Neighbor-locating coloring: graph operations and extremal cardinalities

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Abstract

A $k-$*coloring* of a graph $G = (V, E)$ is a k-partition $\Pi = \{S_1, \ldots, S_k\}$ of V into independent sets, called *colors*. A k-coloring is called *neighbor-locating* if for every pair of vertices u, v belonging to the same color S_i, the set of colors of the neighborhood of u is different from the set of colors of the neighborhood of v. The *neighbor-locating chromatic number*, $\chi_{_{NL}}(G)$, is the minimum cardinality of a neighbor-locating coloring of G.

In this paper, we examine the neighbor-locating chromatic number for various graph operations: the join, the disjoint union and Cartesian product. We also characterize all connected graphs of order $n \geq 3$ with neighbor-locating chromatic number equal either to n or to $n-1$ and determine the neighbor-locating chromatic number of split graphs.

Keywords: Coloring, location, neighbor-location, complete multipartite graph, join graph, split graph, disjoint union, Cartesian product.

1 Introduction

Domination and location in graphs are two important concepts that have received a lot of attention, usually separately, but sometimes also both together. There are mainly two types of location, the metric location and the neighbor location. In this work, we are interested in the neighbor location, and we study this concept in the particular context of a special kind of vertex partitions, called colorings.

In [6], G. Chartrand, E. Salehi and P. Zhang, brought the concept of metric-location to the ambit of vertex partitions. Metric-location and domination, in the context of vertex partitions, is studied in [7]. In [3], there were introduced the so-called *locating colorings* considering partitions formed by independents sets.

Neighbor location in sets was introduced by P. Slater in [8]. In [2], bounds of this parameter are given. In [1], merging the concepts of papers [3,8], there were introduced the so-called *neighbor-locating colorings* and the *neighbor-locating chromatic number*, studying this parameter in some basic families of graphs.

In this work, we examine the neighbor-locating chromatic number for various graph operations. In Section 2, we study the join graph and all connected graphs with order n and neighbor-locating chromatic number equal to n are characterized.

The neighbor-locating chromatic number of all connected split graphs is determined in Section 3. The disjoint union and the Cartesian product are studied in Section 4. Finally, in Section 5, all connected graphs of order $n \geq 3$ with neighbor-locating chromatic number $n-1$ are given.

We introduce now some basic terminology. The distance between vertices $v, w \in V$ of a graph $G = (V, E)$ is denoted by $d(v, w)$. The diameter of G is $\mathrm{diam}(G) = \max\{d(v, w) : v, w \in V\}$. The distance $d(v, S)$ between a vertex $v \in V$ and a set of vertices $S \subseteq V$ is $d(v, S) = \min\{d(v, w) : w \in S\}$. Let $\Pi = \{S_1, \ldots, S_k\}$ be a partition of V. If all the parts of Π are independent sets, then we say that Π is a *coloring* of G and each part is a *color*.

A coloring $\Pi = \{S_1, \ldots, S_k\}$ is called a *(metric-)locating coloring*, an ML-coloring for short, if for every $i \in \{1, \ldots, k\}$ and for every pair of distinct

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vertices $u, v \in S_i$, there exists $j \in \{1, \ldots, k\}$ such that $d(u, S_j) \neq d(v, S_j)$. The *(metric-)locating-chromatic number* $\chi_{_L}(G)$ is the minimum cardinality of an ML-coloring of G.

A coloring $\Pi = \{S_1, \ldots, S_k\}$ is called a *neighbor-locating coloring*, an NL-*coloring* for short, if for every $i \in \{1, \ldots, k\}$ and for every pair of not isolated distinct vertices $u, v \in S_i$, there exists $j \in \{1, \ldots, k\}$ such that either $d(u, S_j) = 1$ and $d(v, S_j) \neq 1$ or $d(u, S_j) \neq 1$ and $d(v, S_j) = 1$ and there may be at most one isolated vertex for each color. The *neighbor-locating chromatic number* $\chi_{_{NL}}(G)$ is the minimum cardinality of an NL-coloring of G. For undefined basic concepts of graph theory we refer the reader to the text [5].

2 Join graphs

A graph $G = (V, E)$ is said to be a *join graph* if it is the join $G_1 \vee G_2$ of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, i.e., if $V = V_1 \cup V_2$ and $E = E_1 \cup E_2 \cup E'$, where $E' = \{v_1 v_2 : v_1 \in V_1, v_2 \in V_2\}$.

As a straightforward consequence of this definition, the following properties hold.

Proposition 2.1 *Let G_1, G_2 and G_3 be three graphs.*

(i) *$G_1 \vee G_2$ is a connected graph of diameter at most 2.*

(ii) $G_1 \vee (G_2 \vee G_3) \cong (G_1 \vee G_2) \vee G_3$.

