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A Plastino<sup>1,3,4</sup> and M C Rocca<sup>1,2,3,5</sup> <sup>1</sup> Departamento de Física, Universidad Nacional de La Plata, Argentina<sup>2</sup> Departamento de Matemática, Universidad Nacional de La Plata, Argentina<sup>3</sup> Consejo Nacional de Investigaciones Científicas y Tecnológicas (IFLP-CCT-CONICET)-C. C. 727, 1900 La Plata— Argentina<sup>4</sup> SThAR—EPFL, Lausanne, Switzerland<sup>5</sup> Author to whom any correspondence should be addressed.**Keywords:** dimensional regularization, ultrahyperfunctions, Wheeler's propagators, Feynman's propagators**Abstract**

The Dimensional Regularization (DR) of Bollini and Giambiagi (BG) can not be defined for all Schwartz Tempered Distributions Explicitly Lorentz Invariant (STDELI)  $\mathcal{S}'_L$ . In this paper we overcome such limitation and show that it can be generalized to all aforementioned STDELI and obtain a product in a ring with zero divisors. For this purpose, we resort to a formula obtained by Bollini and Rocca and demonstrate the existence of the convolution (in Minkowskian space) of such distributions. This is done by following a procedure similar to that used so as to define a general convolution between the Ultradistributions of J Sebastiao e Silva (JSS), also known as Ultrahyperfunctions, obtained by Bollini *et al.* Using the Inverse Fourier Transform we get the ring with zero divisors  $\mathcal{S}'_{LA}$ , defined as  $\mathcal{S}'_{LA} = \mathcal{F}^{-1}\{\mathcal{S}'_L\}$ , where  $\mathcal{F}^{-1}$  denotes the Inverse Fourier Transform. In this manner we effect a dimensional regularization in momentum space (the ring  $\mathcal{S}'_L$ ) via convolution, and a product of distributions in the corresponding configuration space (the ring  $\mathcal{S}'_{LA}$ ). This generalizes the results obtained by BG for Euclidean space. We provide several examples of the application of our new results in Quantum Field Theory (QFT). In particular, the convolution of  $n$  massless Feynman's propagators and the convolution of  $n$  massless Wheeler's propagators in Minkowskian space. The results obtained in this work have already allowed us to calculate the classical partition function of Newtonian gravity, for the first time ever, in the Gibbs' formulation and in the Tsallis' one. It is our hope that this convolution will allow one to quantize non-renormalizable Quantum Field Theories (QFT's).

**1. Introduction**

The problem of defining the product of two distributions (a product in a ring with divisors of zero) is an old one of hard functional analysis.

In QFT the problem of evaluating the product of distributions with coincident point singularities is related to the asymptotic behaviour of loop integrals of propagators.

From a mathematical point of view, practically all definitions of that product lead to limitations on the set of distributions that can be multiplied by each other to give another distribution of the same type.

In fact, Laurent Schwartz showed that he can not define a product of distributions regarded as an algebra, instead of as a ring with divisors of zero.

In [1–4] it was demonstrated that it is possible to define a general convolution between the ultradistributions of JSS [5] (Ultrahyperfunctions). This convolution yields another Ultrahyperfunction. Therefore, we have a product in a ring with zero divisors. Such a ring is the space of Distributions of Exponential Type, or Ultradistributions of Exponential Type, obtained applying the anti-Fourier transform to the space of Tempered Ultradistributions or Ultradistributions of Exponential Type.

We must clarify at this point that the Ultrahyperfunctions are the generalization and extension to the complex plane of the Schwartz Tempered Distributions and the Distributions of Exponential Type. That is, the Temperate Distributions and those of Exponential Type are a subset of the Ultrahyperfunctions.

The problem we then face is that of formulating the convolution between Ultradistributions. This is a complex issue, difficult to manage, even if it has the advantage of allowing one to attempt non-renormalizable QFT's.

Fortunately, we have found that a method similar to that used to define the convolution of Ultradistributions can also be used to define the convolution of Lorentz Invariant distributions using the DR of BG in momentum space. *As a consequence, Ultradistributions need not be used in the calculations of this paper, which considerably simplifies it.* Taking advantage of such Regularization one can also work in configuration space [6]. Thus, one can obtain a convolution of Lorentz Invariant Tempered Distributions in momentum space and the corresponding product in configuration space.

DR is one of the most important advances in theoretical physics and is used in several disciplines of it [7–60].

Our DR generalization happens to be a convolution of STDELI in momentum space and a product in a ring with divisors of zero in configuration space.

It is our hope that this convolution can then be used to treat non-renormalizable QFT's. This is our present goals.

More to the point, let us emphasize that our work is concerned with deeper issues than those regarding QFT axiomatics in Euclidean space and QFT renormalization. Here, we are *generalizing* BG dimensional regularization to *all Schwartz Tempered, explicitly Lorentz invariant, distributions (STDELI)*, something that BG were unable to achieve. This would permit one to deal with non-renormalizable QFT's. Indeed, we do not have to use counterterms in a renormalization process devoted to eliminate infinities. *This is exactly what we do not want to do, since a non-renormalizable theory involves an infinite number of counterterms.* The central purpose of our work is to define a STDELI convolution in order to avoid counterterms. We do not appeal to a simple correlation-functions' convolution (not defined for all STDELI). At the same time, we conserve all extant solutions to the problem of running coupling constants and the renormalization group. The STDELI convolution, once obtained, converts configuration space into a ring with zero-divisors. In it, one has now defined a product between the ring-elements. Thus, any unitary-causal-Lorentz invariant theory quantified in such a manner becomes predictive. The distinction those between renormalizable on not-renormalizable QFT's becomes unnecessary now. With our BG generalization, that uses Laurent's expansions in the dimension, all finite constants of the convolutions become completely determined, eliminating arbitrary choices of finite constants. This is tantamount to eliminating all finite renormalizations of the theory. What is the importance of using only *that term independent of the dimension* in Laurent's expansion? That the result obtained for finite convolutions will coincide with such a term. This translates to configuration space the product-operation in a ring with divisors of zero.

