

Comment on: “Interaction of the magnetic quadrupole moment of a non-relativistic particle with an electric field in a rotating frame. *Ann. Phys.* 412 (2020) 168040”

Francisco M. Fernández*
INIFTA, DQT, Sucursal 4, C.C 16,
1900 La Plata, Argentina

Abstract

We analyze a recent treatment of the interaction of a magnetic quadrupole moment with a radial electric field for a non-relativistic particle in a rotating frame and show that the derivation of the equations in the paper is anything but rigorous. The authors presented eigenvalues and eigenfunctions for two sets of quantum numbers as if they belonged to the same physical problem when they are solutions for two different models. In addition to it, the authors failed to comment on the possibility of multiple solutions for every set of quantum numbers.

In a recent paper [1] the authors studied the interaction of a magnetic quadrupole moment with a radial electric field for a non-relativistic particle in a rotating frame. They solved the Schrödinger equation for a model potential by means of a power-series method and obtained the lowest eigenvalues and eigenfunctions. In this Comment we analyze the derivation of the main equations and discuss their solutions. We will not consider the validity of the model

*fernande@quimica.unlp.edu.ar

or its physical utility, for this reason we skip most of the steps followed by the authors to derive their main equations and outline just those that are relevant for the discussion of the results.

The authors stated that “In the geometric approach, the medium with a disclination has the line element in cylindrical coordinates (in units such that $c = 1$), given by

$$ds^2 = -dt^2 + d\rho^2 + \alpha^2 \rho^2 d\varphi^2 + dz^2, \quad (1)$$

where $-\infty < z < \infty$, $-\infty < t < \infty$, $\rho \geq 0$ and $0 \leq \varphi \leq 2\pi$. In this way they were able to derive a differential operator that they called $\vec{\pi}$ that we do not show here because it is not relevant for present purposes. In order to simplify the differential equation for $\vec{\pi}^2$ the authors stated that “If we consider $\partial_\varphi = i\ell$, $\partial_z = ik \dots$ ” It is not clear if the authors simply believe that those derivative operators are just imaginary numbers and, besides, they did not indicate the possible values of ℓ and k (at this point). Notice that they also failed to say that $\hbar = 1$ in the chosen units.

The authors chose the “static scalar potential”

$$V(\rho) = a_1\rho + a_2\rho^2 - \frac{a_3}{\rho} + \frac{a_4}{\rho^2}, \quad (2)$$

and later stated that “The interaction is time-independent so that one can write $\Psi(t, r, \varphi, z) = e^{-i(\varepsilon t - \ell\varphi - kz)}\psi(\rho)$.” It is worth pointing out that the differential equation for $\vec{\pi}^2$ does not contain a time derivative so that the factor $e^{-i\varepsilon t}$ is unnecessary and the other two factors $e^{i(\ell\varphi + kz)}$ are also unnecessary because the authors had already removed the derivatives ∂_φ and ∂_z in the unorthodox way indicated above.

In this way, and by means of a suitable change of variables, the authors arrived at the eigenvalue equation

$$\left[\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{V_{-2}}{r^2} + \frac{V_{-1}}{r} - V_1 r - r^2 + W \right] \psi(r) = 0, \quad (3)$$

$$V_{-2} = \frac{\ell^2}{\alpha^2} + 2ma_4, \quad V_{-1} = \frac{2ma_3}{\sqrt{\eta}}, \quad V_1 = \frac{2ma_1}{\eta^{3/2}}, \quad W = \frac{\kappa^2}{\eta},$$

where the parameters a_i , η and κ are given in the authors’ paper and will not be shown here. In order to solve this equation the authors proposed the

transformation

$$\begin{aligned}\psi(r) &= e^{-\frac{r^2}{2} - \frac{Cr}{2}} r^{\frac{\ell^2}{\alpha^2} + D} h(r), \\ C &= \frac{2ma_1}{\eta^{3/2}}, D = -\frac{\ell^2}{\alpha^2} \pm \alpha \sqrt{\ell^2 + 2ma_4\alpha^2},\end{aligned}\quad (4)$$

and stated that “the positive sign is physically acceptable”. One can easily verify that the correct behaviour at origin is r^s , where

$$s = \sqrt{V_{-2}} = \frac{1}{|\alpha|} \sqrt{\ell^2 + 2ma_4\alpha^2} \neq \frac{\ell^2}{\alpha^2} + D. \quad (5)$$

This *misprint* is carried out throughout the paper.

