k \leq 3\}.$$

It contains 8 elements. We now have:

$$\begin{aligned} \dim D &= \sum_{j=0}^{3} \dim P(\Bbbk_{\chi^j}) + \sum_{(m,n)\in I} 2 \dim P(V_{m,n}) + 4 \cdot 2 \dim P(V_{i,j}) \\ &\geq \sum_{j=0}^{3} \dim P(\Bbbk_{\chi^j}) + 8 \cdot 2 \cdot 2 + 4 \cdot 2 \cdot 4 = \sum_{j=0}^{3} \dim P(\Bbbk_{\chi^j}) + 64, \end{aligned}$$

a contradiction. Hence, $\dim P(V_{i,j}) = \dim V_{i,j}$ and consequently $P(V_{i,j}) \simeq V_{i,j}$.
(iii) The first assertion is well-known. Since $P(V_{i,j}) \simeq V_{i,j}$ for all $(i,j) \in \Lambda$, we have that $64 = 4 \dim P(\Bbbk_\varepsilon) + 48$, which implies that $\dim P(\Bbbk_\varepsilon) = 4$. □

We describe next the projective cover of the trivial module. Consider the 4-dimensional D-module $P = \Bbbk\{p_1, p_2, p_3, p_4\}$ whose structure is given by the following table:

(5)

$\cdot$	p_1	p_2	p_3	p_4
a	p_1	$-\xi p_2$	ξp_3	p_4
b	p_3	ξp_4	0	0
c	$-p_3$	ξp_4	0	0
d	p_1	ξp_2	$-\xi p_3$	p_4
x	$p_2 + \sqrt{2} p_3$	$-\sqrt{2} p_4$	p_4	0
g	p_1	$-p_2$	$-p_3$	p_4

Set $P_i = \Bbbk\{p_i, \ldots, p_4\}$ for $i = 1,2,3,4$. It is easy to see that $P_4 \simeq \Bbbk_\varepsilon$, $P_3/P_4 \simeq \Bbbk_\chi$, $P_2/P_3 \simeq \Bbbk_{\chi^3}$ and $P_1/P_2 \simeq \Bbbk_\varepsilon$ as D-modules. In particular, $\operatorname{Soc}(P) \simeq \Bbbk_\varepsilon \simeq \operatorname{Top}(P)$.

Lemma 2.11. *P is an indecomposable D-module with composition series given by $P = P_1 \supset P_2 \supset P_3 \supset P_4 \supset \{0\}$.*

Proof. Assume that $P = M \oplus N$ for two submodules M and N of P. Either M or N contains an element w of the form $p_1 + \alpha p_2 + \beta p_3 + \gamma p_4$ with $\alpha, \beta, \gamma \in \Bbbk$. Without loss of generality, suppose that $w \in M$. Then $p_4 = (xb) \cdot w$ and $p_3 + \xi \alpha p_4 = b \cdot w$ belong to M. This implies that $p_3, p_4 \in M$ and, as a consequence, $p_1 + \alpha p_2 \in M$. The element $x \cdot (p_1 + \alpha p_2) = p_2 + \sqrt{2} p_3 - \sqrt{2} \alpha p_4$ belongs to M as well. Hence, $p_1, p_2, p_3, p_4 \in M$ and we are done. The second assertion follows from the preceding discussion. □

Lemma 2.12. *$P \simeq P(\Bbbk_\varepsilon)$ as D-modules.*

Proof. As D is a Frobenius algebra, every projective module is injective; in particular, $P(\Bbbk_\varepsilon)$ is an injective envelope $E(\Bbbk_\lambda)$ for some character λ. Moreover, since D is unimodular, the socle and top of $P(\Bbbk_\varepsilon)$ coincide and we must have that $P(\Bbbk_\varepsilon) \simeq E(\Bbbk_\varepsilon)$. On the other hand, by Lemma 2.11, we know that P is an indecomposable module with $\mathrm{Soc}(P) \simeq \Bbbk_\varepsilon$. Thus, P embeds in $E(\Bbbk_\varepsilon)$, which implies that they are isomorphic, since they have the same dimension. □

Remark 2.13. *Using that $P(\Bbbk_{\chi^\ell}) \simeq P \otimes \Bbbk_{\chi^\ell}$ and (5), one obtains a composition series of $P(\Bbbk_{\chi^\ell})$ by tensoring the given composition series of P with $\Bbbk_{\chi^\ell}$. Set $P_j(\Bbbk_{\chi^\ell}) = P_j \otimes \Bbbk_{\chi^\ell}$ for all $1 \leq \ell \leq 3$ and $1 \leq j \leq 4$. Then, one has that $P_3(\Bbbk_{\chi^\ell})/P_4(\Bbbk_{\chi^\ell}) \simeq \Bbbk_{\chi^{\ell+1}}$, $P_2(\Bbbk_{\chi^\ell})/P_3(\Bbbk_{\chi^\ell}) \simeq \Bbbk_{\chi^{\ell+3}}$ and $P_1(\Bbbk_{\chi^\ell})/P_2(\Bbbk_{\chi^\ell}) \simeq \Bbbk_{\chi^\ell}$ as D-modules.*

As a direct consequence of the results above we have the following theorem.

Theorem 2.14. *The D-modules $P_1(\Bbbk_{\chi^\ell}) = P \otimes \Bbbk_{\chi^\ell}$ and $V_{i,j}$, with $0 \leq \ell \leq 3$ and $(i,j) \in \Lambda$, are the projective covers of the simple D-modules. In particular,*

$$ {}_D D \simeq \sum_{\ell=0}^{3} P_1(\Bbbk_{\chi^\ell}) \oplus \sum_{(i,j)\in\Lambda} V_{i,j}^2. $$
□

Remark 2.15. *For $0 \leq \ell \leq 3$, let $\{p_{i,\ell} = p_i \otimes 1\}_{1\leq i\leq 4}$ be the linear basis of $P_1(\Bbbk_{\chi^\ell})$ constructed from the linear basis $\{p_i\}_{1\leq i \leq 4}$ of P. By (5), the D-module structure of $P_1(\Bbbk_{\chi^\ell})$ can be described explicitly:*

$$ \begin{aligned} a \cdot (p_{i,\ell}) &= (a \cdot p_i) \otimes (a \cdot 1) + (c \cdot p_i) \otimes (b \cdot 1) = \xi^\ell (a \cdot p_i) \otimes 1, \\ b \cdot (p_{i,\ell}) &= (b \cdot p_i) \otimes (a \cdot 1) + (d \cdot p_i) \otimes (b \cdot 1) = \xi^\ell (b \cdot p_i) \otimes 1, \\ c \cdot (p_{i,\ell}) &= (a \cdot p_i) \otimes (c \cdot 1) + (c \cdot p_i) \otimes (d \cdot 1) = \xi^{-\ell} (c \cdot p_i) \otimes 1, \\ d \cdot (p_{i,\ell}) &= (b \cdot p_i) \otimes (c \cdot 1) + (d \cdot p_i) \otimes (d \cdot 1) = \xi^{-\ell} (d \cdot p_i) \otimes 1, \\ x \cdot (p_{i,\ell}) &= (g \cdot p_i) \otimes (x \cdot 1) + (x \cdot p_i) \otimes (1 \cdot 1) = (x \cdot p_i) \otimes 1, \\ g \cdot (p_{i,\ell}) &= (g \cdot p_i) \otimes (g \cdot 1) = (-1)^\ell (g \cdot p_i) \otimes 1. \end{aligned} \tag{6} $$

We end this subsection with the Clebsch-Gordan decomposition of the tensor product of two 2-dimensional simple D-modules.

Proposition 2.16. *Let $V_{i,j}$ and $V_{k,l}$ be 2-dimensional simple left D-modules. Then*

$$ V_{i,j} \otimes V_{k,l} \simeq \begin{cases} P(\Bbbk_{\chi^{i+k-1}}), & \text{if } 2(i+k)+j+l \equiv 0 \mod 4; \\ V_{i+k,j+l} \oplus V_{i+k+3,j+l+2}, & \text{otherwise.} \end{cases} $$

Proof. As D is a quasitriangular Hopf algebra, by Lemma 2.10 (i) and Remark 2.8, we have that $\mathrm{Hom}_D(V_{i,j} \otimes V_{k,l}, \Bbbk_{\chi^t}) = \mathrm{Hom}_D(V_{i,j}, \Bbbk_{\chi^t} \otimes V_{k,l}^*) = \mathrm{Hom}_D(V_{i,j}, V_{-k+1+t,-l+2+2t})$. By Schur's lemma, the latter is non-zero if and only if $2(i+k)+j+l = 0$ and $t = i+k-1$ in $\mathbb{Z}_4$. Also observe that $V_{i,j} \otimes V_{k,l}$ is projective.

If $2(i+k)+j+l \equiv 0 \mod 4$, then $\mathrm{Hom}(V_{i,j} \otimes V_{k,l}, \Bbbk_{\chi^{i+k-1}}) \neq 0$ and, hence, $V_{i,j} \otimes V_{k,l}$ must contain a submodule isomorphic to $P(\Bbbk_{\chi^{i+k-1}})$. As both modules have the same dimension, the inclusion induces an isomorphism.

Now, if $2(i+k)+j+l \not\equiv 0 \mod 4$, then $V_{i,j} \otimes V_{k,l}$ cannot contain a 1-dimensional module. Thus, $V_{i,j} \otimes V_{k,l}$ is isomorphic to a direct sum of two 2-dimensional simple modules. Fix $\{v_1, v_2\}$ and $\{w_1, w_2\}$ linear bases of $V_{i,j}$ and $V_{k,l}$, respectively, and set $u_1 = v_1 \otimes w_1$, $u_2 = v_1 \otimes w_2$, $u_3 = v_2 \otimes w_1$ and $u_4 = v_2 \otimes w_2$. A direct computation shows that the matrices defining the actions $\rho(a)$ and $\rho(g)$ on $V_{i,j} \otimes V_{k,l}$, with respect to the basis $\{u_i\}_{1\leq i\leq 4}$, have the following form

$$ \rho(a) = \begin{pmatrix} \xi^{i+k} & 0 & 0 & \xi^{2k} \\ 0 & \xi^{i+k+3} & 0 & 0 \\ 0 & 0 & \xi^{i+k+3} & 0 \\ 0 & 0 & 0 & \xi^{i+k+2} \end{pmatrix} \text{ and } \rho(g) = \begin{pmatrix} \xi^{j+l} & 0 & 0 & 0 \\ 0 & -\xi^{j+l} & 0 & 0 \\ 0 & 0 & -\xi^{j+l} & 0 \\ 0 & 0 & 0 & \xi^{j+l} \end{pmatrix}. $$

Looking at the eigenspace decomposition with respect to the action of a and g, it follows that necessarily $V_{i,j} \otimes V_{k,l} \simeq V_{i+k,j+l} \oplus V_{i+k+3,j+l+2}$. □

2.4. **Some indecomposable D-modules.** Let A be a finite-dimensional $\Bbbk$-algebra and $V_1, \dots, V_n$ a complete list of non-isomorphic simple left A-modules. The Ext-Quiver of A is the quiver $\mathrm{ExtQ}(A)$ with vertices $1, \dots, n$ and $\dim \mathrm{Ext}^1_A(V_i, V_j)$ arrows from the vertex i to the vertex j. Given a quiver Q with vertices $1, \dots, n$, its separation diagram is the unoriented graph with vertices $1, \dots, n, 1', \dots, n'$ and with an edge from i to j' for each arrow $i \to j$ in Q. The separation diagram of A is the separation diagram of its Ext-Quiver. It is well-known that a finite-dimensional algebra is of finite (tame) representation type if and only if its separation diagram is a disjoint union of finite (affine) Dynkin diagrams.

In this section we compute the separation diagram of D and show that D is of tame representation type. In order to do so, we use the isomorphism of abelian groups between $\mathrm{Ext}^1_A(V_i, V_j)$ and the equivalence classes of extensions $0 \to V_j \to M \to V_i \to 0$ of V_i by V_j.