As examples, we calculate some convolutions of distributions used in QFT. In particular, the convolution of  $n$  massless Feynman's propagators and the convolution of  $n$  massless Wheeler's propagators. For a full discussion about definition and properties of Wheeler's propagators see [61, 62] which in turn are based on Wheeler and Feynman works [63, 64].

The results obtained in this work have already allowed us to calculate the classical partition function of Newtonian gravity, for the first time ever, in the Gibbs' formulation and in the Tsallis' one [65].

Note that we have added an [appendix](#) the by recourse to a simple example, is able to make it explicit how much simpler is our treatment is compared to the habitual DR technique.

## 2. Preliminary materials

### 2.1. Lorentz Invariant Tempered Distributions

In this subsection we give the definitions that we will use in this paper.

We consider first the case on the  $\nu$ -dimensional Minkowskian space  $\mathbf{M}_\nu$ . Let  $S'$  be the space of Schwartz Tempered Distributions [5, 66]. Let be  $g \in S'$ . We say that  $g \in S'_L$  if and only if:

$$g(\rho) = \frac{d^l}{d\rho^l} f(\rho) \quad (2.1. 1)$$

where the derivative is in the sense of distributions,  $l$  is a natural number,  $\rho = k^2 = k_0^2 - k_1^2 - k_2^2 - \dots - k_{\nu-1}^2$ ,  $f$  satisfies:

$$\int_{-\infty}^{\infty} \frac{|f(\rho)|}{(1 + \rho^2)^n} d\rho < \infty, \quad (2.1. 2)$$

and is continuous in  $\mathbf{M}_\nu$ . The exponent  $n$  is a natural number. We say then that  $f \in T_{1L}$ .

In the case of Euclidean space  $\mathbf{R}_\nu$ , let  $g \in \mathcal{S}'$ . We say that  $g \in \mathcal{S}'_R$  if and only if

$$g(k) = \frac{d^l}{dk^l} f(k), \tag{2.1. 3}$$

where  $k^2 = k_0^2 + k_1^2 + k_2^2 - \dots + k_{\nu-1}^2$ , with  $f(k)$  satisfying:

$$\int_0^\infty \frac{|f(k)|}{(1+k^2)^n} dk < \infty, \tag{2.1. 4}$$

and  $f(k)$  is continuous in  $\mathbf{R}_\nu$ . We say then that  $f \in \mathcal{T}_{1R}$ .

We call  $\mathcal{S}'_{LA}$  and  $\mathcal{S}'_{RA}$  the Fourier Anti-transformed Spaces of  $\mathcal{S}'_L$  and  $\mathcal{S}'_R$ , respectively.

**2.2. The Fourier transform in Euclidean space**

The Fourier transform of a spherically symmetric function is given, according to Bochner’s formula, by [67]:

$$f(k, \nu) = \frac{(2\pi)^{\frac{\nu}{2}}}{k^{\frac{\nu-2}{2}}} \int_0^\infty \hat{f}(r, \nu) r^{\frac{\nu}{2}} \mathcal{J}_{\frac{\nu-2}{2}}(kr) dr, \tag{2.2. 1}$$

where  $r^2 = x_0^2 + x_1^2 + \dots + x_{\nu-1}^2$ ;  $k^2 = k_0^2 + k_1^2 + \dots + k_{\nu-1}^2$  and  $\mathcal{J}_{\frac{\nu-2}{2}}$  is the Bessel function of order  $(\nu - 2)/2$ . By the use of the equality

$$\pi \mathcal{J}_{\frac{\nu-2}{2}}(z) = e^{-i\frac{\pi}{4}\nu} \mathcal{K}_{\frac{\nu-2}{2}}(-iz) + e^{i\frac{\pi}{4}\nu} \mathcal{K}_{\frac{\nu-2}{2}}(iz), \tag{2.2. 2}$$

where  $\mathcal{K}$  is the modified Bessel function, (2.2. 1) takes the form:

$$f(k, \nu) = 2 \frac{(2\pi)^{\frac{\nu-2}{2}}}{k^{\frac{\nu-2}{2}}} \int_0^\infty \hat{f}(r, \nu) r^{\frac{\nu}{2}} [e^{-i\frac{\pi}{4}\nu} \mathcal{K}_{\frac{\nu-2}{2}}(-ikr) + e^{i\frac{\pi}{4}\nu} \mathcal{K}_{\frac{\nu-2}{2}}(ikr)] dr. \tag{2.2. 3}$$

By performing the change of variables  $x = r^2$ ,  $\rho = k^2$ , (2.2. 1) and (2.2. 3) can be re-written as:

$$f(\rho, \nu) = \pi \frac{(2\pi)^{\frac{\nu-2}{2}}}{\rho^{\frac{\nu-2}{4}}} \int_0^\infty \hat{f}(x, \nu) x^{\frac{\nu-2}{4}} \mathcal{J}_{\frac{\nu-2}{2}}(\rho^{1/2}x^{1/2}) dx \tag{2.2. 4}$$

$$f(\rho, \nu) = \frac{(2\pi)^{\frac{\nu-2}{2}}}{\rho^{\frac{\nu-2}{4}}} \int_0^\infty \hat{f}(x, \nu) x^{\frac{\nu-2}{4}} [e^{-i\frac{\pi}{4}\nu} \mathcal{K}_{\frac{\nu-2}{2}}(-ix^{1/2}\rho^{1/2}) + e^{i\frac{\pi}{4}\nu} \mathcal{K}_{\frac{\nu-2}{2}}(ix^{1/2}\rho^{1/2})] dx. \tag{2.2. 5}$$

