Entanglement between distant qubits in cyclic *XX* chains

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We evaluate the exact concurrence between any two spins in a cyclic *XX* chain of n spins placed in a uniform transverse magnetic field, at both zero and finite temperature, by means of the Jordan-Wigner transformation plus a number-parity-projected statistics. It is shown that, while at $T=0$ there is always entanglement between any two spins in a narrow field interval before the transition to the aligned state, at low but nonzero temperatures the entanglement remains nonzero for arbitrarily high fields, for any pair separation L, although its magnitude decreases exponentially with increasing field. It is also demonstrated that the associated limit temperatures approach a constant nonzero value in this limit, which decreases as L^{-2} for $L \ll n$, but exhibit special finite-size effects for distant qubits ($L \approx n/2$). Related aspects such as the different behavior of even and odd antiferromagnetic chains, the existence of n ground-state transitions, and the thermodynamic limit $n \to \infty$ are also discussed.

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I. INTRODUCTION

Quantum entanglement denotes those correlations with no classical analog that can be exhibited by composite quantum systems and that constitute one of the most fundamental features of quantum mechanics. It is considered an essential resource in the field of quantum information [1], where it plays a key role in various quantum-information processing tasks such as quantum teleportation [2] and quantum cryptography [3]. It is also playing an increasingly important role in condensed matter physics, providing a new perspective for understanding quantum phase transitions and collective phenomena in strongly correlated systems [4–7].

In particular, there has been considerable interest in investigating entanglement in quantum spin chains with Heisenberg interactions [8,9], since they provide a scalable qubit representation apt for quantum processing tasks [10,11] which can be realized in diverse physical systems. Studies of the pairwise entanglement in the Ising and *XY* models [4,5,12] and in the isotropic Heisenberg model [13–16] at zero and finite temperature and in a transverse uniform field, as well as in diverse *XX*, *XY*, and *XYZ* models for two or a small number of qubits [17–21], have been made. An important result is that the entanglement range may remain finite at a quantum phase transition, limited for instance to first and second neighbors in the Ising model [4,5], in contrast with the behavior of the correlation length, which diverges at these points. Global thermal entanglement has also been studied [22], showing that limit temperatures for pairwise entanglement are lower bounds to those limiting entanglement between global partitions. A fundamental result for finite systems is that there is always a *finite* limit temperature for entanglement, since any mixed state becomes completely separable if it is sufficiently close to the full random state [23,24].

In this work we analyze the entanglement between any two spins in a cyclic chain with nearest-neighbor *XX* coupling in a transverse magnetic field (control parameter) by means of an exact analytic treatment valid for any spin number n and pair separation L, based on the Jordan-Wigner mapping and the use of number-parity-projected statistics for $T>0$. Recent studies in *XX* chains have focused on either chains with a small number of spins [17,21,25], where results were obtained through direct diagonalization, or open chains at zero temperature and field [15]. We will show that the *XX* model offers very interesting properties such as entanglement between any pair (full range) in a finite field interval just before the critical point at $T=0$, which subsists for large fields at low but nonzero temperatures $T<T_L$. Moreover, limit temperatures T_L approach a nonzero limit for large fields, for all separations L. It also displays n ground-state transitions at analytic field values, entailing a stepwise variation of the entanglement range suitable for its use as an entanglement switch. Let us mention that *XX* chains have also been employed for entanglement teleportation [25].

Section II describes the formalism for evaluating the exact concurrence between arbitrary sites at both zero and finite temperature. Section III describes the main physical results, including the ground-state transitions and concurrence in both ferro- and antiferromagnetic systems, and a detail study of the limit temperatures for entanglement. Conclusions are drawn in Sec. IV.

II. FORMALISM

We consider a cyclic chain of n spins with nearest-neighbor *XX* coupling. The Hamiltonian reads

$$H = bS^z - v\sum_{j=1}^{n}(s_j^x s_{j+1}^x + s_j^y s_{j+1}^y) \tag{1a}$$

$$= bS^z - \frac{1}{2}v\sum_{j=1}^{n}(s_j^+ s_{j+1}^- + s_{j+1}^+ s_j^-), \tag{1b}$$

where $s_j^{x,y,z}$ are the spin components (in units of $\hbar$) at site j, $s_j^{\pm}=s_j^x \pm i s_j^y$, $S^z=\Sigma_{j=1}^n s_j^z$ is the total spin along the direction of the transverse magnetic field b, and $n+1\equiv 1$. Our aim is to examine the entanglement between qubits at arbitrary sites i,j ($i\neq j$) in the thermal state

$$\rho(T) = Z^{-1}\exp(-\beta H), \quad \beta = 1/T, \tag{2}$$

where $Z = \mathrm{Tr}\exp(-\beta H)$ and T is the temperature (we set the Boltzmann constant $k=1$). This entanglement is determined by the reduced pair density $\rho_{ij} = \mathrm{Tr}_{n-\{ij\}}\rho(T)$ and can be measured through the concurrence [26]

$$C_{ij} = [2\lambda_M - \mathrm{tr}\, R]_+, \quad R = \sqrt{\rho_{ij}^{1/2}\tilde{\rho}_{ij}\rho_{ij}^{1/2}}, \tag{3}$$

where $[u]_+ \equiv (u+|u|)/2$, λ_M denotes the largest eigenvalue of the Hermitian matrix R, and $\tilde{\rho}_{ij} = 4^2 s_i^y s_j^y \rho_{ij}^* s_i^y s_j^y$ is the spin-flipped density (tr R is the fidelity [1] between $\tilde{\rho}_{ij}$ and ρ_{ij}). The entanglement of formation [27] of the pair is $E_{ij} = -\sum_{\nu=\pm} q_\nu \log_2 q_\nu$, where $q_\pm = (1 \pm \sqrt{1-C_{ij}^2})/2$ and is just an increasing function of C_{ij}, with $E_{ij} = C_{ij} = 1$ (0) for a maximally entangled (separable) pair state.