2.4.1. *2-dimensional (non-simple) indecomposable modules.* Let A be the subalgebra of D generated by a, d and g. Then A is an 8-dimensional commutative algebra given by $A = \Bbbk\langle a, g : \ a^4 = 1 = g^4, ag = ga\rangle$. In particular, all simple A-modules are 1-dimensional.

Definition 2.17. *For $0 \leq \ell \leq 3$, let $M^+_\ell = \Bbbk\{m_1, m_2\}$ be the 2-dimensional D-module whose structure is given by setting $\Bbbk\, m_1 \simeq \Bbbk_{\chi^\ell}$ and*

$$a \cdot m_2 = \xi^{\ell+1}\ m_2, \qquad b \cdot m_2 = 0 = c \cdot m_2,$$
$$g \cdot m_2 = (-1)^{\ell+1} m_2, \qquad x \cdot m_2 = m_1.$$

It is easy to see that M^+_ℓ is an indecomposable D-module that contains a submodule isomorphic to $\Bbbk_{\chi^\ell}$ and verifies that $M^+_\ell / \Bbbk_{\chi^\ell} = \Bbbk_{\chi^{\ell+1}}$. Analogously, for $0 \leq \ell \leq 3$, let $M^-_\ell = \Bbbk\{m_1, m_2\}$ be the 2-dimensional D-module given by $\Bbbk\, m_1 \simeq \Bbbk_{\chi^\ell}$ and

$$a \cdot m_2 = \xi^{\ell-1}\ m_2, \qquad b \cdot m_2 = \frac{\sqrt{2}}{2}\xi^{\ell-1} m_1, \qquad c \cdot m_2 = \frac{\sqrt{2}}{2}(-\xi)^{\ell+1} m_1,$$
$$g \cdot m_2 = (-1)^{\ell-1} m_2, \qquad x \cdot m_2 = m_1.$$

Then, M^-_ℓ is an indecomposable module that contains a submodule isomorphic to $\Bbbk_{\chi^\ell}$ and satisfies that $M^-_\ell / \Bbbk_{\chi^\ell} = \Bbbk_{\chi^{\ell-1}}$.

Observe that, for all $0 \leq \ell \leq 3$, the submodule $P_3(\Bbbk_{\chi^\ell})$ of $P_1(\Bbbk_{\chi^\ell})$ is isomorphic to M^+_ℓ.

Lemma 2.18. *Fix $0 \leq \ell \leq 3$.*

(i) *Let M be a 2-dimensional indecomposable D-module that contains a submodule isomorphic to $\Bbbk_{\chi^\ell}$. Then $M \simeq M^+_\ell$ or $M \simeq M^-_\ell$.*

(ii)
$$\dim \mathrm{Ext}^1_D(\Bbbk_{\chi^k}, \Bbbk_{\chi^\ell}) = \begin{cases} 1, & \text{if } k = \ell \pm 1; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. (i) Write $\lambda = \chi^\ell$. We must have that $M \simeq \Bbbk_\lambda \oplus \Bbbk_\mu$ as A-modules, with μ some character on A. That is, M has a linear basis $\{m_1, m_2\}$ such that $\Bbbk m_1 \simeq \Bbbk_\lambda$ (as D-module) and $z \cdot m_2 = \mu(z)\, m_2$, for any $z \in A$. Since $b^2 = 0$ and $x^2 = g^2 - 1$, it follows that

$$b \cdot m_2 = \alpha\ m_1 \qquad \text{and} \qquad x \cdot m_2 = \beta\ m_1,$$

for some $\alpha, \beta \in \Bbbk$. As $a^2 b = c$, we have that $c \cdot m_2 = \lambda(a)^2 \alpha\, m_1$. Moreover, using the relation $bx - \xi xb = \sqrt{2}\xi(a - gd)$, it follows that $\mu(g) = \mu(a)^2$. Thus, μ defines a character $\widetilde{\mu}$ on D by taking $\widetilde{\mu}(x) = \widetilde{\mu}(b) = \widetilde{\mu}(c) = 0$ and $\widetilde{\mu}|_A = \mu$.

Observe that α and β are not simultaneously zero. Indeed, if $\alpha = \beta = 0$, then $M \simeq \Bbbk_\lambda \oplus \Bbbk_{\widetilde{\mu}}$ as D-modules, a contradiction since M is indecomposable. Furthermore, the relation $ax + \xi xa = \sqrt{2}\xi(b + gc)$ yields that

$$\beta(\lambda(a) + \xi\widetilde{\mu}(a)) = 2\sqrt{2}\xi\alpha. \tag{7}$$

Hence, $\beta \neq 0$.

Moreover, from the relation $xg = -gx$, we get that $\widetilde{\mu}(g) = -\lambda(g)$, since otherwise, we would get that $\beta = 0$. Thus, $\widetilde{\mu}(a)^2 = (-1)^{\ell+1}$. This implies that $\widetilde{\mu}(a) = \pm\xi^{\ell+1} = \xi^{\ell\pm1}$, and consequently $\widetilde{\mu} = \chi^{\ell\pm1}$.

If $\widetilde{\mu} = \chi^{\ell+1}$, then $\alpha = 0$, by (7). In this case, M is an indecomposable module isomorphic to M_ℓ^+. Denote this module by $M_\ell^+(\beta)$.

If $\widetilde{\mu} = \chi^{\ell-1}$, then $\lambda(a)\beta = \sqrt{2}\xi\alpha$, by (7). Thus, M is isomorphic to M_ℓ^- and (i) follows. Denote this module by $M_\ell^-(\beta)$.

(ii) By the preceding discussion, we have that $\dim \operatorname{Ext}^1_D(\Bbbk_{\chi^k}, \Bbbk_{\chi^\ell}) = 0$ if $k \neq \ell \pm 1$. On the other hand, if M is a non-trivial extension of $\Bbbk_{\chi^\ell}$ by $\Bbbk_{\chi^{\ell\pm1}}$, then $M = M_\ell^\pm(\beta)$ for some $\beta \in \Bbbk^\times$. Assume $M_\ell^\pm(\beta) \simeq M_\ell^\pm(\beta')$ as extensions, with $\beta, \beta' \in \Bbbk^\times$. Let $\{m_1, m_2\}$ and $\{m_1', m_2'\}$ be the linear bases of $M_\ell^\pm(\beta)$ and $M_\ell^\pm(\beta')$, respectively, as defined above; and write $\varphi : M_\ell^\pm(\beta) \to M_\ell^\pm(\beta')$ for the isomorphism. Then, we must have that $\varphi(m_1) = m_1'$ and $\varphi(m_2) = \gamma\, m_1' + \eta\, m_2'$ for some $\eta \neq 0$. Moreover, since $\varphi(x \cdot m_2) = \beta\varphi(m_1) = \beta\, m_1'$ equals $x \cdot \varphi(m_2) = \eta\, \beta'\, m_1'$, it follows that $\beta = \eta\beta'$. This implies that $\dim \operatorname{Ext}^1_D(\Bbbk_{\chi^{\ell\pm1}}, \Bbbk_{\chi^\ell}) = 1$ and the lemma is proved. ☐

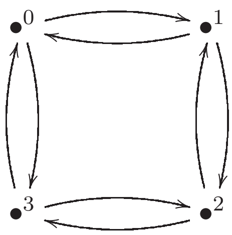
Lemma 2.19. (i) $\dim \operatorname{Ext}^1_D(V_{i,j}, V_{k,\ell}) = 0$ *for all* $(i,j), (k,\ell) \in \Lambda$.

(ii) $\dim \operatorname{Ext}^1_D(V_{i,j}, \Bbbk_{\chi^\ell}) = 0 = \dim \operatorname{Ext}^1_D(\Bbbk_{\chi^\ell}, V_{i,j})$ *for all* $(i,j) \in \Lambda$ *and* $\ell \in \mathbb{Z}_4$.

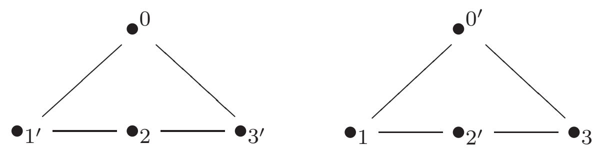
Proof. The proof follows easily from the fact that $V_{i,j}$ is projective for all $(i,j) \in \Lambda$. ☐

Corollary 2.20. *D is of tame representation type.*

Proof. Denote by i the vertex corresponding to the character χ^i for all $0 \leq i \leq 3$. Lemma 2.18 implies that $\operatorname{ExtQ}(D)$ contains the quiver



Thus, the separation diagram of D contains the quiver $A_3^{(1)} \coprod A_3^{(1)}$



Moreover, by Lemma 2.19, $\operatorname{ExtQ}(D)$ consists of the quiver above and twelve isolated points representing the simple modules $V_{i,j}$. Hence, D is of tame representation type. ☐

2.4.2. *3-dimensional indecomposable modules.* In this subsection we describe the 3-dimensional indecomposable modules as we did in the previous subsection for dimension 2.

Remark 2.21. *Let M be a non-simple indecomposable D-module. As the simple 2-dimensional D-modules $V_{i,j}$ are projective and injective for all $(i,j) \in \Lambda$, they cannot be contained in any submodule of a quotient module of M. In particular,* $\operatorname{Soc}(M)$ *and* $\operatorname{Top}(M)$ *consist of direct sums of* 1*-dimensional modules.*

Furthermore, if $0 \subsetneq \operatorname{Soc}(M) \subsetneq \operatorname{Soc}^2(M) \subsetneq \cdots \subsetneq \operatorname{Soc}^n(M) = M$ *is the socle series of M, then* $\operatorname{Soc}(M/\operatorname{Soc}^i(M))$ *does not contain any simple projective and injective module. Indeed, assume that* $\operatorname{Soc}(M/\operatorname{Soc}^i(M)) = \operatorname{Soc}^{i+1}(M)/\operatorname{Soc}^i(M)$ *contains a simple projective and injective module S. Then, by the injectivity, S is a direct summand of* $M/\operatorname{Soc}^i(M)$*. Thus, it is a quotient of* $M/\operatorname{Soc}^i(M)$ *and consequently also of M. Since S is also projective, it must be a direct summand of M, a contradiction because M is indecomposable.*

Next, we define some 3-dimensional indecomposable D-modules.

Definition 2.22. *For $0 \leq \ell \leq 3$, let $N_\ell = \Bbbk\{n_1, n_2, n_3\}$ be the 3-dimensional D-module whose structure is determined by setting $\Bbbk\, n_1 \simeq \Bbbk_{\chi^\ell}$, $\Bbbk\, n_2 \simeq \Bbbk_{\chi^{\ell+2}}$ and*

$$a \cdot n_3 = \xi^{\ell+1}\, n_3, \qquad b \cdot n_3 = \frac{\sqrt{2}}{2}\xi^{\ell+1} n_2, \qquad c \cdot n_3 = -\frac{\sqrt{2}}{2}(-\xi)^{\ell+1} n_2,$$
$$g \cdot n_3 = (-1)^{\ell+1}\, n_3, \qquad x \cdot n_3 = n_1 + n_2.$$

It holds that N_ℓ is an indecomposable D-module with socle isomorphic to $\Bbbk_{\chi^\ell} \oplus \Bbbk_{\chi^{\ell+2}}$. Moreover, one has that $N_\ell/(\Bbbk_{\chi^\ell} \oplus \Bbbk_{\chi^{\ell+2}}) \simeq \Bbbk_{\chi^{\ell+1}}$, $N_\ell/\Bbbk_{\chi^\ell} \simeq M^-_{\ell+2}$ and $N_\ell/\Bbbk_{\chi^{\ell+2}} \simeq M^+_\ell$.

Recall from Remark 2.13 that for $0 \leq \ell \leq 3$, there is a 3-dimensional indecomposable D-module $P_2(\Bbbk_{\chi^\ell})$, which is the unique maximal submodule of $P_1(\Bbbk_{\chi^\ell})$.