**2.3. The Fourier transform in Minkowskian space**

For the Minkowskian case we have, according to [2]

$$f(\rho, \nu) = (2\pi)^{\frac{\nu-2}{2}} \int_{-\infty}^\infty \hat{f}(x, \nu) \left\{ e^{\frac{i\pi(\nu-2)}{4}} \frac{(x+i0)^{\frac{\nu-2}{4}}}{(\rho+i0)^{\frac{\nu-2}{4}}} \mathcal{K}_{\frac{\nu-2}{2}}[-i(x+i0)^{1/2}(\rho+i0)^{1/2}] + e^{\frac{i\pi(2-\nu)}{4}} \frac{(x-i0)^{\frac{\nu-2}{4}}}{(\rho-i0)^{\frac{\nu-2}{4}}} \mathcal{K}_{\frac{\nu-2}{2}}[i(x-i0)^{1/2}(\rho-i0)^{1/2}] \right\} dx, \tag{2.3. 1}$$

where  $\rho = k_0^2 - k_1^2 - k_2^2 \dots - k_{\nu-1}^2$ . The corresponding inversion formula is then given by [2]:

$$\hat{f}(x, \nu) = \frac{1}{(2\pi)^{\frac{\nu+2}{2}}} \int_{-\infty}^\infty f(\rho, \nu) \left\{ e^{\frac{i\pi(\nu-2)}{4}} \frac{(\rho+i0)^{\frac{\nu-2}{4}}}{(x+i0)^{\frac{\nu-2}{4}}} \mathcal{K}_{\frac{\nu-2}{2}}[-i(x+i0)^{1/2}(\rho+i0)^{1/2}] + e^{\frac{i\pi(2-\nu)}{4}} \frac{(\rho-i0)^{\frac{\nu-2}{4}}}{(x-i0)^{\frac{\nu-2}{4}}} \mathcal{K}_{\frac{\nu-2}{2}}[i(x-i0)^{1/2}(\rho-i0)^{1/2}] \right\} d\rho. \tag{2.3. 2}$$

Equation (2.3. 1) is the generalization of Bochner’s formula (2.2. 1) to Minkowskian Space.

**2.4. An original example**

As an example not previously published of this formula we will calculate the Fourier anti-transform of the Dirac’s delta  $\delta(\rho)$  in four dimensions. For this, we make use of the formula given in [68]:

$$\mathcal{K}_1(z) = -\frac{1}{z} + \sum_{k=0}^\infty \frac{\left(\frac{z}{2}\right)^{(1+2k)}}{k!(1+k)!} \left[ \ln\left(\frac{z}{2}\right) - \frac{1}{2}\psi(k+1) - \psi(k+2) \right], \tag{2.4. 1}$$

where  $\psi'(z) = \frac{d[\ln \Gamma(z)]}{dz}$ . Then:

$$\hat{f}(x, 4) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \delta(\rho) \left\{ i \frac{(\rho + i0)^{\frac{1}{2}}}{(x + i0)^{\frac{1}{2}}} \mathcal{K}_1[-i(x + i0)^{1/2}(\rho + i0)^{1/2}] - i \frac{(\rho - i0)^{\frac{\nu-2}{4}}}{(x - i0)^{\frac{\nu-2}{4}}} \mathcal{K}_1[i(x - i0)^{1/2}(\rho - i0)^{1/2}] \right\} d\rho. \tag{2.4. 2}$$

After a simple calculation we obtain:

$$\hat{f}(x, 4) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \delta(\rho) \left[ \frac{1}{x + i0} + \frac{1}{x - i0} \right] d\rho \tag{2.4. 3}$$

and finally:

$$\hat{f}(x, 4) = \frac{1}{(2\pi)^3} \left[ \frac{1}{x + i0} + \frac{1}{x - i0} \right]. \tag{2.4. 4}$$

### 3. The convolution in Euclidean space

#### 3.1. The generalization of dimensional regularization in configuration space to the euclidean space

The expression for the convolution of two spherically symmetric functions was deduced in [6]

$(h(k, \nu) = (f * g)(k, \nu))$ :

$$h(k, \nu) = \frac{2^{4-\nu} \pi^{\frac{\nu-1}{2}}}{\Gamma\left(\frac{\nu-1}{2}\right) k^{\nu-2}} \iint_0^{\infty} f(k_1, \nu) g(k_2, \nu) \times [4k_1^2 k_2^2 - (k^2 - k_1^2 - k_2^2)^2]_+^{\frac{\nu-3}{2}} k_1 k_2 dk_1 dk_2. \tag{3.1. 1}$$

However, BG did not obtain a product in a ring with divisors of zero, which we will do now. Consider here that  $f$  and  $g$  belong to  $S'_R$ . With the change of variables  $\rho = k^2, \rho_1 = k_1^2, \rho_2 = k_2^2$  takes the form:

$$h(\rho, \nu) = \frac{2^{2-\nu} \pi^{\frac{\nu-1}{2}}}{\Gamma\left(\frac{\nu-1}{2}\right) \rho^{\frac{\nu-2}{2}}} \iint_0^{\infty} f(\rho_1, \nu) g(\rho_2, \nu) \times [4\rho_1 \rho_2 - (\rho - \rho_1 - \rho_2)^2]_+^{\frac{\nu-3}{2}} d\rho_1 d\rho_2. \tag{3.1. 2}$$

Let  $\mathfrak{B}$  be a vertical band contained in the complex  $\nu$ -plane  $\mathfrak{W}$ . Integral (3.1. 2) is an analytic function of  $\nu$  defined in the domain  $\mathfrak{B}$ . Then, according to the method of [1],  $h(\nu, \rho)$  can be analytically continued to other parts of  $\mathfrak{W}$ . In particular, near the dimension  $\nu_0$  we have the Laurent's expansion:

$$h(\rho, \nu) = \sum_{m=-1}^{\infty} h^{(m)}(\rho) (\nu - \nu_0)^m. \tag{3.1. 3}$$

Here,  $\nu_0$  is the dimension of the considered space. In particular,  $\nu_0 = 4$  is the dimension that we will consider. We now define the convolution product as the  $(\nu - \nu_0)$ -independent term of the Laurent's expansion. (3.1. 3):

$$h_{\nu_0}(\rho) = h^{(0)}(\rho). \tag{3.1. 4}$$

Thus, in the ring with zero divisors  $S'_{RA}$ , we have defined a product of distributions.