The authors rewrote the resulting equation for $h(r)$

$$\begin{aligned}h''(r) + \left(\frac{2\sqrt{V_{-2}} + 1}{r} - 2r \right) h'(r) - V_1 h'(r) + \frac{2V_{-1} - V_1 (2\sqrt{V_{-2}} + 1)}{2r} h(r) + \\ \left(\frac{V_1^2}{4} - 2\sqrt{V_{-2}} - 2 + W \right) h(r) = 0,\end{aligned}\quad (6)$$

as the biconfluent Heun equation

$$H''(s) + \left(\frac{1+a}{s} - 2s - b \right) H'(s) + \left(c - 2 - a - \frac{b|a+1| + d}{2s} \right) H(s) = 0, \quad (7)$$

where the parameters a , b , c and d are given in the authors' paper and here we only show the correct expression for a

$$a = 2\sqrt{V_{-2}} = \frac{2}{|\alpha|} \sqrt{\ell^2 + 2ma_4\alpha^2}. \quad (8)$$

Since $a > 0$ then $|a+1| = a+1$ that greatly facilitates the calculation (it seems that the authors did not realize this fact).

In order to solve the Heun equation the authors tried the power-series

$$H(s) = \sum_{n=0} c_n s^{n+p}, \quad (9)$$

and concluded that “from the coefficient of s^{p-2} , we see that $p = 0$ or $p = -a$ ”. This analysis is unnecessary after having discussed the behaviour of $h(r)$ at origin from which it follows that the physically acceptable solution is in fact $p = 0$. However, the authors commented on this point: “For the sake of this

paper, we shall consider only the solutions $p = 0$ from Eq. (27)”. It seems that the authors believe that the other solution $p = -a$, already discarded previously, is suitable. The coefficients c_j satisfy the three-term recurrence relation

$$\begin{aligned} c_{j+2} &= A_j c_{j+1} + B_j c_j = 0, \quad j = -1, 0, 1, \dots, \quad c_{-1} = 0, \\ A_j &= \frac{2b(j+1) + b(a+1) + d}{2(j+2)(j+2+a)}, \quad B_j = \frac{a-c+2j+2}{(j+2)(j+2+a)}. \end{aligned} \quad (10)$$

In order to obtain polynomial solutions the authors chose the conditions $c - a = 2(n_0 + 1)$ and $c_{n_0+1} = 0$. From the former they obtained an expression for the energy $\mathcal{E}_{n_0, \ell}$ and the latter tells us that not all the model parameters are independent. For example, the authors decided to obtain a in terms of the other parameters, n_0 and ℓ ; that is to say $a_{n_0, \ell}$. In this way the authors stated that “ a_4 should be considered as $a_{4, n_0, \ell}$ ”. Consequently, the potential (2) should be written as $V_{n_0, \ell}(\rho)$ because it changes with the quantum numbers through a_4 . As a result each pair $\mathcal{E}_{n_0, \ell}, \psi_{n_0, \ell}(\rho)$ obtained from the authors’ procedure corresponds to some model potential $V_{n_0, \ell}(\rho)$. Such quantum-mechanical models are known as quasi-exactly solvable or conditionally solvable and some variants of this model, even more general ones, have already been treated before in a much more rigorous way [2, 3]. However, the authors presented their explicit results $\mathcal{E}_{1, \ell}, \psi_{1, \ell}(\rho)$ and $\mathcal{E}_{2, \ell}, \psi_{2, \ell}(\rho)$ as if they were energies and states of the same model and as if the problem was exactly solvable.