Theorem 2.2 *Let G and H be two graphs. Then,*

$$\chi_{_{NL}}(G \vee H) = \chi_{_{NL}}(G) + \chi_{_{NL}}(H).$$

Let $r, n_1, \ldots, n_r, n$ integers such that $2 \leq r$, $1 \leq n_1 \leq \ldots \leq n_r$ and $n = n_1 + \ldots + n_r$. Then, the complete r-partite graph $K_{n_1,\ldots,n_r}$ is the graph $\overline{K_{n_1}} \vee \ldots \vee \overline{K_{n_r}}$, where $\overline{K_{n_i}}$ denotes the empty graph of order n_i, i.e., the complement of the complete graph of order n_i. Notice that complete graphs, complete bipartite graphs and stars are some examples of complete multipartite graphs.

Theorem 2.3 *Let G be a connected graph of order $n \geq 3$. Then, $\chi_{_{NL}}(G) = n$ if and only if G is a complete multipartite graph.*

3 Split graphs

A *split graph* is a graph such that the vertices can be partitioned into a clique and an independent set. When every vertex in the independent set is adjacent

to every vertex in the clique it is said to be a *complete split graph*. Observe that complete split graphs are examples of complete multipartite graphs that we just studied in the previous section. We have taken a step further and we have studied the neighbor-locating chromatic number in general split graphs.

For any connected split graph $G = (V, E)$ we can assume that there are two subsets U and W of V in such a way that (i) $V = U \cup W$, $U \cap W = \emptyset$; (ii) $G[U]$ is a complete graph; (iii) $G[W]$ is an empty graph; (iv) W is a maximal independent set, i.e., for each vertex $u \in U$ there exists a vertex $w \in W$ such that $uw \in E$.

For every $X \subseteq U$, we consider the set $\mathcal{P}(X) = \{w \in W : N(w) = X\}$ and we denote $\rho(G) = \max\{|X| + |\mathcal{P}(X)| : X \subseteq U\}$. Observe that $\rho(G) \geq |U|$.

Theorem 3.1 *If $G = (V, E)$ is a connected split graph, then*

$$\chi_{_{NL}}(G) = \begin{cases} \rho(G), & \text{if } \mathcal{P}(X) = \emptyset \text{ for all } X \subseteq U \text{ s.t. } |X| = |U| - 1 \\ \max(|U| + 1, \rho(G)), & \text{if } \mathcal{P}(X) \neq \emptyset \text{ for some } X \subseteq U \text{ s.t. } |X| = |U| - 1 \end{cases}$$

Remark 3.2 The value of the neighbor-locating chromatic number obtained for general split graphs fits with some known results, such as $\chi_{_{NL}}(S_{1,n-1}) = n$ or $\chi_{_{NL}}(G) = n$, if G is a complete split graph.

4 Disjoint union and Cartesian product

The disjoint union, $G + H$, of graphs G and H such that $V(G) \cap V(H) = \emptyset$ is a graph where the vertex set is $V(G) \cup V(H)$ and the edge set is $E(G) \cup E(H)$. In this section, we study the relationship between $\chi_{_{NL}}(G + H)$, $\chi_{_{NL}}(G)$ and $\chi_{_{NL}}(H)$.

Theorem 4.1 *Let G, H be two graphs. If $\chi_{_{NL}}(G) = k$ and $\chi_{_{NL}}(H) = h$, with $k \geq h$, then $k \leq \chi_{_{NL}}(G + H) \leq k + h$.*

Theorem 4.2 *Let k, h integers numbers where $k \geq h$. Then, for every $l \in [k, k + h]$, there exist graphs G and H such that $\chi_{_{NL}}(G) = k$, $\chi_{_{NL}}(H) = h$ and $\chi_{_{NL}}(G + H) = l$.*

Theorem 4.3 *Let G, H be two graphs and let $\chi_{_{NL}}(G) = k$, $\chi_{_{NL}}(H) = h$. If H has no isolated vertices, then $k \leq \chi_{_{NL}}(G + H) \leq k + h - 1$.*

Theorem 4.4 *Let G be a graph and let $\chi_{_{NL}}(G) = k$. If G contains a universal vertex, then $\chi_{_{NL}}(G + G) \leq k + 1$.*

The Cartesian product, $G \square H$, of graphs G and H is a graph such that the vertex set is $V(G) \times V(H)$ and two vertices (u, u') and (v, v') are adjacent

in $G\Box H$ if and only if either $u = v$ and u' is adjacent to v' in H, or $u' = v'$ and u is adjacent to v in G. Now, we study the relationship between $\chi_{NL}(G\Box H)$, $\chi_{NL}(G)$ and $\chi_{NL}(H)$.