Lemma 2.23. *Let N be a 3-dimensional indecomposable D-module. Then $N \simeq P_2(\Bbbk_{\chi^\ell})$ or $N \simeq N_\ell$ for some $0 \leq \ell \leq 3$.*

Proof. By Remark 2.21, $\operatorname{Soc}(N)$ contains only 1-dimensional modules. Assume first that $\operatorname{Soc}(N) = \Bbbk_\lambda$ for some D-character λ. Then N embeds in the injective hull $E(\Bbbk_\lambda)$ of $\Bbbk_\lambda$, which is isomorphic to the projective cover $P_1(\Bbbk_\lambda)$, because D is unimodular. Thus, N must be isomorphic to the unique maximal submodule $P_2(\Bbbk_\lambda)$ of $P_1(\Bbbk_\lambda)$.

Suppose now that $\operatorname{Soc}(N) = \Bbbk_\lambda \oplus \Bbbk_\mu$ for some D-characters λ, μ. Then N fits into an exact sequence

$$0 \to \Bbbk_\lambda \oplus \Bbbk_\mu \to N \to \Bbbk_\tau \to 0, \tag{8}$$

for some D-character τ. In particular, $N \simeq \Bbbk_\lambda \oplus \Bbbk_\mu \oplus \Bbbk_\tau$ as A-modules. Let $\{n_1, n_2, n_3\}$ be a linear basis of N such that $\Bbbk n_1 \simeq \Bbbk_\lambda$, $\Bbbk n_2 \simeq \Bbbk_\mu$ and

$$a \cdot n_3 = \tau(a) n_3, \qquad d \cdot n_3 = \tau(d) n_3, \qquad g \cdot n_3 = \tau(g) n_3 = \tau(a)^2 n_3,$$
$$b \cdot n_3 = \beta_1 n_1 + \beta_2 n_2, \qquad c \cdot n_3 = \gamma_1 n_1 + \gamma_2 n_2, \qquad x \cdot n_3 = \theta_1 n_1 + \theta_2 n_2.$$

As $a^2 b = c$, we have that $\lambda(a)^2 \beta_1 = \gamma_1$ and $\mu(a)^2 \beta_2 = \gamma_2$. Also, using the relations $cg = -gc$, $bg = -gb$, $xg = -gx$ and $ax + \xi xa = \sqrt{2}\xi(b + gc)$, we obtain the equalities

$$\beta_1(\lambda(g) + \tau(g)) = 0, \qquad \beta_2(\mu(g) + \tau(g)) = 0,$$
$$\gamma_1(\lambda(g) + \tau(g)) = 0, \qquad \gamma_2(\mu(g) + \tau(g)) = 0,$$
$$\theta_1(\lambda(g) + \tau(g)) = 0, \qquad \theta_2(\mu(g) + \tau(g)) = 0,$$
$$\theta_1(\lambda(a) + \xi\tau(a)) = 2\sqrt{2}\xi\beta_1, \qquad \theta_2(\mu(a) + \xi\tau(a)) = 2\sqrt{2}\xi\beta_2.$$

If $\tau(g) \neq -\lambda(g)$ and $\tau(g) \neq -\mu(g)$, then $\beta_i = \gamma_i = \theta_i = 0$ and consequently $N \simeq \Bbbk_\lambda \oplus \Bbbk_\mu \oplus \Bbbk_\tau$ as D-modules, a contradiction. If $\tau(g) = -\lambda(g)$ but $\tau(g) \neq -\mu(g)$, then $\beta_2 = \gamma_2 = \theta_2 = 0$ and this implies that $N = L \oplus \Bbbk_\mu$ with $L = \Bbbk\{n_1, n_3\}$. Analogously, N is decomposable if $\tau(g) = -\mu(g)$ but $\tau(g) \neq -\lambda(g)$. Hence, $-\tau(g) = \lambda(g) = \mu(g)$ and thus $\lambda(a) = \pm\mu(a) = \pm\xi\tau(a)$ or $\lambda(a) = \pm\mu(a) = \mp\xi\tau(a)$. The same reasoning shows that $\theta_1 \neq 0 \neq \theta_2$ since otherwise N would be decomposable. So, we may assume that $\theta_1 = \theta_2 = 1$.

Moreover, $\lambda = -\mu$ since otherwise N is also decomposable. Indeed, if $\lambda = \mu$, we have that $\beta_1 = \beta_2$, from which follows that $N \simeq \Bbbk\{n_1\} \oplus \Bbbk\{v, n_3\}$ as D-modules with $v = n_1 + n_2$.

Set $\lambda = \chi^\ell$ for some $0 \leq \ell \leq 3$. Then $\mu = \chi^{\ell+2}$ and $\tau = \chi^{\ell\pm 3} = \chi^{\ell\mp 1}$. From the paragraph above it follows that $\beta_1 = 0$ for $\tau(a) = \chi^{\ell+1}(a) = \xi^{\ell+1}$, and $\beta_2 = 0$ for $\tau(a) = \chi^{\ell-1}(a) = \xi^{\ell-1}$. If $\tau = \chi^{\ell+1}$, then $\beta_2 = \frac{\sqrt{2}}{2}\xi^{\ell+1}$ and $\gamma_2 = -\frac{\sqrt{2}}{2}(-\xi)^{\ell+1}$. In such a case, $N \simeq N_\ell$. If $\tau = \chi^{\ell-1}$, the same argument shows that $N \simeq N_{\ell+2}$ and the lemma is proved. □

Remark 2.24. *Observe that $N_\ell^* \simeq P_2(\Bbbk_{\chi^{-\ell-1}})$ as D-modules, since N_ℓ^* is a 3-dimensional indecomposable module with socle $\Bbbk_{\chi^{-\ell-1}}$.*

3. The category ${}^{\mathcal{K}}_{\mathcal{K}}\mathcal{YD}$

Using the equivalence ${}_D\mathcal{M} \simeq {}^{\mathcal{K}}_{\mathcal{K}}\mathcal{YD}$, we determine in this section the simple and some indecomposable objects of ${}^{\mathcal{K}}_{\mathcal{K}}\mathcal{YD}$, and describe their braidings. Note that, by [2, Proposition 2.2.1], one has ${}^{\mathcal{K}}_{\mathcal{K}}\mathcal{YD} \simeq {}^{\mathcal{A}}_{\mathcal{A}}\mathcal{YD}$ with $\mathcal{A} = \mathcal{A}_4'' = \mathcal{K}^*$.

3.1. **Simple objects and projective covers in ${}^{\mathcal{K}}_{\mathcal{K}}\mathcal{YD}$.** Our intention is to describe the simple D-modules and their projective covers as left Yetter-Drinfeld modules over $\mathcal{K}$. To achieve our goal, we simply need to describe the coaction, since the action is given by the restriction to $\mathcal{K}$ of the action of D.

Proposition 3.1. *For $0 \leq j \leq 3$, set $\Bbbk_{\chi^j} = \Bbbk v$. Then $\Bbbk_{\chi^j} \in {}^{\mathcal{K}}_{\mathcal{K}}\mathcal{YD}$ with its structure given by*

$$a \cdot v = \xi^j v, \qquad b \cdot v = c \cdot v = 0, \qquad d \cdot v = \xi^{-j} v \quad \text{and} \quad \delta(v) = a^{2j} \otimes v.$$

Proof. Since $\Bbbk_{\chi^j}$ is 1-dimensional, we must have that $\delta(v) = h \otimes v$ for some $h \in G(\mathcal{K}) = \{1, a^2\}$. As $f \cdot v = \langle f, h\rangle v$ for all $f \in \mathcal{K}^*$ and $\langle g, a^2\rangle = -1$, it follows that $\delta(v) = a^{2j} \otimes v$. □

Proposition 3.2. *The braiding of $\Bbbk_{\chi^j}$ is given by $c(v \otimes v) = (-1)^j v \otimes v$.*

Proof. Just apply formula (3) to Proposition 3.1. □

Now we describe the Yetter-Drinfeld structure of the 2-dimensional simple modules.

Proposition 3.3. *For $(i,j) \in \Lambda$, set $V_{i,j} = \Bbbk\{e_1, e_2\}$, and write $\lambda_1 = \xi^i$ and $\lambda_2 = \xi^j$. Then $V_{i,j} \in {}^{\mathcal{K}}_{\mathcal{K}}\mathcal{YD}$ with its action given by*

$$\begin{aligned}
&a \cdot e_1 = \lambda_1 e_1, && b \cdot e_1 = 0, && c \cdot e_1 = 0, && d \cdot e_1 = \lambda_1^{-1} e_1,\\
&a \cdot e_2 = -\xi\lambda_1 e_2, && b \cdot e_2 = \lambda_1^2 e_1, && c \cdot e_2 = e_1, && d \cdot e_2 = \xi\lambda_1^{-1} e_2,
\end{aligned}$$

and its coaction by

$$\begin{aligned}
&\delta(e_1) = 1 \otimes e_1 - 2\lambda_1 ac \otimes e_2, && \delta(e_2) = a^2 \otimes e_2, && \text{for } \lambda_2 = 1,\\
&\delta(e_1) = a^2 \otimes e_1 + 2\lambda_1 ab \otimes e_2, && \delta(e_2) = 1 \otimes e_2, && \text{for } \lambda_2 = -1,\\
&\delta(e_1) = d \otimes e_1 + (\lambda_1^3 - \xi\lambda_1)c \otimes e_2, && \delta(e_2) = a \otimes e_2 + \frac{1}{2}(\lambda_1 + \xi\lambda_1^3) b \otimes e_1, && \text{for } \lambda_2 = \xi,\\
&\delta(e_1) = a \otimes e_1 + (\lambda_1^3 + \xi\lambda_1)b \otimes e_2, && \delta(e_2) = d \otimes e_2 + \frac{1}{2}(\lambda_1 - \xi\lambda_1^3) c \otimes e_1, && \text{for } \lambda_2 = -\xi.
\end{aligned}$$

Proof. Let $\{v_i\}_{1\leq i\leq 8}$ be a basis of $\mathcal{K}$ and $\{v^i\}_{1\leq i\leq 8}$ its dual basis. Recall that $\delta(v) = \sum_{i=1}^{8} v_i \otimes v^i \cdot v$ for all $v \in V_{i,j}$. Then, by the isomorphism from Lemma 2.3, we have that

$$\begin{aligned}
\delta(e_1) &= \sum_{i=0}^{3} (g^i)^* \otimes g^i \cdot e_1 + \sum_{i=0}^{3} (xg^i)^* \otimes xg^i \cdot e_1\\
&= \sum_{i=0}^{3} (g^i)^* \otimes \lambda_2^i e_1 + \sum_{i=0}^{3} (xg^i)^* \otimes (-\lambda_2)^i x_{21} e_2,\\
\delta(e_2) &= \sum_{i=0}^{3} (g^i)^* \otimes g^i \cdot e_2 + \sum_{i=0}^{3} (xg^i)^* \otimes xg^i \cdot e_2\\
&= \sum_{i=0}^{3} (g^i)^* \otimes (-\lambda_2)^i e_2 + \sum_{i=0}^{3} (xg^i)^* \otimes \lambda_2^i x_{12} e_1,
\end{aligned} \tag{9}$$

where $(g^i)^* = \frac{1}{4}(1 + \xi^i a + (-\xi)^i d + (-1)^i a^2)$, $(xg^i)^* = \frac{1}{4\sqrt{2}\xi}((-\xi)^i b + \xi^i c + ab + (-1)^i ac)$ for all $0 \leq i \leq 3$, and $x_{21} = \sqrt{2}\xi(\lambda_1^3 - \lambda_1\lambda_2)$, $x_{12} = \frac{\sqrt{2}}{2}\xi(\lambda_1 + \lambda_1^3\lambda_2)$.