The fact that the model potential depends on the quantum numbers when we force that kind of truncation condition is not the only feature of the approach that they failed to mention. Another important point is that the condition $c_{n_0+1} = 0$ is a nonlinear equation that may have more than one solution. In order to illustrate this point we substitute $2(n_0 + 1) + a$ for c in $c_{n_0+1} = 0$ and solve for a . For example, when $n_0 = 1$ we obtain

$$a_{1, \ell}^{\pm} = \frac{4 - 2b^2 - bd \pm \sqrt{b^4 - 8b^2 - 8bd + 16}}{b^2}. \quad (11)$$

Of course, we should choose a real, positive root. When $n_0 = 2$, a is a root of a

cubic polynomial

$$a^3b^3 + a^2b(9b^2 + 3bd - 32) + a(23b^3 + 18b^2d + 3b(d^2 - 48) - 32d) + 15b^3 + 23b^2d + b(9d^2 - 112) + d(d^2 - 48) = 0. \quad (12)$$

If, for a given n_0 there are more than one real positive root $a_{n_0,\ell}$ then we would have eigenvalues and eigenfunctions for more than one potential $V_{n_0,\ell}$ for such pair of quantum numbers.

The problem of multiple solutions emerging from a truncation condition was also overlooked by Bakke [4] in his calculation of bound states for a Coulomb-type potential induced by the interaction between a moving electric quadrupole moment and a magnetic field.

Summarizing: we have clearly seen that the derivation of the equations in the paper by Hassanabadi et al [1] is anything but rigorous. They presented eigenvalues and eigenfunctions for two sets of quantum numbers as if they belonged to the same physical problem when they are solutions for two different models. In addition to it, the authors failed to realize the possibility of multiple solutions for every set of quantum numbers.

Addendum

In what follows we analyze the reply to present Comment. With respect to our criticism about the wrong behaviour at origin the authors stated that “It is a typo in one part of the article only and has no effect on the results”. However, it is worth noticing that their wavefunctions (39) and (43) already exhibit the wrong behaviour at origin (and we suppose that they are part of their results). The authors appear to believe that the expressions $\partial_\varphi = i\ell$ and $\partial_z = ik$ are correct.

The authors stated that they verified the correctness of their results with other methods, such as “Quasi-Exactly-Solvable method and Ansatz method”; unfortunately, they did not give any reference and we do not know what they

exactly mean by such names. They seem to be something different from the Frobenius method.

The main point is that the results given by the Frobenius method, followed by a suitable truncation of the series, are not wrong by themselves. What is wrong is the interpretation of such results. In the Comment we said (or, at least, tried to) that the authors obtained eigenvalues for different model potentials and presented them as if they were the spectrum of a single problem. In this Addendum we expand on this issue and show revealing results.

For present discussion we rewrite the eigenvalue equation as

$$u''(x) + \frac{1}{x}u(x) - \frac{\gamma^2}{x^2}u(x) - \frac{a}{x}u(x) - bxu(x) - x^2u(x) + Wu(x) = 0, \quad (13)$$

where γ , a and b are real model parameters that have nothing to do with the parameters in the equations discussed in the paper and Comment. Only the form of the equation is the same. This eigenvalue equation has square integrable solutions

$$\int_0^\infty |u(x)|^2 x dx < \infty, \quad (14)$$

for all $-\infty < a, b < \infty$ for an infinite number of discrete values of $W(a, b)$. Such eigenvalues satisfy the Hellmann-Feynman theorem [6]

$$\frac{\partial W}{\partial a} = \left\langle \frac{1}{x} \right\rangle > 0, \quad \frac{\partial W}{\partial b} = \langle x \rangle > 0. \quad (15)$$

In what follows we try to solve the eigenvalue equation (13) by means of the Frobenius method and the ansatz

$$u(x) = x^s \exp\left(-\frac{b}{2}x - \frac{x^2}{2}\right) P(x), \quad P(x) = \sum_{j=0}^{\infty} c_j x^j, \quad s = |\gamma|. \quad (16)$$