#### 3.2. Example

As an example of the use of (2.2. 1) and (3.1. 1), we evaluate the convolution of a massless propagator with a propagator corresponding to a scalar particle of mass  $m$ . The result of this convolution, using this formula, is given in [69]. It is:

$$h(k, \nu) = 2^{\nu-2} \pi^{\frac{\nu}{2}} m^{\nu-4} \frac{\Gamma\left(\frac{\nu-2}{2}\right) \Gamma\left(\frac{4-\nu}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} F\left(1, \frac{4-\nu}{2}; \frac{\nu}{2}; -\frac{k^2}{m^2}\right). \tag{3.2. 1}$$

Now we use the equality:

$$\begin{aligned} & \Gamma\left(\frac{4-\nu}{2}\right) F\left(1, \frac{4-\nu}{2}; \frac{\nu}{2}; -\frac{k^2}{m^2}\right) \\ &= \Gamma\left(\frac{4-\nu}{2}\right) - \frac{2}{\nu} \Gamma\left(\frac{6-\nu}{2}\right) \frac{k^2}{m^2} F\left(1, \frac{6-\nu}{2}; \frac{2+\nu}{2}; -\frac{k^2}{m^2}\right). \end{aligned} \tag{3.2. 2}$$

After a tedious calculation, we obtain the corresponding Laurent's expansion of  $h(k, \nu)$ :

$$h(k, \nu) = -\frac{8\pi^2}{\nu - 4} + 4\pi^2(C + 2 - \ln 4 - \ln \pi - \ln m^2) - 2\pi^2 \frac{k^2}{m^2} F\left(1, 1; 3; -\frac{k^2}{m^2}\right) + \sum_{s=1}^{\infty} a_s (\nu - 4)^s, \tag{3.2. 3}$$

where  $C$  is Euler's constant with changed sign  $C = -0.577\ 215\ 664\ 90$ . Thus, we have

$$\frac{1}{k^2} * \frac{1}{k^2 + m^2} = 4\pi^2(C + 2 - \ln 4 - \ln \pi - \ln m^2) - 2\pi^2 \frac{k^2}{m^2} F\left(1, 1; 3; -\frac{k^2}{m^2}\right). \tag{3.2. 4}$$

## 4. The convolution in Minkowskian space

### 4.1. The generalization of dimensional regularization in configuration space to the Minkowskian space

In this section we repeat the efforts of the preceding one for Minkowskian space.

The generalization of the Bochner's formula to Minkowskian space has been obtained in [2]. The corresponding expression for  $\nu = 2n$  is:

$$h(\rho, \nu) = \frac{\pi^{\frac{\nu-3}{2}}}{2^{\nu-1}} e^{\frac{i\pi(2-\nu)}{2}} \Gamma\left(\frac{3-\nu}{2}\right) \iint_{-\infty}^{\infty} f(\rho_1, \nu) g(\rho_2, \nu) \times \left\{ (\rho - i0)^{-\frac{1}{2}} \left[ \frac{(\rho - \rho_1 - \rho_2)^2 - 4\rho_1\rho_2}{\rho} + i0 \right]^{\frac{\nu-3}{2}} + e^{i\pi(\nu-2)} \times (\rho + i0)^{-\frac{1}{2}} \left[ \frac{(\rho - \rho_1 - \rho_2)^2 - 4\rho_1\rho_2}{\rho} - i0 \right]^{\frac{\nu-3}{2}} \right\} d\rho_1 d\rho_2 \tag{4.1. 1}$$

$$h(\rho, \nu) = (f * g)(\rho, \nu).$$

When  $\nu = 2n + 1$  we obtain:

$$h(\rho, \nu) = -\frac{i\pi^{\frac{\nu-3}{2}}}{2^{\nu-1}\Gamma\left(\frac{\nu-3}{2}\right)} \iint_{-\infty}^{\infty} f(\rho_1, \nu) g(\rho_2, \nu) \left[ \frac{(\rho - \rho_1 - \rho_2)^2 - 4\rho_1\rho_2}{\rho} \right]^{\frac{\nu-3}{2}} \{ (\rho - i0)^{-\frac{1}{2}} \times \left[ \psi\left(\frac{\nu-1}{2}\right) + \frac{i\pi}{2} + \ln \left[ \frac{(\rho - \rho_1 - \rho_2)^2 - 4\rho_1\rho_2}{\rho} + i0 \right] \right] - (\rho + i0)^{-\frac{1}{2}} \left[ \psi\left(\frac{\nu-1}{2}\right) + \frac{i\pi}{2} + \ln \left[ -\frac{(\rho - \rho_1 - \rho_2)^2 - 4\rho_1\rho_2}{\rho} + i0 \right] \right] \} d\rho_1 d\rho_2. \tag{4.1. 2}$$

For the Minkowskian case one can also employ Laurent's expansion

$$h(\rho, \nu) = \sum_{m=-1}^{\infty} h^{(m)}(\rho) (\nu - \nu_0)^m \tag{4.1. 3}$$

and therefore, again, we have for the convolution the result:

$$h_{\nu_0}(\rho) = h^{(0)}(\rho). \tag{4.1. 4}$$

Thus, in the ring with zero divisors  $S'_{LA}$  we have defined a product of distributions.

### 4.2. Examples

As an example of the use of (4.1. 1) we will consider the convolution of two Dirac's  $\delta$ -distributions,  $\delta(\rho)$ . The result is

$$h(\rho, \nu) = \frac{\pi^{\frac{\nu-3}{2}}}{2^{\nu-1}} e^{\frac{i\pi(2-\nu)}{2}} \Gamma\left(\frac{3-\nu}{2}\right) [(\rho - i0)^{-\frac{1}{2}} (\rho + i0)^{\frac{\nu-3}{2}} + e^{i\pi(\nu-2)} (\rho + i0)^{-\frac{1}{2}} (\rho - i0)^{\frac{\nu-3}{2}}]. \tag{4.2. 1}$$

Simplifying terms we obtain:

$$h(\rho, \nu) = \frac{\pi^{\frac{\nu-3}{2}}}{2^{\nu-1}} e^{\frac{i\pi(2-\nu)}{2}} \Gamma\left(\frac{3-\nu}{2}\right) [\rho_+^{\frac{\nu-4}{2}} + e^{\frac{i\pi(\nu-2)}{2}} \rho_-^{\frac{\nu-4}{2}}]. \tag{4.2. 2}$$

Thus, in four dimensions:

$$h_4(\rho) = \delta(\rho) * \delta(\rho) = \frac{\pi}{2} \text{Sgn}(\rho). \tag{4.2. 3}$$

Note that this convolution does not make sense in a four-dimensional Euclidean space, since in that case  $\delta(\rho) \equiv 0$ .