Theorem 4.5 *Let G, H be connected graphs, then*

$$\chi_{NL}(G\Box H) \leq \chi_{NL}(G)\,\chi_{NL}(H).$$

Note that for $G \cong H \cong K_2$ the above inequality is attained.

Theorem 4.6 *If G is a graph with a universal vertex and H is a connected graph, then $\chi_{NL}(G\Box H) \leq \chi_{NL}(G) + |V(H)| - 1$.*

5 Extremal graphs: case $\chi_{NL}(G) = n - 1$

In Section 2, we have characterized all the connected graphs with neighbor-locating chromatic number equal to the order of the graph. Now, we determine the connected graphs of order n and neighbor-locating chromatic number equal to $n - 1$.

- Let $\mathcal{H}$ denote the set of all connected graphs G or order $n \geq 3$ such that $H \cong G - v$ is a complete multipartite graph for some vertex v of G.
- Let V_1, V_2, ..., V_k, $k \geq 2$, the partite sets of H, where $|V_i| = n_i$ and $a_i = |N(v) \cap V_i|$ $(1 \leq i \leq k)$.
- Let $\mathcal{F}$ denote the set of all graphs $G \in \mathcal{H}$ that satisfy al least one of the following properties:

(1) For each integer $i \in \{1, \ldots, k\}$, we have $a_i \in \{0, n_i\}$ and there are at least two distinct integers $j, j' \in \{1, \ldots, k\}$ for which $a_j = a_{j'} = 0$.

(2) There is exactly one integer $j \in \{1, \ldots, k\}$ such that $0 < a_j < n_j$, and $a_j = n_j - 1$ for this integer j.

- Let $\mathcal{G}$ denote the set of all graphs $G = G_n \vee 2K_2$ where G_n is a complete multipartite graph G_n of order $n - 4 \geq 1$.

Theorem 5.1 *Let G be a connected graph of order $n \geq 5$. Then, $\chi_{NL}(G) = n - 1$ if and only if $G \in \mathcal{F} \cup \mathcal{G}$.*

The proof of this Theorem is based on the following lemmas and previous results.

Theorem 5.2 **([4])** *Let G be a connected graph of order $n \geq 4$. Then, $\chi_{L}(G) = n - 1$ if and only if $G \in \mathcal{F} \cup \mathcal{G}$.*

If a graph G_1 is an induced subgraph of a graph G_2, we write $G_1 \prec G_2$.

Lemma 5.3 *Let G be a connected graph of order $n \geq 5$. If $\chi_{_{NL}}(G) = n-1$ and $2K_2 \prec G$, then $G \in \mathcal{G}$.*

Lemma 5.4 ([4]) *Let G be a connected graph of order $n \geq 4$. If $\chi_{_L}(G) = n-1$ and $2K_2 \not\prec G$, then $G \in \mathcal{F}$.*

Lemma 5.5 *Let G be a graph of order $n \geq 5$ and diameter 3 such that $2K_2 \not\prec G$. If $\chi_{_{NL}}(G) = n-1$, then $\chi_{_L}(G) = n-1$.*

Lemma 5.6 *Let G be a connected graph of order $n \geq 3$.*

(i) *If $\mathrm{diam}(G) \leq 2$, then $\chi_{_L}(G) = \chi_{_{NL}}(G)$.*

(ii) *If $\mathrm{diam}(G) \geq 4$, then $\chi_{_{NL}}(G) \leq n-2$.*

References

[1] L. Alcón, M. Gutiérrez, C. Hernando, M. Mora and I. M. Pelayo: *Neighbor-locating colorings in graphs.* Preprint.

[2] J. Cáceres, C. Hernando, M. Mora, I. M. Pelayo and M. L. Puertas: *Locating-dominating codes: bounds and extremal cardinalities.* Appl. Math. Comput., **220** (2013), 38–45.

[3] G. Chartrand, D. Erwin, M. A. Henning, P. J. Slater and P. Zhang: *The locating-chromatic number of a graph.* Bull. Inst. Combin. Appl., **36** (2002), 89–101.

[4] G. Chartrand, D. Erwin, M. A. Henning, P. J. Slater and P. Zhang: *Graphs of order n with locating-chromatic number $n-1$.* Discrete Math., **269** (2003), 65–79.

[5] G. Chartrand, L. Lesniak, P. Zhang: Graphs and Digraphs, (5th edition), CRC Press, Boca Raton, Florida, (2011).

[6] G. Chartrand, E. Salehi and P. Zhang: *The partition dimension of a graph.* Aequationes Mathematicae, **59** (2000), 45–54.

[7] C. Hernando, M. Mora and I. M. Pelayo: *Metric-locating-dominating partitions in graphs.* Submitted. (https://arxiv.org/abs/1711.01086).

[8] P. J. Slater, *Dominating and reference sets in a graph*, J. Math. Phys. Sci., **22** (1988), 445–455.