Computing the formulae (9) for each $0 \leq i,j \leq 3$, we obtain the explicit coactions presented in the statement of the proposition. □

Next, we describe the braiding of the simple modules $V_{i,j}$ in ${}^{\mathcal{K}}_{\mathcal{K}}\mathcal{YD}$. We use a matrix-like notation to state it in a compact form. Its proof follows by a direct computation using formula (3) and Proposition 3.3.

Proposition 3.4. *Set $\lambda_1 = \xi^i$ and $\lambda_2 = \xi^j$. The braiding of $V_{i,j} \in {}^{\mathcal{K}}_{\mathcal{K}}\mathcal{YD}$ is given by the following formulae:*

(i) *For $j = 0$, $i \in \{1,3\}$:*

$$c(\left\{ \begin{matrix} e_1 \\ e_2 \end{matrix} \right\} \otimes \left\{ \begin{matrix} e_1 & e_2 \end{matrix} \right\}) = \left\{ \begin{matrix} e_1 \otimes e_1 & e_2 \otimes e_1 + 2e_1 \otimes e_2 \\ -e_1 \otimes e_2 & e_2 \otimes e_2 \end{matrix} \right\}.$$

(ii) *For $j = 2$, $i \in \{0,2\}$:*

$$c(\left\{ \begin{matrix} e_1 \\ e_2 \end{matrix} \right\} \otimes \left\{ \begin{matrix} e_1 & e_2 \end{matrix} \right\}) = \left\{ \begin{matrix} e_1 \otimes e_1 & -e_2 \otimes e_1 + 2e_1 \otimes e_2 \\ e_1 \otimes e_2 & e_2 \otimes e_2 \end{matrix} \right\}.$$

(iii) *For $j = 1$ and i arbitrary:*

$$c(\left\{ \begin{matrix} e_1 \\ e_2 \end{matrix} \right\} \otimes \left\{ \begin{matrix} e_1 & e_2 \end{matrix} \right\}) = \left\{ \begin{matrix} \lambda_1^3 e_1 \otimes e_1 & \xi\lambda_1^3 e_2 \otimes e_1 + (\lambda_1^3 - \xi\lambda_1) e_1 \otimes e_2 \\ \lambda_1 e_1 \otimes e_2 & -\xi\lambda_1 e_2 \otimes e_2 + \frac{1}{2}(\lambda_1^3 + \xi\lambda_1) e_1 \otimes e_1 \end{matrix} \right\}.$$

(iv) *For $j = 3$ and i arbitrary:*

$$c(\left\{ \begin{matrix} e_1 \\ e_2 \end{matrix} \right\} \otimes \left\{ \begin{matrix} e_1 & e_2 \end{matrix} \right\}) = \left\{ \begin{matrix} \lambda_1 e_1 \otimes e_1 & -\xi\lambda_1 e_2 \otimes e_1 + (\lambda_1 + \xi\lambda_1^3) e_1 \otimes e_2 \\ \lambda_1^3 e_1 \otimes e_2 & \xi\lambda_1^3 e_2 \otimes e_2 + \frac{1}{2}(\lambda_1 - \xi\lambda_1^3) e_1 \otimes e_1 \end{matrix} \right\}.$$

□

Remark 3.5. *All the braidings given by Proposition 3.4 are not of diagonal type. See the Appendix of the first* `arXiv` *version of this paper.*

We end this section with the description of the projective covers of the 1-dimensional modules as objects in ${}^{\mathcal{K}}_{\mathcal{K}}\mathcal{YD}$ and its braidings.

Recall that the module $P_1(\Bbbk_{\chi^j})$ is isomorphic to $P(\Bbbk_{\chi^j})$, for $0 \leq j \leq 3$.

Proposition 3.6. *$P_1(\Bbbk_{\chi^j}) \in {}^{\mathcal{K}}_{\mathcal{K}}\mathcal{YD}$ with its action given by (5), (6); and its coaction by*

$$\begin{aligned} \delta(p_{1,j}) &= (a^2)^j \otimes p_{1,j} - \frac{\xi\sqrt{2}}{2}(a^2)^j ac \otimes (p_{2,j} + \sqrt{2}p_{3,j}), \\ \delta(p_{2,j}) &= (a^2)^{j+1} \otimes p_{2,j} + \xi(a^2)^j ab \otimes p_{4,j}, \\ \delta(p_{3,j}) &= (a^2)^{j+1} \otimes p_{3,j} - \frac{\xi\sqrt{2}}{2}(a^2)^j ab \otimes p_{4,j}, \\ \delta(p_{4,j}) &= (a^2)^j \otimes p_{4,j}. \end{aligned}$$

Proof. By the same reason presented in the beginning of the proof of the Proposition 3.3, we obtain that

$$\begin{aligned} \delta(p_{1,j}) &= \sum_{i=0}^{3} (g^i)^* \otimes g^i \cdot p_{1,j} + \sum_{i=0}^{3} (xg^i)^* \otimes xg^i \cdot p_{1,j} \\ &= \sum_{i=0}^{3} (g^i)^* \otimes ((-1)^j)^i p_{1,j} + \sum_{i=0}^{3} (xg^i)^* \otimes ((-1)^{j+1})^i (p_{2,j} + \sqrt{2}p_{3,j}) \\ &= (a^2)^j \otimes p_{1,j} - \frac{\xi\sqrt{2}}{2}(a^2)^j ac \otimes (p_{2,j} + \sqrt{2}p_{3,j}); \\ \delta(p_{2,j}) &= \sum_{i=0}^{3} (g^i)^* \otimes g^i \cdot p_{2,j} + \sum_{i=0}^{3} (xg^i)^* \otimes xg^i \cdot p_{2,j} \\ &= \sum_{i=0}^{3} (g^i)^* \otimes ((-1)^{j+1})^i p_{2,j} + \sum_{i=0}^{3} (xg^i)^* \otimes (-\sqrt{2}((-1)^j)^i) p_{4,j} \end{aligned}$$

$$
\begin{aligned}
&= (a^2)^{j+1} \otimes p_{2,j} + \xi (a^2)^j ab \otimes p_{4,j};\\
\delta(p_{3,j}) &= \sum_{i=0}^{3} (g^i)^* \otimes g^i \cdot p_{3,j} + \sum_{i=0}^{3} (xg^i)^* \otimes xg^i \cdot p_{3,j}\\
&= \sum_{i=0}^{3} (g^i)^* \otimes ((-1)^{j+1})^i p_{3,j} + \sum_{i=0}^{3} (xg^i)^* \otimes ((-1)^j)^i p_{4,j}\\
&= (a^2)^{j+1} \otimes p_{3,j} - \frac{\xi\sqrt{2}}{2} (a^2)^j ab \otimes p_{4,j};\\
\delta(p_{4,j}) &= \sum_{i=0}^{3} (g^i)^* \otimes g^i \cdot p_{4,j} + \sum_{i=0}^{3} (xg^i)^* \otimes xg^i \cdot p_{4,j} = \sum_{i=0}^{3} (g^i)^* \otimes ((-1)^j)^i p_{4,j}\\
&= (a^2)^j \otimes p_{4,j}.
\end{aligned}
$$

□

The following result holds by a straightforward computation using (3) and the coactions given in Proposition 3.6.

Proposition 3.7. *Fix* $0 \le j \le 3$. *The braiding of* $P_1(\Bbbk_{\chi^j}) \in {}^{\mathcal{K}}_{\mathcal{K}}\mathcal{YD}$ *is given by the formulae:*

$$
c(p_{1,j} \otimes \begin{Bmatrix} p_{1,j} \\ p_{2,j} \\ p_{3,j} \\ p_{4,j} \end{Bmatrix}) = \begin{Bmatrix} (-1)^j p_{1,j} \\ p_{2,j} \\ p_{3,j} \\ (-1)^j p_{4,j} \end{Bmatrix} \otimes p_{1,j} + \frac{\sqrt{2}}{2} \begin{Bmatrix} -p_{3,j} \\ (-1)^j p_{4,j} \\ 0 \\ 0 \end{Bmatrix} \otimes (p_{2,j} + \sqrt{2} p_{3,j}),
$$

$$
c(p_{2,j} \otimes \begin{Bmatrix} p_{1,j} \\ p_{2,j} \\ p_{3,j} \\ p_{4,j} \end{Bmatrix}) = \begin{Bmatrix} p_{1,j} \\ (-1)^{j+1} p_{2,j} \\ (-1)^{j+1} p_{3,j} \\ p_{4,j} \end{Bmatrix} \otimes p_{2,j} + \begin{Bmatrix} (-1)^{j+1} p_{3,j} \\ -p_{4,j} \\ 0 \\ 0 \end{Bmatrix} \otimes p_{4,j},
$$

$$
c(p_{3,j} \otimes \begin{Bmatrix} p_{1,j} \\ p_{2,j} \\ p_{3,j} \\ p_{4,j} \end{Bmatrix}) = \begin{Bmatrix} p_{1,j} \\ (-1)^{j+1} p_{2,j} \\ (-1)^{j+1} p_{3,j} \\ p_{4,j} \end{Bmatrix} \otimes p_{3,j} + \frac{\sqrt{2}}{2} \begin{Bmatrix} (-1)^j p_{3,j} \\ p_{4,j} \\ 0 \\ 0 \end{Bmatrix} \otimes p_{4,j},
$$

$$
c(p_{4,j} \otimes \begin{Bmatrix} p_{1,j} \\ p_{2,j} \\ p_{3,j} \\ p_{4,j} \end{Bmatrix}) = \begin{Bmatrix} (-1)^j p_{1,j} \\ p_{2,j} \\ p_{3,j} \\ (-1)^j p_{4,j} \end{Bmatrix} \otimes p_{4,j}.
$$

□

4. Nichols algebras in ${}^{\mathcal{K}}_{\mathcal{K}}\mathcal{YD}$

In this section we determine all finite-dimensional Nichols algebras of simple modules over $\mathcal{K}$. They consist of exterior algebras of dimension 2 and 8-dimensional algebras with triangular braiding. Indeed, since all objects in ${}^{\mathcal{K}}_{\mathcal{K}}\mathcal{YD}$ can be described as objects in the category of Yetter-Drinfeld modules over the pointed Hopf algebra $\mathcal{K}^* = \mathcal{A}_4''$, by [14] it follows that the associated braiding is triangular. To the best of our knowledge, these 8-dimensional Nichols algebras constitute new examples. They are isomorphic to quantum linear spaces as algebras, but not as coalgebras since the braiding differs; in our case, the braiding is not diagonal. It remains an open question if they are twist equivalent and in such a case, in which category.

We begin by studying the Nichols algebras of the 1-dimensional simple modules and their projective covers.

Lemma 4.1. *Let $0 \leq j \leq 3$. The Nichols algebras $\mathfrak{B}(\Bbbk_{\chi^j})$ associated with $\Bbbk_{\chi^j} = \Bbbk x$ are:*

$$\mathfrak{B}(\Bbbk_{\chi^j}) = \begin{cases} \Bbbk[x], & \text{for } j = 0, 2, \\ \Bbbk[x]/(x^2) = \bigwedge \Bbbk_{\chi^j}, & \text{for } j = 1, 3. \end{cases}$$

Proof. Immediate, since the braiding $c = (-1)^j \tau$, where τ represents the usual flip. □

Corollary 4.2. *Let $W = \Bbbk_{\chi^{j_1}} \oplus \cdots \oplus \Bbbk_{\chi^{j_t}}$ be a direct sum of 1-dimensional modules with $j_s \in \{1, 3\}$ for all $1 \leq s \leq t$. Then $\mathfrak{B}(W) = \bigwedge W \simeq \mathfrak{B}(\Bbbk_{\chi^{j_1}}) \underline{\otimes} \cdots \underline{\otimes} \mathfrak{B}(\Bbbk_{\chi^{j_t}})$.*

Proof. If $j_s \in \{1, 3\}$ for all $1 \leq s \leq t$, then the braiding on $W \otimes W$ is $-\tau$ and therefore $\mathfrak{B}(W) = \bigwedge W$. The last assertion follows from [6, Theorem 2.2], because $c^2_{W \otimes W} = \mathrm{id}_{W \otimes W}$. Indeed, if $v \in \Bbbk_{\chi^{j_r}}$ and $w \in \Bbbk_{\chi^{j_s}}$, then $c(v \otimes w) = (a^2)^{j_r} w \otimes v = (-1)^{j_r j_s} w \otimes v$. □

Lemma 4.3. *Let $0 \leq j \leq 3$. Then $\mathfrak{B}(P(\Bbbk_{\chi^j}))$ is infinite-dimensional.*

Proof. In all cases, the braiding on $P(\Bbbk_{\chi^j}) \otimes P(\Bbbk_{\chi^j})$ contains an eigenvector of eigenvalue 1. The claim then follows by Remark 1.2. Indeed, by Proposition 3.7 we have that $c(p_{4,j} \otimes p_{4,j}) = p_{4,j} \otimes p_{4,j}$ for $j = 0, 2$, and $c(p_{3,j} \otimes p_{3,j}) = p_{3,j} \otimes p_{3,j}$ for $j = 1, 3$. □

Before we describe the Nichols algebras associated with 2-dimensional simple modules, we analyze the Nichols algebras of non-simple indecomposable modules. It turns out that they are all infinite-dimensional.