As a second example we calculate the convolution  $\delta(\rho - m^2) * \delta(\rho - m^2)$ . In this case we have

$$h(\rho, \nu) = \frac{\pi^{\frac{\nu-3}{2}}}{2^{\nu-1}} e^{\frac{i\pi(2-\nu)}{2}} \Gamma\left(\frac{3-\nu}{2}\right) \left[ (\rho - i0)^{-\frac{1}{2}} (\rho - 2m^2 + i0)^{\frac{\nu-3}{2}} + e^{i\pi(\nu-2)} (\rho + i0)^{-\frac{1}{2}} (\rho - 2m^2 - i0)^{\frac{\nu-3}{2}} \right]. \tag{4.2. 4}$$

When  $\nu = 4$  we obtain

$$\begin{aligned} & \delta(\rho - m^2) * \delta(\rho - m^2) \\ &= \frac{\pi}{4} \left[ (\rho - i0)^{-\frac{1}{2}} (\rho - 2m^2 + i0)^{\frac{1}{2}} + e^{i\pi(\nu-2)} (\rho + i0)^{-\frac{1}{2}} (\rho - 2m^2 - i0)^{\frac{1}{2}} \right]. \end{aligned} \tag{4.2. 5}$$

## 5. The convolution of n massless Feynman's propagators

### 5.1. The Minkowskian space case

Let us now calculate the convolution of n massless Feynman's propagators ( $n \geq 2$ ). For this purpose we take into account that

$$\mathcal{F}^{-1}\{f_1 * f_2 * \dots * f_n\} = (2\pi)^{(n-1)\nu} \hat{f}_1 \hat{f}_2 \dots \hat{f}_n \tag{5.1. 1}$$

According to [66], we have

$$\mathcal{F}^{-1}\{(\rho + i0)^{-1}\} = \frac{e^{-\frac{i\pi}{2}(\nu-1)}}{(2\pi)^\nu} 2^{(\nu-2)} \pi^{\frac{\nu}{2}} \Gamma\left(\frac{\nu}{2} - 1\right) (x - i0)^{1-\frac{\nu}{2}}, \tag{5.1. 2}$$

and therefore,

$$\begin{aligned} & \mathcal{F}^{-1}\{(\rho + i0)^{-1} * (\rho + i0)^{-1} * \dots * (\rho + i0)^{-1}\} \\ &= (2\pi)^{(n-1)\nu} \frac{e^{-\frac{i\pi}{2}(\nu-1)n}}{(2\pi)^{\nu n}} 2^{(\nu-2)n} \pi^{\frac{\nu n}{2}} \left[ \Gamma\left(\frac{\nu}{2} - 1\right) \right]^n (x - i0)^{n(1-\frac{\nu}{2})}. \end{aligned} \tag{5.1. 3}$$

Using again [66] we have now

$$\begin{aligned} & \mathcal{F}\{(x - i0)^{n(1-\frac{\nu}{2})}\} \\ &= \frac{e^{-\frac{i\pi}{2}(\nu-1)}}{\Gamma\left[n\left(\frac{\nu}{2} - 1\right)\right]} 2^{\nu+2n(1-\frac{\nu}{2})} \pi^{\frac{\nu}{2}} \Gamma\left[\frac{\nu}{2} + n\left(1 - \frac{\nu}{2}\right)\right] (\rho + i0)^{n(\frac{\nu}{2}-1)-\frac{\nu}{2}}, \end{aligned} \tag{5.1. 4}$$

with which we obtain

$$\begin{aligned} & (\rho + i0)^{-1} * (\rho + i0)^{-1} * \dots * (\rho + i0)^{-1} \\ &= \frac{e^{-\frac{i\pi}{2}(n-1)(\nu-1)}}{\Gamma\left[n\left(\frac{\nu}{2} - 1\right)\right]} \pi^{\frac{\nu}{2}(n-1)} \left[ \Gamma\left(\frac{\nu}{2} - 1\right) \right]^n \Gamma\left[\frac{\nu}{2} + n\left(1 - \frac{\nu}{2}\right)\right] (\rho + i0)^{n(\frac{\nu}{2}-1)-\frac{\nu}{2}}. \end{aligned} \tag{5.1. 5}$$

We have then, for the convolution of n massless Feynman's propagators, the result

$$\begin{aligned} & i(\rho + i0)^{-1} * i(\rho + i0)^{-1} * \dots * i(\rho + i0)^{-1} \\ &= \frac{e^{\frac{i\pi}{2}[n-(n-1)(\nu-1)]}}{\Gamma\left[n\left(\frac{\nu}{2} - 1\right)\right]} \pi^{\frac{\nu}{2}(n-1)} \left[ \Gamma\left(\frac{\nu}{2} - 1\right) \right]^n \Gamma\left[\frac{\nu}{2} + n\left(1 - \frac{\nu}{2}\right)\right] (\rho + i0)^{n(\frac{\nu}{2}-1)-\frac{\nu}{2}}. \end{aligned} \tag{5.1. 6}$$

After a tedious calculation we obtain the corresponding Laurent's expansion around  $\nu = 4$ :

$$\begin{aligned} & i(\rho + i0)^{-1} * i(\rho + i0)^{-1} * \dots * i(\rho + i0)^{-1} = \frac{2i\pi^{2(n-1)}\rho^{n-2}}{[\Gamma(n)]^2(\nu - 4)} \\ &+ \frac{i\pi^{2(n-1)}\rho^{n-2}}{\Gamma(n)\Gamma(n-1)} \left[ \ln(\rho + i0) - i\pi + \ln(\pi) + \frac{n}{n-1}\psi(1) - \frac{n}{n-1}\psi(n) \right. \\ &\left. - \psi(n-1) \right] + \sum_{m=1}^{\infty} a_m(\rho)(\nu - 4)^m. \end{aligned} \tag{5.1. 7}$$