The expansion coefficients c_j satisfy the three-term recurrence relation

$$\begin{aligned} c_{j+2} &= A_j c_{j+1} + B_j c_j, \quad j = -1, 0, 1, 2, \dots, \quad c_{-1} = 0, \quad c_0 = 1, \\ A_j &= \frac{2a + b(2j + 2s + 3)}{2(j+2)[j+2(s+1)]}, \quad B_j = \frac{4(2j + 2s - W + 2) - b^2}{4(j+2)[j+2(s+1)]}. \end{aligned} \quad (17)$$

If the truncation condition $c_{n+1} = c_{n+2} = 0$, $c_n \neq 0$, $n = 0, 1, \dots$, has physically acceptable solutions for a , b and W then we obtain exact eigenfunctions because

$c_j = 0$ for all $j > n$. This truncation condition is equivalent to $B_n = 0$, $c_{n+1} = 0$ or

$$W_s^{(n)} = 2(n + s + 1) - \frac{b^2}{4}, \quad c_{n+1}(a, b) = 0, \quad (18)$$

where the second condition determines a relationship between the parameters a and b . On setting $W = W_s^{(n)}$ the coefficient B_j takes a simpler form:

$$B_j = \frac{2(j - n)}{(j + 2)[j + 2(s + 1)]}. \quad (19)$$

Notice that the truncation condition does not provide all the solutions but only those for which the parameters a and b exhibit certain relations. The reason is that this problem is not exactly solvable, as the authors appear to believe, but quasi-exactly solvable or conditionally solvable (see [7–10] and, in particular, the remarkable review [11] and references therein for more details).

As an illustrative example we consider the eigenvalue equation (13) with $b = 1$. In this case $c_{n+1}(a, 1) = 0$ is a polynomial function of a of degree $n + 1$ and it can be proved that all the roots $a_s^{(n,i)}$, $i = 1, 2, \dots, n + 1$, are real [7, 8]. For convenience we arrange the roots so that $a_s^{(n,i)} > a_s^{(n,i+1)}$ and stress the point that all of them correspond to the same eigenvalue $W_s^{(n,i)} = W_s^{(n)}$. It is important to realize that the eigenvalue $W_s^{(n)}$ is common to a set of different quantum-mechanical problems because the potential depends on a . The origin of the authors' misconception can be traced back to this obvious fact. For example, the eigenvalues $\mathcal{E}_{n_0,l}$ obtained by them correspond to different quantum-mechanical problems and are, consequently, meaningless. The polynomial solutions

$$u_s^{(n,i)}(x) = x^s \exp\left(-\frac{x^2}{2}\right) P_s^{(n,i)}(x), \quad P_s^{(n,i)}(x) = \sum_{j=0}^n c_{j,s}^{(n,i)} x^j, \quad s = |\gamma|, \quad (20)$$

share the same eigenvalue $W_s^{(n)}$ and also correspond to different quantum-mechanical problems.

The actual eigenvalues $W_{\nu,s}(a)$, $\nu = 0, 1, \dots$, $W_{\nu,s} < W_{\nu+1,s}$, of equation (13) (for a given value of b) are curves in the $a - W$ plane. It follows from the Hellmann-Feynman theorem (15) that $(a_s^{(n,i)}, W_s^{(n)})$ is a point on the curve

$W_{i-i,s}(a)$. In order to verify this fact we need the actual eigenvalues $W_{\nu,s}(a)$ that we have to obtain by means of a suitable approximate method because the eigenvalue equation (13) is not exactly solvable [8, 11]. Here, we resort to the well known Rayleigh-Ritz variational method that is known to yield upper bounds to all the eigenvalues [12] and, for simplicity, choose the non-orthogonal basis set of Gaussian functions $\left\{ \varphi_{j,s}(x) = x^{s+j} \exp\left(-\frac{x^2}{2}\right), j = 0, 1, \dots \right\}$.