Remark 4.4. *Let $V \in {}^{\mathcal{K}}_{\mathcal{K}}\mathcal{YD}$ be a finite-dimensional module such that $\dim \mathfrak{B}(V) < \infty$. Since taking the Nichols algebra defines a functor between the category of braided vector spaces and the category of braided Hopf algebras, see [2], it follows that $\dim \mathfrak{B}(W) < \infty$ for all $W \in \mathrm{Soc}(V)$ or $W \in \mathrm{Top}(V)$. Furthermore, let $0 \subseteq \mathrm{Soc}(V) \subseteq \mathrm{Soc}^2(V) \subseteq \cdots \subseteq \mathrm{Soc}^n(V) = V$ be a socle series of V. Then $\dim \mathfrak{B}(V/\mathrm{Soc}^i(V))$, $\dim \mathfrak{B}(\mathrm{Soc}^i(V))$ and $\dim \mathfrak{B}(\mathrm{Soc}(V/\mathrm{Soc}^i(V)))$ are finite for all $1 \leq i \leq n$.*

Theorem 4.5. *Let $M \in {}^{\mathcal{K}}_{\mathcal{K}}\mathcal{YD}$ be a finite-dimensional non-simple indecomposable module. Then $\mathfrak{B}(M)$ is infinite-dimensional.*

Proof. We prove the claim by induction on $\dim M$. Assume first that $\dim M = 2$. By Lemma 2.18 (*i*), we have that $M \simeq M_\ell^+$ or $M \simeq M_\ell^-$ for some $0 \leq \ell \leq 3$. Since $\mathrm{Soc}(M_\ell^+) = \Bbbk_{\chi^\ell}$, $\mathrm{Top}(M_\ell^+) = \Bbbk_{\chi^{\ell+1}}$ and $\mathrm{Soc}(M_\ell^-) = \Bbbk_{\chi^\ell}$, $\mathrm{Top}(M_\ell^-) = \Bbbk_{\chi^{\ell-1}}$, by Lemma 4.1 and Remark 4.4, it follows that $\mathfrak{B}(M)$ is infinite-dimensional.

Assume now that $\dim M = n \geq 3$ and suppose that $\dim \mathfrak{B}(N)$ is infinite for all indecomposable module of dimension less than n. By Remark 2.21, $\mathrm{Soc}(M)$ consists of 1-dimensional modules. Let $\bar{N}$ be a simple module contained in $\mathrm{Soc}(M/\mathrm{Soc}(M))$ and denote by N the corresponding submodule of M. Also $\dim \bar{N} = 1$ by Remark 2.21. If $\mathrm{Soc}(M) = \Bbbk_\lambda$, then N is an indecomposable module of dimension 2. The previous paragraph implies that $\dim \mathfrak{B}(N)$ is infinite and consequently $\dim \mathfrak{B}(M)$ is infinite. Assume that $\mathrm{Soc}(M)$ contains more than one simple module and let $\Bbbk_\lambda \subset \mathrm{Soc}(M)$. If $N/\Bbbk_\lambda$ is semisimple, then N contains an indecomposable module of dimension 2 and whence $\dim \mathfrak{B}(N/\Bbbk_\lambda)$ is infinite. This implies again that $\dim \mathfrak{B}(N)$ and $\dim \mathfrak{B}(M)$ are both infinite. If $N/\Bbbk_\lambda$ is not semisimple, then it contains an indecomposable module of dimension less than n. By induction, $\dim \mathfrak{B}(N/\Bbbk_\lambda)$ is infinite and the theorem follows. □

Remark 4.6. *Let $V \in {}^{\mathcal{K}}_{\mathcal{K}}\mathcal{YD}$ be such that $\dim \mathfrak{B}(V)$ is finite. Then by Theorem 4.5, V is necessarily semisimple. In these notes we analyse only Nichols algebras over simple modules, since the case of semisimple modules demands much more work to be carried out. A first approach could be done by studying the Yetter-Drinfeld submodules $\mathrm{ad}^n(V)(W)$ of a given Nichols algebra $\mathfrak{B}(V \oplus W)$ with V and W simple modules, see [7] for details. A direct computation shows that $\mathfrak{B}(V \oplus W)$ is infinite-dimensional for $V = \Bbbk_\chi$, $W = V_{3,1}$, $V_{3,3}$, and $V = \Bbbk_{\chi^3}$, $W = V_{2,1}$, $V_{2,3}$. In fact, $\mathrm{ad}(\Bbbk_\chi)(V_{3,1}) \simeq V_{0,3}$, $\mathrm{ad}(\Bbbk_\chi)(V_{3,3}) \simeq V_{0,1}$, $\mathrm{ad}(\Bbbk_{\chi^3})(V_{2,1}) \simeq V_{1,3}$ and $\mathrm{ad}(\Bbbk_{\chi^3})(V_{2,3}) \simeq V_{1,1}$.*

Now, we analyze the Nichols algebras associated with 2-dimensional simple modules.

Lemma 4.7. *Let $\Lambda' = \Lambda \smallsetminus \{(2,1),(3,1),(2,3),(3,3)\}$. Then $\mathfrak{B}(V_{i,j})$ is infinite-dimensional for all $(i,j) \in \Lambda'$.*

Proof. In all cases, the braiding of $V_{i,j}$ contains an eigenvector $w \otimes w$ of eigenvalue 1, hence the lemma follows by Remark 1.2. Indeed, for $(i,j) = (1,1)$ or $(1,3)$, the element $w = e_1 + \sqrt{2}\xi e_2$ do the job. For the other cases, take $w = e_1$. □

Next, we describe the Nichols algebras associated with the pairs in $\Lambda \backslash \Lambda'$ by generators and relations. It turns out that all of them are isomorphic to algebras associated with quantum linear planes.

Recall that every graded Hopf algebra in ${}^{\mathcal{K}}_{\mathcal{K}}\mathcal{YD}$ satisfies the Poincaré duality [2, Proposition 3.2.2], that is, for $R = \bigoplus_{i=0}^{N} R^i$ with $R^N \neq \{0\}$, it holds that $\dim R^i = \dim R^{N-i}$.

Proposition 4.8. *$\mathfrak{B}(V_{2,1})$ is the algebra generated by the elements x, y satisfying the following relations*

$$x^2 = 0, \qquad xy + \xi yx = 0, \qquad y^4 = 0. \tag{10}$$

In particular $\dim \mathfrak{B}(V_{2,1}) = 8$.

Proof. Write $x = e_1$, $y = e_2$ for the linear generators of $V_{2,1}$. Then, by Proposition 3.4, we have that

$$c(\left\{ \begin{matrix} x \\ y \end{matrix} \right\} \otimes \left\{ \begin{matrix} x & y \end{matrix} \right\}) = \left\{ \begin{matrix} -x \otimes x & -\xi y \otimes x + (\xi - 1) x \otimes y \\ -x \otimes y & \xi y \otimes y - \frac{1}{2}(1+\xi) x \otimes x \end{matrix} \right\}.$$

Hence, the relations (10) must hold in $\mathfrak{B}(V_{2,1})$. Indeed, the first two ones are easily checked since they are primitive elements of degree 2. Let us focus in the last one; we show that it is also primitive, modulo relations in degree 2. Since

$$\Delta(y^2) = (y \otimes 1 + 1 \otimes y)(y \otimes 1 + 1 \otimes y) = y^2 \otimes 1 + (1+\xi) y \otimes y - \frac{1}{2}(1+\xi) x \otimes x + 1 \otimes y^2,$$

we get that

$$\begin{aligned} \Delta(y^3) &= (y \otimes 1 + 1 \otimes y)(y^2 \otimes 1 + (1+\xi) y \otimes y - \frac{1}{2}(1+\xi) x \otimes x + 1 \otimes y^2) \\ &= y^3 \otimes 1 + \frac{1}{2}(1+\xi) xy \otimes x + \xi y^2 \otimes y - \frac{1}{2}(1+\xi) x \otimes xy + \xi y \otimes y^2 + 1 \otimes y^3 \end{aligned}$$

because $c(y \otimes y^2) = (x^2 - y^2) \otimes y + \frac{1}{2}(1-\xi)(yx - xy) \otimes x$. Thus, as $c(y \otimes xy) = -\xi xy \otimes y + \frac{1}{2}(1+\xi) x^2 \otimes x$ and $c(y \otimes y^3) = \xi(-y^3 + x^2 y + yx^2 - xyx) \otimes y + \frac{1}{2}(1+\xi)(y^2 x - x^3 - yxy + xy^2) \otimes x$, it follows that

$$\Delta(y^4) = (y \otimes 1 + 1 \otimes y)\Delta(y^3) = y^4 \otimes 1 + 1 \otimes y^4.$$

Hence, we have a graded braided Hopf algebra epimorphism $\pi : T(V_{2,1})/I \twoheadrightarrow \mathfrak{B}(V_{2,1})$, where I is the two-sided ideal generated by the relations (10). Set $R = T(V_{2,1})/I$, then clearly $R = \bigoplus_{i=0}^{4} R^i$ with $R^4 \neq 0$, $R^0 \simeq \Bbbk$ and $R^1 \simeq V_{2,1}$. By the Poincaré duality, we have that π is injective in degree 0, 1, 3 and 4. In order to prove that π is an isomorphism, it remains to show that π is injective in degree 2. This is equivalent to check that the relations in degree 2 in the Nichols algebra are just $x^2 = 0$ and $xy + \xi yx = 0$, which follows by a direct computation using the braiding. □

The proof of the next three propositions follows the same lines of Proposition 4.8. Thus, for their proof we only show that the defining relations hold in the Nichols algebra. We also write $x = e_1$, $y = e_2$ for the linear generators of each $V_{i,j}$ in its respective proof.

Proposition 4.9. *$\mathfrak{B}(V_{2,3})$ is the algebra generated by the elements x, y satisfying the following relations:*

$$x^2 = 0, \qquad xy - \xi yx = 0, \qquad y^4 = 0.$$

In particular, $\dim \mathfrak{B}(V_{2,3}) = 8$.