The independent  $\nu - 4$  term is the result of the convolution in four dimensions

$$\begin{aligned}
 & [i(\rho + i0)^{-1} * i(\rho + i0)^{-1} * \dots * i(\rho + i0)^{-1}]_{\nu_0=4} \\
 &= \frac{i\pi^{2(n-1)}\rho^{n-2}}{\Gamma(n)\Gamma(n-1)} \left[ \ln(\rho + i0) - i\pi + \ln(\pi) + \frac{n}{n-1}\psi(1) - \frac{n}{n-1}\psi(n) \right. \\
 & \left. - \psi(n-1) \right].
 \end{aligned} \tag{5.1. 8}$$

### 5.2. The Euclidean space case

Let us now calculate the convolution of  $n$  massless Feynman's propagators ( $n \geq 2$ ) in Euclidean space, using again (5.1. 1). According to [66], we obtain

$$\mathcal{F}^{-1}\{k^{-2}\} = \frac{1}{(2\pi)^\nu} 2^{(\nu-2)} \pi^{\frac{\nu}{2}} \Gamma\left(\frac{\nu}{2} - 1\right) r^{2-\nu} \tag{5.2. 1}$$

For  $n$  propagators we have then

$$\begin{aligned}
 & \mathcal{F}^{-1}\{k^{-2} * k^{-2} * \dots * k^{-2}\} \\
 &= \frac{(2\pi)^{(n-1)\nu}}{(2\pi)^{\nu n}} 2^{(\nu-2)n} \pi^{\frac{\nu n}{2}} \left[ \Gamma\left(\frac{\nu}{2} - 1\right) \right]^n r^{n(2-\nu)}.
 \end{aligned} \tag{5.2. 2}$$

Appealing again to [66], we can evaluate the corresponding Fourier Transform

$$\begin{aligned}
 & \mathcal{F}\{r^{n(2-\nu)}\} \\
 &= \frac{1}{\Gamma\left[n\left(\frac{\nu}{2} - 1\right)\right]} 2^{\nu+2n\left(1-\frac{\nu}{2}\right)} \pi^{\frac{\nu}{2}} \Gamma\left[\frac{\nu}{2} + n\left(1 - \frac{\nu}{2}\right)\right] k^{n(\nu-2)-\nu}.
 \end{aligned} \tag{5.2. 3}$$

Thus,

$$\begin{aligned}
 & k^{-2} * k^{-2} * \dots * k^{-2} \\
 &= \frac{\pi^{\frac{\nu}{2}(n-1)}}{\Gamma\left[n\left(\frac{\nu}{2} - 1\right)\right]} \left[ \Gamma\left(\frac{\nu}{2} - 1\right) \right]^n \Gamma\left[\frac{\nu}{2} + n\left(1 - \frac{\nu}{2}\right)\right] k^{n(\nu-2)-\nu}.
 \end{aligned} \tag{5.2. 4}$$

Let  $\rho = k^2$ . We have then for the convolution of  $n$  massless Feynman's propagators the result

$$\begin{aligned}
 & \rho^{-1} * \rho^{-1} * \dots * \rho^{-1} \\
 &= \frac{\pi^{\frac{\nu}{2}(n-1)}}{\Gamma\left[n\left(\frac{\nu}{2} - 1\right)\right]} \left[ \Gamma\left(\frac{\nu}{2} - 1\right) \right]^n \Gamma\left[\frac{\nu}{2} + n\left(1 - \frac{\nu}{2}\right)\right] \rho^{n\left(\frac{\nu}{2}-1\right)-\frac{\nu}{2}}.
 \end{aligned} \tag{5.2. 5}$$

By recourse to Laurent's expansion we obtain

$$\begin{aligned}
 \rho^{-1} * \rho^{-1} * \dots * \rho^{-1} &= \frac{2(-1)^{n-1}\pi^{2(n-1)}\rho^{n-2}}{[\Gamma(n)]^2(\nu-4)} \\
 &+ \frac{(-1)^{n-1}\pi^{2(n-1)}\rho^{n-2}}{\Gamma(n)\Gamma(n-1)} \left[ \ln(\rho) + \ln(\pi) + \frac{n}{n-1}\psi(1) - \frac{n}{n-1}\psi(n) \right. \\
 & \left. - \psi(n-1) \right] + \sum_{m=1}^{\infty} a_m(\rho)(\nu-4)^m.
 \end{aligned} \tag{5.2. 6}$$

The result of the convolution in four dimensions is then

$$\begin{aligned}
 & [\rho^{-1} * \rho^{-1} * \dots * \rho^{-1}]_{\nu_0=4} \\
 &= \frac{(-1)^{n-1}\pi^{2(n-1)}\rho^{n-2}}{\Gamma(n)\Gamma(n-1)} \left[ \ln(\rho) + \ln(\pi) + \frac{n}{n-1}\psi(1) - \frac{n}{n-1}\psi(n) \right. \\
 & \left. - \psi(n-1) \right].
 \end{aligned} \tag{5.2. 7}$$

We emphasize that the results of this section are completely original.



## 6. The convolution of massless Wheeler’s propagators

### 6.1. The convolution of two massless Wheeler’s propagators

The Wheeler’s massless propagator is given by (note that this propagator can not be defined in Euclidean space)

$$W(\rho) = \frac{i}{2} \left[ \frac{1}{\rho + i0} + \frac{1}{\rho - i0} \right], \tag{6.1. 1}$$

and can be written in the form:

$$W(\rho) = \frac{i}{\rho + i0} - \pi\delta(\rho). \tag{6.1. 2}$$

Therefore, we have

$$W(\rho) * W(\rho) = \frac{i}{\rho + i0} * \frac{i}{\rho + i0} - 2\pi\delta(\rho) * \frac{i}{\rho + i0} + \pi^2\delta(\rho) * \delta(\rho). \tag{6.1. 3}$$

After a long and tedious calculation, using (4.1. 1) we obtain

$$\begin{aligned} -2\pi\delta(\rho) * \frac{i}{\rho + i0} &= \frac{-i\pi^{\frac{\nu-1}{2}}}{2^{\nu-2}} e^{i\pi(\frac{3-\nu}{2})} \Gamma\left(\frac{3-\nu}{2}\right) \Gamma(\nu-2) \Gamma(3-\nu) \\ &\times \{ [1 + e^{i\pi(\nu-2)}][1 - e^{-i\pi(3-\nu)}] H(\rho) \rho^{\frac{\nu}{2}-1} \\ &+ 2e^{i\pi(\frac{\nu-3}{2})} [e^{i\pi(\nu-2)} - 1] H(-\rho) (-\rho)^{\frac{\nu}{2}-2} \}. \end{aligned} \tag{6.1. 4}$$