In order to facilitate the variational calculations we choose $s = 0$ in what follows. Figure 1 shows several eigenvalues $W_0^{(n)}$ given by the truncation condition (red points) and the lowest actual eigenvalues $W_{\nu,0}(a)$ obtained from the variational method (blue lines). We see that there are solutions to the eigenvalue equation (13) for all values of a , that each $W_{\nu,0}(a)$ is a continuous function of a that satisfies the Hellmann-Feynman theorem (15) and that each pair $\left(a_0^{(n,i)}, W_0^{(n)}\right)$ is a point on those curves as argued above. Any vertical line starting from a given value of a will pass through no more than one red point. It means that the truncation condition yields only one eigenvalue and just for a particular model potential. We realize that the eigenvalues obtained by Hassanabadi et al [1] have no physical meaning unless one connects the points $\left(a_s^{(n,i)}, W_s^{(n)}\right)$ properly.

References

- [1] H. Hassanabadi, M. de Montigny, and M. Hosseinpour, Interaction of the magnetic quadrupole moment of a non-relativistic particle with an electric field in a rotating frame, *Ann. Phys.* 412 (2020) 168040.
- [2] A. DeSousa Dutra, Exact solutions of the Schrödinger equation for Coulombian atoms in the presence of some anharmonic potentials, *Phys. Lett. A* 131 (1988) 319-321.
- [3] S. Bera, B. Chakrabarti, and T. K. Das, Application of conditional shape invariance symmetry to obtain the eigen-spectrum of the mixed potential $V(r) = ar + br^2 + \frac{c}{r} + \frac{l(l+1)}{r^2}$, *Phys. Lett. A* 381 (2017) 1356-1361.

- [4] K. Bakke, Bound states for a Coulomb-type potential induced by the interaction between a moving electric quadrupole moment and a magnetic field, *Ann. Phys.* 341 (2014) 86-93
- [5] H. Hassanabadi, M. Hosseinpour, and M. de Montigny, Reply to “Comment on: ‘Interaction of the magnetic quadrupole moment of a non-relativistic particle with an electric field in a rotating frame. *Ann. Phys.* 412 (2020) 168040’ ”, *Ann. Phys.* xxx (2020) xxx.
- [6] R. P. Feynman, Forces in Molecules, *Phys. Rev.* 56 (1939) 340-343.
- [7] M. S. Child, S-H. Dong, and X-G. Wang, Quantum states of a sextic potential: hidden symmetry and quantum monodromy, *J. Phys. A* 33 (2000) 5653-5661.
- [8] P. Amore and F. M. Fernández, On some conditionally solvable quantum-mechanical problems, *Phys. Scr.*, in press, DOI: 10.1088/1402-4896/abb252 (2020)arXiv:2007.03448 [quant-ph].
- [9] F. M. Fernández, The rotating harmonic oscillator revisited, 2020. arXiv:2007.11695 [quant-ph].
- [10] F. M. Fernández, The truncated Coulomb potential revisited, 2020. arXiv:2008.01773 [quant-ph].
- [11] A. V. Turbiner, One-dimensional quasi-exactly solvable Schrodinger equations, *Phys. Rep.* 642 (2016) 1-71. arXiv:1603.02992v2.
- [12] F. L. Pilar, *Elementary Quantum Chemistry*, McGraw-Hill, New York, (1968).

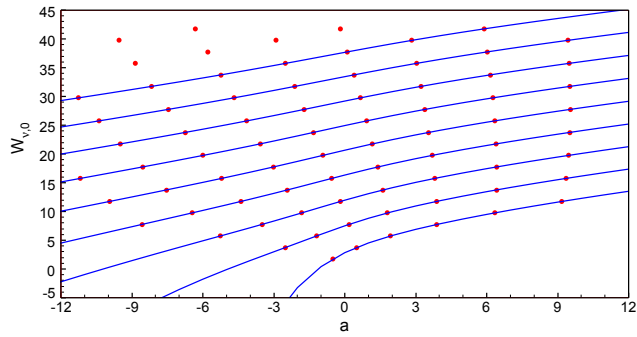


Figure 1: Eigenvalues $W_0^{(n)}(a, 1)$ from the truncation condition (red points) and $W_{\nu,0}(a)$ obtained by means of the variational method (blue lines)