Proof. Using the braiding given by Proposition 3.4, we see that

$$\begin{aligned}\Delta(x^2) &= x^2\otimes 1 + 1\otimes x^2,\\ \Delta(xy) &= xy\otimes 1 - \xi x\otimes y + \xi y\otimes x + 1\otimes xy,\\ \Delta(yx) &= yx\otimes 1 + y\otimes x - x\otimes y + 1\otimes yx,\\ \Delta(y^2) &= y^2\otimes 1 + (1-\xi)y\otimes y + \frac{1}{2}(\xi-1)x\otimes x + 1\otimes y^2.\end{aligned}$$

Thus, the relations $x^2 = 0$ and $xy - \xi yx = 0$ must hold in $\mathfrak{B}(V_{2,3})$, since both elements are primitive of degree 2. Let us check that the relation $y^4 = 0$ also holds. Since $c(y\otimes y^2) = (x^2 - y^2)\otimes y + \frac{1}{2}(1+\xi)(yx - xy)\otimes x$, we have that

$$\Delta(y^3) = y^3\otimes 1 + \frac{1}{2}(1-\xi)xy\otimes x - \xi y^2\otimes y + \frac{1}{2}(\xi-1)x\otimes xy - \xi y\otimes y^2 + 1\otimes y^3.$$

Then,

$$\Delta(y^4) = \Delta(y^3)(y\otimes 1 + 1\otimes y) = y^4\otimes 1 + 1\otimes y^4,$$

because $c(y\otimes y^3) = -\xi(-y^3 + x^2y + yx^2 - xyx)\otimes y + \frac{1}{2}(1-\xi)(y^2x - x^3 - yxy + xy^2)\otimes x$ and $c(y\otimes xy) = \xi xy\otimes y + \frac{1}{2}(1-\xi)x^2\otimes x$. Hence, the relation $y^4 = 0$ must hold in $\mathfrak{B}(V_{2,3})$. □

Proposition 4.10. *$\mathfrak{B}(V_{3,1})$ is the algebra generated by the elements x, y satisfying the following relations:*

$$x^2 - 2y^2 = 0, \qquad xy + yx = 0, \qquad y^4 = 0.$$

In particular, $\dim \mathfrak{B}(V_{3,1}) = 8$.

Proof. By Proposition 3.4, we get that

$$\Delta(x^2) = x^2\otimes 1 + (1+\xi)x\otimes x + 1\otimes x^2, \qquad \Delta(xy) = xy\otimes 1 + \xi x\otimes y - y\otimes x + 1\otimes xy,$$
$$\Delta(yx) = yx\otimes 1 + y\otimes x - \xi x\otimes y + 1\otimes yx, \quad \Delta(y^2) = y^2\otimes 1 + \frac{1}{2}(1+\xi)x\otimes x + 1\otimes y^2.$$

From this formulae it follows that the relations $x^2 - 2y^2 = 0$ and $xy + yx = 0$ hold in $\mathfrak{B}(V_{3,1})$. Using that $c(y\otimes y^2) = (y^2 - x^2)\otimes y - \frac{1}{2}(1+\xi)(xy + yx)\otimes x$, we have that

$$\Delta(y^3) = y^3\otimes 1 - \frac{1}{2}(1+\xi)xy\otimes x - y^2\otimes y - \frac{1}{2}(1-\xi)x\otimes xy + y\otimes y^2 + 1\otimes y^3,$$

and consequently $\Delta(y^4) = y^4\otimes 1 + 1\otimes y^4$, because $c(y\otimes y^3) = (yx^2 + xyx - y^3 + x^2y)\otimes y + \frac{1}{2}(1+\xi)(yxy - x^3 + xy^2 + y^2x)\otimes x$ and $c(y\otimes xy) = \xi xy\otimes y + \frac{1}{2}(1-\xi)x^2\otimes x$. Hence, the relation $y^4 = 0$ also holds in $\mathfrak{B}(V_{3,1})$. □

Proposition 4.11. *$\mathfrak{B}(V_{3,3})$ is the algebra generated by the elements x, y satisfying the following relations:*

$$x^2 - 2y^2 = 0, \qquad xy + yx = 0, \qquad y^4 = 0.$$

In particular, $\dim \mathfrak{B}(V_{3,3}) = 8$.

Proof. Using the braiding given in Proposition 3.4, we have that

$$\Delta(x^2) = x^2\otimes 1 + (1-\xi)x\otimes x + 1\otimes x^2, \qquad \Delta(xy) = xy\otimes 1 - \xi x\otimes y - y\otimes x + 1\otimes xy,$$
$$\Delta(yx) = yx\otimes 1 + y\otimes x + \xi x\otimes y + 1\otimes yx, \quad \Delta(y^2) = y^2\otimes 1 + \frac{1}{2}(1-\xi)x\otimes x + 1\otimes y^2.$$

This gives us that the relations $x^2 - 2y^2 = 0$ and $xy + yx = 0$ must hold in $\mathfrak{B}(V_{3,3})$. Since $c(y\otimes y^2) = (y^2 - x^2)\otimes y + \frac{1}{2}(\xi-1)(xy + yx)\otimes x$, it follows that

$$\Delta(y^3) = y^3\otimes 1 + \frac{1}{2}(\xi-1)xy\otimes x - y^2\otimes y - \frac{1}{2}(1+\xi)x\otimes xy + y\otimes y^2 + 1\otimes y^3,$$

and consequently $\Delta(y^4) = y^4 \otimes 1 + 1 \otimes y^4$, because $c(y \otimes y^3) = (yx^2 + xyx - y^3 + x^2y) \otimes y + \frac{1}{2}(1-\xi)(yxy - x^3 + xy^2 + y^2x) \otimes x$ and $c(y \otimes xy) = -\xi xy \otimes y + \frac{1}{2}(1+\xi)x^2 \otimes x$. □

We end this section with the characterization of the finite-dimensional Nichols algebras over indecomposable objects in ${}^{\mathcal{K}}_{\mathcal{K}}\mathcal{YD}$.

Proof of Theorem A. Let V be an indecomposable module such that $\mathfrak{B}(V)$ is finite-dimensional. Then by Theorem 4.5, V is necessarily simple. The claim then follows by Lemmata 4.1 and 4.7, and Propositions 4.8, 4.9, 4.10 and 4.11. Clearly, Nichols algebras over distinct families are pairwise non-isomorphic, since they are generated by the set of primitive elements which are non-isomorphic as Yetter–Drinfeld modules. □

5. Hopf algebras over $\mathcal{K}$

In this last section we determine all finite-dimensional Hopf algebras H such that $H_{[0]} = \mathcal{K}$ and the corresponding infinitesimal braiding is a simple object in ${}^{\mathcal{K}}_{\mathcal{K}}\mathcal{YD}$ under the assumption that the diagram is a Nichols algebra. That is, the graded algebra with respect to the standard filtration is $\operatorname{gr} H = \bigoplus_{i\geq 0} H_{[i]}/H_{[i-1]} \simeq \mathfrak{B}(R(1))\#\mathcal{K}$ with $R(1)$ isomorphic to a simple object in ${}^{\mathcal{K}}_{\mathcal{K}}\mathcal{YD}$.

Next, we show that the bosonizations of the Nichols algebras associated with the simple modules $\Bbbk_{\chi^\ell}$ with $\ell = 1, 3$ and $V_{2,1}$, $V_{2,3}$ do not admit deformations.

Recall that, for $v \in V = R(1)$, the formula given by the bosonization yields

$$\Delta(v\#1) = v^{(1)}\#(v^{(2)})_{(-1)} \otimes (v^{(2)})_{(0)}\#1 = v\#1 \otimes 1\#1 + 1\#v_{(-1)} \otimes v_{(0)}\#1.$$

We also write $v = v\#1$ for all $v \in V$, and $k = 1\#k$ for all $k \in \mathcal{K}$.

Proposition 5.1. *Let H be a finite-dimensional Hopf algebra over $\mathcal{K}$ such that its infinitesimal braiding V is isomorphic to $\Bbbk_{\chi^\ell}$ with $\ell = 1$ or 3. Assume that the diagram is a Nichols algebra. Then $H \simeq (\bigwedge \Bbbk_{\chi^\ell})\#\mathcal{K}$.*

Proof. Write $\bigwedge \Bbbk_{\chi^\ell} = \Bbbk[x]/(x^2)$. As $\operatorname{gr} H \simeq (\bigwedge \Bbbk_{\chi^\ell})\#\mathcal{K}$ with $\ell = 1$ or 3, we have to prove that the defining relation $x^2 = 0$ of $\bigwedge \Bbbk_{\chi^\ell}$ remains homogeneous in H. Since $\delta(x) = a^2 \otimes x$, we obtain that

$$\Delta(x^2) = (x \otimes 1 + a^2 \otimes x)^2 = x^2 \otimes 1 + 1 \otimes x^2 + (a^2 \cdot x + x)a^2 \otimes x = x^2 \otimes 1 + 1 \otimes x^2,$$

which implies that $x^2 = 0$ in H, because $\mathcal{P}(\mathcal{K}) = \{0\}$. □

Proposition 5.2. *Let H be a finite-dimensional Hopf algebra over $\mathcal{K}$ such that its infinitesimal braiding V is isomorphic either to $V_{2,1}$ or $V_{2,3}$. Assume that the diagram is a Nichols algebra. Then $H \simeq \mathfrak{B}(V)\#\mathcal{K}$.*

Proof. We know that $\operatorname{gr} H \simeq \mathfrak{B}(V)\#\mathcal{K}$ with V isomorphic either to $V_{2,1}$ or $V_{2,3}$. As in the proof of Proposition 5.1, we prove that the homogeneous relations also hold in H.

Assume first that $V \simeq V_{2,1}$. Then, $\mathfrak{B}(V_{2,1})\#\mathcal{K}$ is the algebra generated by the elements x, y, a, b, c, d with x, y satisfying the relations of $\mathfrak{B}(V_{2,1})$ (10), a, b, c, d satisfying the relations of $\mathcal{K}$ (1), and all together satisfying the following relations:

$$\begin{aligned} &ax = -xa, \quad ay = \xi ya + xc, \quad bx = -xb, \quad by = \xi yb + xd, \\ &cx = -xc, \quad cy = -\xi yc + xa, \quad dx = -xd, \quad dy = -\xi yd + xb. \end{aligned} \tag{11}$$

As $\Delta(x) = x \otimes 1 + d \otimes x + (\xi - 1)c \otimes y$ and $\Delta(y) = y \otimes 1 + a \otimes y - \frac{\xi+1}{2}b \otimes x$, we have that $\Delta(xy + \xi yx) = (xy + \xi yx) \otimes 1 + 1 \otimes (xy + \xi yx)$ and $\Delta(y^4) = y^4 \otimes 1 + 1 \otimes y^4$. Since $\mathcal{P}(H) = \{0\}$, it follows that the relations $xy + \xi yx = 0$ and $y^4 = 0$ hold in H.

On the other hand, $\Delta(x^2) = x^2 \otimes 1 + a^2 \otimes x^2 + (\xi - 1)ab \otimes (xy + \xi yx) = x^2 \otimes 1 + a^2 \otimes x^2$, that is, x^2 is a $(1, a^2)$-primitive element in $H_{[1]}$. As $P_{1,a^2}(H_{[1]}) = P_{1,a^2}(\mathcal{K}) = \Bbbk\{1 - a^2, ab\}$, we must have that

$$x^2 = \mu_1(1 - a^2) + \mu_2 ab \qquad \text{for some } \mu_1, \mu_2 \in \Bbbk.$$

But, by (11),

$$0 = ax^2 - x^2a = a(\mu_1(1-a^2) + \mu_2 ab) - (\mu_1(1-a^2) + \mu_2 ab)a = \mu_2(1+\xi)c,$$
$$0 = bx^2 - x^2b = b(\mu_1(1-a^2) + \mu_2 ab) - (\mu_1(1-a^2) + \mu_2 ab)b = 2\mu_1 c,$$

which implies that $\mu_1 = \mu_2 = 0$. Therefore, the relation $x^2 = 0$ also holds in H and consequently, $H \simeq \operatorname{gr} H$.

For $V \simeq V_{2,3}$, the proof follows the same lines as for $V \simeq V_{2,1}$. □

Hereafter, we define two Hopf algebras $\mathfrak{A}_{3,1}(\mu)$ and $\mathfrak{A}_{3,3}(\mu)$, which are constructed by deforming the relations on the Nichols algebras $\mathfrak{B}(V_{3,1})$ and $\mathfrak{B}(V_{3,3})$ over $\mathcal{K}$, respectively, and show that they are liftings of the corresponding bosonizations.