This last equation can be re-written in the form

$$\begin{aligned} -2\pi\delta(\rho) * \frac{i}{\rho + i0} &= \frac{\pi^{\frac{\nu+3}{2}} e^{\frac{i\pi}{2}(3-\nu)} \cos \pi\left(\frac{\nu-2}{2}\right)}{2^{\nu-4} \Gamma\left(\frac{\nu-1}{2}\right) \sin \pi\left(\frac{\nu-3}{2}\right) \sin \pi\nu} \\ &\left\{ \cos \pi\left(\frac{\nu-2}{2}\right) H(\rho) \rho^{\frac{\nu}{2}-1} - e^{i\pi(\nu-2)} H(-\rho) (-\rho)^{\frac{\nu}{2}-2} \right\}. \end{aligned} \tag{6.1. 5}$$

For the first convolution of (6.1. 3), we have from (5.1. 6), with  $n = 2$

$$\frac{i}{\rho + i0} * \frac{i}{\rho + i0} = \frac{e^{i\frac{\pi}{2}(3-\nu)} \pi^{\frac{\nu}{2}}}{\Gamma(\nu-2)} \left[ \Gamma\left(\frac{\nu}{2} - 1\right) \right]^2 \Gamma\left(2 - \frac{\nu}{2}\right) (\rho + i0)^{\frac{\nu}{2}-2}. \tag{6.1. 6}$$

This equation can be re-written in the form:

$$\frac{i}{\rho + i0} * \frac{i}{\rho + i0} = \frac{e^{i\frac{\pi}{2}(3-\nu)} \pi^{\frac{\nu-3}{2}} \cos\left(\frac{\nu-3}{2}\right)}{2^{\nu} \Gamma\left(\frac{\nu-1}{2}\right) \sin \pi\nu} (\rho + i0)^{\frac{\nu}{2}-2}. \tag{6.1. 7}$$

When  $\nu = 4$ , the sum of (6.1. 5) and (6.1. 7) has as a result

$$\frac{i}{\rho + i0} * \frac{i}{\rho + i0} - 2\pi\delta(\rho) * \frac{i}{\rho + i0} = \pi^3 H(-\rho) \tag{6.1. 8}$$

Using now (4.2. 3), we find

$$W(\rho) * W(\rho) = \frac{\pi^3}{2}. \tag{6.1. 9}$$

This result was obtained in the [3], formula (6.12) using the convolution of even Tempered Ultradistributions. The coincidence of (6.1. 9) with (6.12) of [3] confirms the validity of the results obtained in section 6 of this paper. We emphasize that the present results are obtained in a manner considerably simpler to that of [3].

### 6.2. The convolution of n massless Wheeler’s propagators

According to [66], we have

$$\mathcal{F}^{-1}\{(\rho + i0)^{-1}\} = \frac{e^{-\frac{i\pi}{2}(\nu-1)}}{(2\pi)^\nu} 2^{(\nu-2)} \pi^{\frac{\nu}{2}} \Gamma\left(\frac{\nu}{2} - 1\right) (x - i0)^{1-\frac{\nu}{2}}, \tag{6.2. 1}$$

$$\mathcal{F}^{-1}\{(\rho - i0)^{-1}\} = \frac{e^{\frac{i\pi}{2}(\nu-1)}}{(2\pi)^\nu} 2^{(\nu-2)} \pi^{\frac{\nu}{2}} \Gamma\left(\frac{\nu}{2} - 1\right) (x + i0)^{1-\frac{\nu}{2}}, \tag{6.2. 2}$$

Thus,

$$\mathcal{F}^{-1}\{W(\rho)\} = \frac{i\pi^{\frac{\nu}{2}}}{(2\pi)^\nu} 2^{(\nu-2)} \Gamma\left(\frac{\nu}{2} - 1\right) \sin\left(\frac{\pi\nu}{2}\right) x_+^{1-\frac{\nu}{2}}, \tag{6.2.3}$$

As a consequence we obtain for n Wheeler’s propagators

$$\begin{aligned} &\mathcal{F}^{-1}\{W(\rho) * W(\rho) * \dots * W(\rho)\} \\ &= \frac{i^n \pi^{\frac{n\nu}{2}}}{(2\pi)^{n\nu}} 2^{n(\nu-2)} \left[\Gamma\left(\frac{\nu}{2} - 1\right)\right]^n \sin^n\left(\frac{\pi\nu}{2}\right) x_+^{n(1-\frac{\nu}{2})}, \end{aligned} \tag{6.2.4}$$

Resorting again to [66] we have:

$$\begin{aligned} \mathcal{F}\{x_+^{n(1-\frac{\nu}{2})}\} &= \pi^{\frac{\nu}{2}-1} 2^{(1-n)\nu+2n} \Gamma\left(n + 1 - \frac{n\nu}{2}\right) \Gamma\left[n - \frac{(n-1)\nu}{2}\right] \\ &\otimes \frac{1}{2} \{e^{-i\pi[n-(n-1)\frac{\nu}{2}]}(\rho - i0)^{(n-1)\frac{\nu}{2}-n} + e^{i\pi[n-(n-1)\frac{\nu}{2}]}(\rho + i0)^{(n-1)\frac{\nu}{2}-n}\} \end{aligned} \tag{6.2.5}$$

Using (6.2.5) we obtain finally:

$$\begin{aligned} &W(\rho) * W(\rho) * \dots * W(\rho) \\ &= \frac{i^n \pi^{\frac{n\nu}{2}(n-1)-1}}{2} \left[\Gamma\left(\frac{\nu}{2} - 1\right)\right]^n \Gamma\left(n + 1 - \frac{n\nu}{2}\right) \Gamma\left[n - \frac{(n-1)\nu}{2}\right] \sin^n\left(\frac{\pi\nu}{2}\right) \\ &\otimes \{e^{i\pi[n-(n-1)\frac{\nu}{2}]}(\rho + i0)^{(n-1)\frac{\nu}{2}-n} + e^{-i\pi[n-(n-1)\frac{\nu}{2}]}(\rho - i0)^{(n-1)\frac{\nu}{2}-n}\} \end{aligned} \tag{6.2.6}$$

We see that formula (6.2.6) has a zero of order  $n - 2$  for  $\nu \geq 4$ ,  $\nu$  even, and consequently cancels for those dimensions when  $n \geq 3$ . Thus we can affirm that for  $\nu = 4$

$$W(\rho) * W(\rho) * \dots * W(\rho) = 0 \tag{6.2.7}$$

when  $n \geq 3$ .