Definition 5.3. *For $\mu \in \Bbbk$, let $\mathfrak{A}_{3,1}(\mu)$ be the algebra generated by the elements x, y, a, b, c, d satisfying the relations* (1) *and the following ones:*

$$\begin{aligned}
&ax = -\xi xa, \qquad ay = -ya - xc, \qquad bx = -\xi xb, \qquad by = -yb - xd,\\
&cx = \xi xc, \qquad cy = -yc + xa, \qquad dx = \xi xd, \qquad dy = -yd + xb, \qquad (12)\\
&x^2 - 2y^2 = \mu(1-a^2), \qquad xy + yx = \xi\mu ac, \qquad y^4 = -\mu y^2(1-a^2) - \frac{\mu^2}{2}(1-a^2).
\end{aligned}$$

$\mathfrak{A}_{3,1}(\mu)$ is a Hopf algebra with coalgebra structure and antipode determined by (2) and:

$$\Delta(x) = x\otimes 1 + d\otimes x + (\xi-1)c\otimes y, \qquad \Delta(y) = y\otimes 1 + a\otimes y + \frac{1}{2}(-\xi-1)b\otimes x,$$
$$\mathcal{S}(x) = -ax - (1+\xi)cy, \qquad \mathcal{S}(y) = -dy + \frac{1}{2}(\xi-1)bx, \qquad \varepsilon(x) = \varepsilon(y) = 0.$$

Remark 5.4. *Clearly, $\mathfrak{A}_{3,1}(0) \simeq \mathfrak{B}(V_{3,1})\#\mathcal{K}$. Also note that $\mathfrak{A}_{3,1}(\mu)$ is the quotient of the algebra $T(V_{3,1})\otimes\mathcal{K}$ by the two-sided ideal generated by the relations* (12)*; denote this ideal by $J_{3,1}$. Furthermore, formulae* (4) *hold.*

Definition 5.5. *For $\mu \in \Bbbk$, let $\mathfrak{A}_{3,3}(\mu)$ be the Hopf algebra defined by $\mathfrak{A}_{3,1}(\mu) = \mathfrak{A}_{3,3}(\mu)$ as algebra but with its coalgebra structure determined by the same counit and comultiplication for the generators a, b, c, d, but*

$$\Delta(x) = x\otimes 1 + a\otimes x + (\xi+1)b\otimes y, \qquad \varepsilon(x) = 0,$$
$$\Delta(y) = y\otimes 1 + d\otimes y + \frac{1}{2}(1-\xi)c\otimes x, \qquad \varepsilon(y) = 0.$$

In particular, we have that

$$\mathcal{S}(x) = -dx - (\xi-1)by, \qquad \mathcal{S}(y) = -ay + \frac{1}{2}(1+\xi)cx.$$

Remark 5.6. *As before, $\mathfrak{A}_{3,3}(0) \simeq \mathfrak{B}(V_{3,3})\#\mathcal{K}$ and $\mathfrak{A}_{3,3}(\mu)$ is the quotient of the algebra $T(V_{3,3})\otimes\mathcal{K}$ by the two-sided ideal generated by the relations* (12).

In the next lemma we show that the algebras $\mathfrak{A}_{3,1}(\mu)$ and $\mathfrak{A}_{3,3}(\mu)$ are finite-dimensional Hopf algebras over $\mathcal{K}$.

Lemma 5.7. *Let $\mu \in \Bbbk$. The Hopf algebras $\mathfrak{A}_{3,1}(\mu)$ and $\mathfrak{A}_{3,3}(\mu)$ are finite-dimensional and $(\mathfrak{A}_{3,1}(\mu))_{[0]} \simeq \mathcal{K} \simeq (\mathfrak{A}_{3,3}(\mu))_{[0]}$.*

Proof. We prove the assertion for $\mathfrak{A}_{3,1}(\mu)$, being the proof for $\mathfrak{A}_{3,3}(\mu)$ completely analogous. First note that, by Remark 5.4, $\mathfrak{A}_{3,1}(\mu) = T(V_{3,1})\otimes\mathcal{K}/J_{3,1}$. Also note that $T(V_{3,1})\otimes\mathcal{K}$ is a graded algebra with the gradation defined as usual on $T(V_{3,1})$ and all the elements in $\mathcal{K}$ being of degree 0.

Let $\mathfrak{A}_0$ be the subalgebra generated by the coalgebra C linearly spanned by a, b, c, d. Then $\mathfrak{A}_0$ is a Hopf subalgebra of $\mathfrak{A}_{3,1}(\mu)$. We claim that $\mathfrak{A}_0$ is isomorphic to $\mathcal{K}$. Indeed, consider the Hopf algebra map $\varphi : \mathcal{K} \to \mathfrak{A}_{3,1}(\mu)$ given by the composition $\mathcal{K} \hookrightarrow T(V_{3,1})\otimes\mathcal{K} \twoheadrightarrow T(V_{3,1})\otimes\mathcal{K}/J_{3,1}$. Clearly, $\mathfrak{A}_0 \simeq \varphi(\mathcal{K})$. Since $\dim\mathcal{K} = 8$, to prove that $\varphi(\mathcal{K}) \simeq \mathcal{K}$ it is enough to show that $\dim\varphi(\mathcal{K}) > 4$. As the relations in $J_{3,1}$ do not involve relations in degree 0, it follows that $\varphi(C) \simeq C$. Moreover, since the elements a, b, c, d are linearly independent in $\mathcal{K}$, they are also so in $\mathfrak{A}_{3,1}(\mu)$. Hence, $\varphi(\mathcal{K}) \simeq \mathcal{K}$ and the claim is proved.

Set $\mathfrak{A}_1 = \mathfrak{A}_0 + \mathcal{K}\{x, y\}$, $\mathfrak{A}_2 = \mathfrak{A}_1 + \mathcal{K}\{xy, y^2\}$, $\mathfrak{A}_3 = \mathfrak{A}_2 + \mathcal{K}\{xy^2, y^3\}$ and $\mathfrak{A}_4 = \mathfrak{A}_3 + \mathcal{K}\{xy^3\}$. Then $\{\mathfrak{A}_n\}_{0\leq n\leq 3}$ is a coalgebra filtration of $\mathfrak{A}_{3,1}(\mu)$. In particular, $(\mathfrak{A}_{3,1}(\mu))_0 \subseteq \mathcal{K}$ and consequently $(\mathfrak{A}_{3,3}(\mu))_{[0]} = \mathcal{K}$; that is, $\mathfrak{A}_{3,1}(\mu)$ is a Hopf algebra over $\mathcal{K}$. Hence, $\mathfrak{A}_{3,1}(\mu)$ is a finite-dimensional Hopf algebra which is free over $\mathcal{K}$. So, 8 divides $\dim \mathfrak{A}_{3,1}(\mu)$. Besides, $\mathfrak{A}_{3,1}$ is a $\mathcal{K}$-module with a set of generators $\{1, x, y, xy, y^2, xy^2, y^3, xy^3\}$. Thus, $\dim \mathfrak{A}_{3,1}(\mu) \leq 8 \dim \mathcal{K} = 64$ and the lemma is proved. □

Now, we show that the algebras $\mathfrak{A}_{3,1}(\mu)$ and $\mathfrak{A}_{3,3}(\mu)$ are liftings of $\mathfrak{B}(V_{3,1})\#\mathcal{K}$ and $\mathfrak{B}(V_{3,3})\#\mathcal{K}$ for all $\mu \in \Bbbk$, respectively.

Lemma 5.8. $\operatorname{gr} \mathfrak{A}_{3,1}(\mu) \simeq \mathfrak{B}(V_{3,1})\#\mathcal{K}$ *and* $\operatorname{gr} \mathfrak{A}_{3,3}(\mu) \simeq \mathfrak{B}(V_{3,3})\#\mathcal{K}$.

Proof. It is enough to show that $\dim \mathfrak{A}_{3,1}(\mu), \dim \mathfrak{A}_{3,3}(\mu) \geq 64$, since by the proof of Lemma 5.7 we have that $\operatorname{gr} \mathfrak{A}_{3,1}(\mu) \simeq R_{3,1}\#\mathcal{K}$ and $\operatorname{gr} \mathfrak{A}_{3,3}(\mu) \simeq R_{3,3}\#\mathcal{K}$, where $R_{3,1}$ and $R_{3,3}$ are K-modules linearly spanned by the set $\{1, x, y, xy, y^2, xy^2, y^3, xy^3\}$. We show that the set $B = \{x^i y^j a^k b^l : 0 \leq i, l \leq 1, 0 \leq j, k \leq 3\}$ is linearly independent by using appropriate representations. As $\mathfrak{A}_{3,1}(\mu)$ and $\mathfrak{A}_{3,3}(\mu)$ have the same algebra structure, we prove it only for $\mathfrak{A}_{3,1}(\mu)$.

For λ a 4-th root of unity, consider the 8-dimensional representation W_λ of $\mathfrak{A}_{3,1}(\mu)$ determined by the following matrices on a fixed basis of W_λ:

$$\rho_1(a) = \begin{pmatrix} \mathbf{a} & \mathbf{0} \\ \mathbf{0} & -\xi\mathbf{a} \end{pmatrix}, \qquad \rho_1(b) = \begin{pmatrix} \mathbf{0} & \mathbf{b}_{12} \\ \mathbf{b}_{21} & \mathbf{0} \end{pmatrix}, \qquad \rho_1(x) = \begin{pmatrix} \mathbf{0} & \mathbf{x} \\ \mathrm{id}_4 & \mathbf{0} \end{pmatrix}, \qquad \rho_1(y) = \begin{pmatrix} \mathbf{y}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{y}_{22} \end{pmatrix},$$

where

$$\mathbf{a} = \begin{pmatrix} \lambda & 0 & \mu(\lambda^3-\lambda) & 0 \\ 0 & -\lambda & 0 & \mu(\lambda-\lambda^3) \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \qquad \mathbf{b}_{12} = \begin{pmatrix} 0 & \xi\mu(\lambda^3-\lambda) & 0 & \xi\mu^2(\lambda-\lambda^3) \\ 0 & 0 & 0 & 0 \\ 0 & 2\xi\lambda^3 & 0 & \xi\mu(\lambda-\lambda^3) \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{b}_{21} = \begin{pmatrix} 0 & -\lambda^3 & 0 & \mu(\lambda^3-\lambda) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda^3 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix} \mu(1-\lambda^2) & 0 & \mu^2(\lambda^2-1) & 0 \\ 0 & \mu(1-\lambda^2) & 0 & \mu^2(\lambda^2-1) \\ 2 & 0 & \mu(\lambda^2-1) & 0 \\ 0 & 2 & 0 & \mu(\lambda^2-1) \end{pmatrix},$$

$$\mathbf{y}_{11} = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2}\mu^2(\lambda^2-1) \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \mu(\lambda^2-1) \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad \mathbf{y}_{22} = \begin{pmatrix} 0 & \mu\lambda^2 & 0 & \frac{1}{2}\mu^2(1-\lambda^2) \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & \mu \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Assume that

$$\sum_{0\leq i,l\leq 1, 0\leq j,k\leq 3} f_{i,j,k,l}\, x^i y^j a^k b^l = 0, \tag{13}$$

for some $f_{i,j,k,l} \in \Bbbk$. Applying (13) to the first vector of the fixed basis, we get that:

$$f_{i,j,0,0} + \lambda f_{i,j,1,0} + \lambda^2 f_{i,j,2,0} + \lambda^3 f_{i,j,3,0} = 0, \quad \text{for all } \ 0 \leq i \leq 1, 0 \leq j \leq 3.$$

Since this equation must hold for any 4-th root of unity λ, it follows that $f_{i,j,k,0} = 0$ for all $0 \leq i \leq 1$ and $0 \leq j, k \leq 3$. To prove that the remaining coefficients are zero, we need another representation.