## 7. Discussion

In QFT, when we use perturbative expansions, we are dealing with products of distributions in configuration space or, what is the same, with convolutions of distributions in momentum space.

In four earlier papers [1–4] we have demonstrated the existence of the convolution of JSS Ultradistributions. This convolution allows us to treat non renormalizable QFT’s, but has the disadvantage of being extremely complex.

Following a procedure similar to those of the previously mentioned papers, we defined the convolution of Lorentz Invariant Temperatd Distributions using the DR of BG.

Using this convolution we have obtained, for example, the convolution of n massless Feynman’s propagators both in Minkowskian and Euclidean spaces and the convolution of two massless Wheeler’s propagators.

It is our hope that this convolution will allow one to treat non-renormalizable QFT’s.

## Appendix

The purpose of this appendix is to compare the generalization of the DR obtained in this paper with the usual BG DR and show the differences between them. For this we consider the convolution of two massless propagators in Euclidean space. We start then with the usual formula for the convolution in four dimensions:

$$[\rho^{-1} * \rho^{-1}]_{\nu=4} = \int \frac{d^4 p}{\vec{p}^2 (\vec{p} - \vec{k})^2} \tag{A.1}$$

The generalization of the previous convolution to  $\nu$  dimensions is

$$\rho^{-1} * \rho^{-1} = \int \frac{d^\nu p}{\vec{p}^2 (\vec{p} - \vec{k})^2} \tag{A.2}$$

Using the Feynman’s parameters:

$$\frac{1}{AB} = \int_0^1 \frac{dx}{[Ax + B(1-x)]^2}, \tag{A.3}$$

we can write the convolution as:

$$\rho^{-1} * \rho^{-1} = \int d^\nu p \int_0^1 \frac{dx}{[(\vec{p} - \vec{k})^2 x + \vec{p}^2 (1-x)]^2} = \int_0^1 dx \int \frac{d^\nu p}{[(\vec{p} - \vec{k})^2 x + \vec{p}^2 (1-x)]^2}, \tag{A.4}$$

or more simply:

$$\rho^{-1} * \rho^{-1} = \int_0^1 dx \int \frac{d^\nu p}{[(\vec{p} - \vec{k}x)^2 + \vec{k}^2 x(1-x)]^2} \quad (\text{A.5})$$

Making the change of variable:  $\vec{s} = \vec{p} - \vec{k}x$  and calling  $a = \vec{k}^2 x(1-x)$  we obtain:

$$\rho^{-1} * \rho^{-1} = \int_0^1 dx \int \frac{d^\nu s}{(s^2 + a)^2}, \quad (\text{A.6})$$

equivalently:

$$\rho^{-1} * \rho^{-1} = \frac{2\pi^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} \int_0^1 dx \int_0^\infty \frac{s^{\nu-1}}{(s^2 + a)^2} ds \quad (\text{A.7})$$

Making the change of variable  $y = s^2$  we have:

$$\rho^{-1} * \rho^{-1} = \frac{\pi^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} \int_0^1 dx \int_0^\infty \frac{y^{\frac{\nu}{2}-1}}{(y + a)^2} dy \quad (\text{A.8})$$

Using [68] we can calculate the previous integral. The result is:

$$\rho^{-1} * \rho^{-1} = \pi^{\frac{\nu}{2}} \Gamma\left(2 - \frac{\nu}{2}\right) \int_0^1 a^{\frac{\nu}{2}-2} dx, \quad (\text{A.9})$$

or in an equivalent way:

$$\rho^{-1} * \rho^{-1} = \pi^{\frac{\nu}{2}} \Gamma\left(2 - \frac{\nu}{2}\right) \rho^{\frac{\nu}{2}-2} \int_0^1 [x(1-x)]^{\frac{\nu}{2}-2} dx \quad (\text{A.10})$$

By recourse again to the results given in [68] we have:

$$\rho^{-1} * \rho^{-1} = \frac{\pi^{\frac{\nu}{2}} \left[\Gamma\left(\frac{\nu}{2} - 1\right)\right]^2 \rho^{\frac{\nu}{2}-2}}{\Gamma(\nu - 2)} \Gamma\left(2 - \frac{\nu}{2}\right) \quad (\text{A.11})$$

We notice now that (A.11) can be written in the form:

$$\rho^{-1} * \rho^{-1} = \frac{2\pi^2}{4 - \nu} - \pi^2 [\ln \rho + \ln \pi - \psi(2)] + \sum_{k=1}^{\infty} a_k (\nu - 4)^k \quad (\text{A.12})$$

We see that the four-dimensional convolution is not univocally defined from the  $\nu$ -dimensional convolution since there are several ways to choose its finite part.

If we resort to our generalization and select the independent term of  $\nu - 4$  we get for the four-dimensional convolution:

$$[\rho^{-1} * \rho^{-1}]_{\nu=4} = -\pi^2 [\ln \rho + \ln \pi - \psi(2)] \quad (\text{A.13})$$

which coincides for  $n = 2$  with our result given in (5.2.7).

We should note that the calculation made in this appendix is more complex than the one obtained with our generalization of the DR for  $n$  massless propagators. In fact, if we wanted to evaluate the convolution of  $n$  massless propagators with the usual method of DR used in this appendix, we would have to perform a very long calculation that would involve a large number of integrals. Even if we were able to obtain the correct result, the four-dimensional convolution would not be completely determined.

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