For λ a 4-th root of unity, consider now the 16-dimensional representation U_λ given by the following matrices on a fixed basis of U_λ:

$$\rho_2(a) = \begin{pmatrix} \mathbf{a} & \mathbf{0} & \mu(\mathbf{d}-\mathbf{a}) & \mathbf{0} & \mathbf{0} & \xi\mu(\mathbf{c}-\mathbf{b}) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{a} & \mathbf{0} & \mu(\mathbf{a}-\mathbf{d}) & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{a} & \mathbf{0} & \mathbf{0} & 2\xi\mathbf{c} & \mathbf{0} & -\xi\mu(\mathbf{b}+\mathbf{c}) \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{a} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{c} & \mathbf{0} & \mu\mathbf{c} & -\xi\mathbf{a} & \mathbf{0} & \xi\mu(\mathbf{a}-\mathbf{d}) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \xi\mathbf{a} & \mathbf{0} & \xi\mu(\mathbf{d}-\mathbf{a}) \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{c} & \mathbf{0} & \mathbf{0} & \xi\mathbf{a} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\xi\mathbf{a} \end{pmatrix},$$

$$\rho_2(b)=\begin{pmatrix} \mathbf{b} & \mathbf{0} & -\mu\mathbf{b} & \mathbf{0} & \mathbf{0} & \xi\mu(\mathbf{d}-\mathbf{a}) & \mathbf{0} & \xi\mu^2(\mathbf{a}-\mathbf{d})\\ \mathbf{0} & -\mathbf{b} & \mathbf{0} & \mu\mathbf{b} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & -\mathbf{b} & \mathbf{0} & \mathbf{0} & 2\xi\mathbf{d} & \mathbf{0} & \xi\mu(\mathbf{a}-\mathbf{d})\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{b} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & -\mathbf{d} & \mathbf{0} & \mu(\mathbf{d}-\mathbf{a}) & -\xi\mathbf{b} & \mathbf{0} & \xi\mu\mathbf{b} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \xi\mathbf{b} & \mathbf{0} & -\xi\mu\mathbf{b}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{d} & \mathbf{0} & \mathbf{0} & \xi\mathbf{b} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\xi\mathbf{b} \end{pmatrix}, \qquad \rho_2(x)=\begin{pmatrix} \mathbf{0}_8 & \mathbf{x}\\ \mathrm{id}_8 & \mathbf{0}_8 \end{pmatrix},$$

$$\rho_2(y)=\begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{\mu^2}{2}(\mathbf{a}^2-\mathrm{id}_2) & \xi\mu\mathbf{ac} & \mathbf{0} & -\xi\mu^2\mathbf{ab} & \mathbf{0}\\ \mathrm{id}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \xi\mu\mathbf{ac} & \mathbf{0} & -\xi\mu^2\mathbf{ab}\\ \mathbf{0} & \mathrm{id}_2 & \mathbf{0} & \mu(\mathbf{a}^2-\mathrm{id}_2) & \mathbf{0} & \mathbf{0} & \xi\mu\mathbf{ac} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathrm{id}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \xi\mu\mathbf{ac}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mu\mathbf{a}^2 & \mathbf{0} & \frac{\mu^2}{2}(\mathrm{id}_2-\mathbf{a}^2)\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathrm{id}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathrm{id}_2 & \mathbf{0} & \mu\,\mathrm{id}_2\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathrm{id}_2 & \mathbf{0} \end{pmatrix},$$

where $\mathbf{a}=\begin{pmatrix} \lambda & 0\\ 0 & -\xi\lambda \end{pmatrix}$, $\mathbf{b}=\begin{pmatrix} 0 & \lambda^2\\ 0 & 0 \end{pmatrix}$, $\mathbf{c}=\begin{pmatrix} 0 & 1\\ 0 & 0 \end{pmatrix}$, $\mathbf{d}=\begin{pmatrix} \lambda^3 & 0\\ 0 & \xi\lambda^3 \end{pmatrix}$ and

$$\mathbf{x}=\begin{pmatrix} \mu(\mathrm{id}_2-\mathbf{a}^2) & \mathbf{0} & \mu^2(\mathbf{a}^2-\mathrm{id}_2) & \mathbf{0}\\ \mathbf{0} & \mu(\mathrm{id}_2-\mathbf{a}^2) & \mathbf{0} & \mu^2(\mathbf{a}^2-\mathrm{id}_2)\\ 2\,\mathrm{id}_2 & \mathbf{0} & \mu(\mathbf{a}^2-\mathrm{id}_2) & \mathbf{0}\\ \mathbf{0} & 2\,\mathrm{id}_2 & \mathbf{0} & \mu(\mathbf{a}^2-\mathrm{id}_2) \end{pmatrix}.$$

Applying the residual equation of (13) to the second vector of the fixed basis, we get that

$$\lambda^2(f_{i,j,0,1}+\lambda f_{i,j,1,1}+\lambda^2 f_{i,j,2,1}+\lambda^3 f_{i,j,3,1})=0, \quad \text{for all } \ 0\le i\le 1, 0\le j\le 3,$$

implying that $f_{i,j,k,1}=0$ for all $0\le i\le 1, 0\le j,k\le 3$. Therefore, B is a linearly independent set and $\mathfrak{A}_{3,1}(\mu)$ is a lifting of $\mathfrak{B}(V_{3,1})\#\mathcal{K}$, for all $\mu\in\Bbbk$. □

Proposition 5.9. *Let H be a finite-dimensional Hopf algebra over $\mathcal{K}$ such that its infinitesimal braiding is isomorphic to $V_{3,1}$ or $V_{3,3}$. Assume that the diagram is a Nichols algebra. Then $H\simeq\mathfrak{A}_{3,1}(\mu)$ or $H\simeq\mathfrak{A}_{3,3}(\mu)$ for some $\mu\in\Bbbk$, respectively.*

Proof. We have that $\operatorname{gr} H\simeq\mathfrak{B}(V)\#\mathcal{K}$ with $V\simeq V_{3,1}$ or $V\simeq V_{3,3}$. Recall that $\mathfrak{B}(V)\#\mathcal{K}$ is the algebra generated by x,y,a,b,c,d, where x,y are the generators of $\mathfrak{B}(V)$, a,b,c,d are the generators of $\mathcal{K}$, and they all together satisfy the first two rows of relations (12).

We prove the claim for $V\simeq V_{3,1}$. The proof for $V\simeq V_{3,3}$ follows the same lines. As $\Delta(x)=x\otimes 1+d\otimes x+(\xi-1)c\otimes y$ and $\Delta(y)=y\otimes 1+a\otimes y-\dfrac{\xi+1}{2}b\otimes x$, we obtain that

$$\Delta(x^2-2y^2)=(x^2-2y^2)\otimes 1+a^2\otimes(x^2-2y^2) \qquad \text{and} \tag{14}$$

$$\Delta(xy+yx)=(xy+yx)\otimes 1+1\otimes(xy+yx)-\xi ac\otimes(x^2-2y^2). \tag{15}$$

By (14), we get that $x^2-2y^2\in P_{1,a^2}(H_{[1]})=P_{1,a^2}(\mathcal{K})=\Bbbk\{1-a^2,ab\}$. Then, in H,

$$x^2-2y^2=\mu_1(1-a^2)+\mu_2 ab \qquad \text{for some } \mu_1,\mu_2\in\Bbbk.$$

Thus, (15) can be rewritten as

$$\Delta(xy+yx-\xi\mu_1 ac)=(xy+yx-\mu_1\xi ac)\otimes 1+1\otimes(xy+yx-\mu_1\xi ac)-\xi\mu_2 ac\otimes ab.$$

However, a tedius calculation on $H_{[1]}$ shows that μ_2 must be 0 in the last equation. Hence,

$$xy+yx=\xi\mu_1 ac \qquad \text{and} \qquad x^2-2y^2=\mu_1(1-a^2) \qquad \text{for some } \mu_1\in\Bbbk.$$

Finally, observe that the element $R:=y^4+\mu_1 y^2(1-a^2)+\dfrac{1}{2}\mu_1^2(1-a^2)$ satisfies $\Delta(R)=R\otimes 1+1\otimes R$, which implies that $R=0$ in H.

Since the defining relations of $\mathfrak{A}_{3,1}(\mu_1)$ hold in H, there exists a surjective Hopf algebra map from H to $\mathfrak{A}_{3,1}(\mu_1)$. As both algebras have the same dimension, the proposition follows. □

We end the paper with the proof of our second main theorem.

Proof of Theorem B. By Theorem A, we have that $\operatorname{gr} H \simeq \mathfrak{B}(V)\#\mathcal{K}$ with V isomorphic to $\Bbbk_{\chi}$, $\Bbbk_{\chi^3}$, $V_{2,1}$, $V_{2,3}$, $V_{3,1}$ or $V_{3,3}$.

If $V \simeq \Bbbk_{\chi}$, $\Bbbk_{\chi^3}$, $V_{2,1}$ or $V_{2,3}$, then $H \simeq \mathfrak{B}(V)\#\mathcal{K}$ by Propositions 5.1 and 5.2. If $V \simeq V_{3,1}$ or $V \simeq V_{3,3}$, then by Proposition 5.9 it follows that $H \simeq \mathfrak{A}_{3,1}(\mu)$ or $H \simeq \mathfrak{A}_{3,3}(\mu)$ for some $\mu \in \Bbbk$, respectively.

Conversely, it is clear that the algebras listed in items (i), (ii) and (iii) are liftings of Hopf algebras over $\mathcal{K}$. The Hopf algebras $\mathfrak{A}_{3,1}(\mu)$ and $\mathfrak{A}_{3,3}(\mu)$ are also liftings by Lemma 5.8.

Finally, two algebras from different families are not isomorphic as Hopf algebras since their infinitesimal braidings are not isomorphic as Yetter–Drinfeld modules. □

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References

[1] N. Andruskiewitsch and J. Cuadra, *On the structure of (co-Frobenius) Hopf algebras*, J. Noncommutative Geometry **7** (2013), Issue 1, pp. 83–104.
[2] N. Andruskiewitsch and M. Graña, *Braided Hopf algebras over non-abelian finite groups*, Colloquium on Operator Algebras and Quantum Groups, Bol. Acad. Nac. Cienc. **63** (1999), 45–78.
[3] N. Andruskiewitsch and H.-J. Schneider, *Pointed Hopf algebras*, New directions in Hopf algebras, 1–68, Math. Sci. Res. Inst. Publ., 43, Cambridge Univ. Press, Cambridge, 2002.
[4] M. Beattie, *Duals of pointed Hopf algebras*, J. Algebra **262** (2003), 54–76.
[5] G. A. García and C. Vay, *Hopf algebras of dimension 16*, Algebr. Represent. Theory, **13** (2010), 383–405.
[6] M. Graña, *A freeness theorem for Nichols algebras*, J. Algebra **231** (2000), no. 1, 235–257.
[7] I. Heckenberger and H-J. Schneider, *Nichols algebras over groups with finite root system of rank two I*, J. Algebra **324** (2010) 3090–3114.
[8] I. Kaplansky, *Bialgebras*, Lecture Notes in Mathematics (1975). University of Chicago.
[9] M. Lorenz, *Representations of finite-dimensional Hopf algebras*, J. Algebra **188** (1997), no. 2, 476–505.
[10] S. Montgomery, *Hopf Algebras and their Actions on Rings*, CMBS Reg. Conf. Ser. in Math. **82**, Amer. Math. Soc., 1993.
[11] D. E. Radford, *On the coradical of a finite-dimensional Hopf algebra*, Proc. Amer. Math. Soc. **53** (1975), 9–15.
[12] ———, *Hopf Algebras*, Knots and Everything **49**, World Scientific, 2011.
[13] D. Ştefan, *Hopf algebras of low dimension*, J. Algebra **211** (1999), 343–361.
[14] S. Ufer, *Triangular braidings and pointed Hopf algebras*, J. Pure and Applied Alg. **210**, Issue 2 (2007), 307–320.

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