Since H commutes with S^z and is invariant under translation and inversion, ρ_{ij} will commute with the pair spin component $S_{ij}^z = s_i^z + s_j^z$ and its elements will depend just on the separation $|i-j|$. Hence, in the standard basis of S_{ij}^z eigenstates, it must be of the form

$$\rho_{ij} = \begin{pmatrix} p_L^+ & 0 & 0 & 0 \\ 0 & p_L & \alpha_L & 0 \\ 0 & \alpha_L & p_L & 0 \\ 0 & 0 & 0 & p_L^- \end{pmatrix}, \quad L = |i-j|, \tag{4}$$

where $p_L^+ + 2p_L + p_L^- = 1$, $p_L^+ - p_L^- = 2\langle s_i^z \rangle$, and

$$p_L^+ = \left\langle \left(s_i^z + \frac{1}{2}\right)\left(s_j^z + \frac{1}{2}\right) \right\rangle, \quad \alpha_L = \langle s_i^+ s_j^- \rangle. \tag{5}$$

Here $\langle O \rangle \equiv \mathrm{Tr}\rho(T)O$ denotes the thermal average of O and $\langle s_i^z \rangle = \langle S_z \rangle / n$ is the intensive magnetization. ρ_{ij} commutes as well with the total spin of the pair $(S^{ij})^2 = \mathbf{S}^{ij}\cdot\mathbf{S}^{ij}$, its eigenstates being the standard triplet states and singlets $|\uparrow\uparrow\rangle$, $|\downarrow\downarrow\rangle$, and $(|\uparrow\downarrow\rangle \pm |\downarrow\uparrow\rangle)/\sqrt{2}$, with eigenvalues $p_L^\pm$, $p_L \pm \alpha_L$. The pair entanglement is obviously driven by the mixing coefficient α_L. The concurrence (3) becomes

$$C_L = 2[|\alpha_L| - \sqrt{p_L^+ p_L^-}]_+, \tag{6}$$

so that ρ_{ij} is entangled if and only if $|\alpha_L| > \sqrt{p_L^+ p_L^-}$. This condition also follows from the Positive Partial Transpose (PPT) criterion [28].

A. Exact energy levels

By means of the Jordan-Wigner transformation to fermion operators $c_j^\dagger = s_j^+ \exp(-i\pi\sum_{k=1}^{j-1} s_k^+ s_k^-)$ [8], we may rewrite H exactly as a bilinear form in $c_j^\dagger, c_j$ for each value of the spin or fermion number parity,

$$P \equiv \exp(i\pi N), \quad N = \sum_{j=1}^{n} c_j^\dagger c_j = S^z + n/2.$$

The result for $P = \sigma = \pm 1$ is [8]

$$H_\sigma = \sum_{j=1}^{n} b\left(c_j^\dagger c_j - \frac{1}{2}\right) - v\left(\frac{1}{2} - \delta_{jn}\delta_{\sigma 1}\right)(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j)$$
$$= \sum_{k\in K_\sigma} \lambda_k\left(c_k'^\dagger c_k' - \frac{1}{2}\right), \quad \lambda_k = b - v\cos\omega_k, \tag{7}$$

where the fermion operators $c_k'^\dagger$ are related to $c_j^\dagger$ by a parity-dependent discrete Fourier transform

$$c_j^\dagger = \frac{1}{\sqrt{n}}\sum_{k\in K_\sigma} e^{i\omega_k j} c_k'^\dagger, \quad \omega_k = 2\pi k/n, \tag{8}$$

$$K_\sigma = \left\{-\left[\frac{1}{2}n\right] + \frac{1}{2}\delta_{\sigma 1}, \ldots, \left[\frac{1}{2}(n-1)\right] + \frac{1}{2}\delta_{\sigma 1}\right\} \tag{9}$$

with $[\cdots]$ denoting integer part. The index k is then half integer (integer) for $\sigma = 1$ (-1).

The 2^n energies are then $\sum_{k\in K_\sigma}(N_k - 1/2)\lambda_k$, where $N_k = 0, 1$ and $\sigma = (-1)^{\sum_k N_k}$. Note that the single-fermion energies λ_k depend on the global parity σ and are degenerate $(\lambda_k = \lambda_{-k})$ for $|k| \neq 0, n/2$. It is also apparent from (7) that the spectrum of H is independent of the sign of b, and for even n also of the sign of v, as $\cos\omega_{k'} = -\cos\omega_k$ for $k' = n/2 - k$ and k' belongs to the same parity as k if n is even. This is also evident from (1), since for even n the sign of v can be inverted by a local transformation $s_j^{x,y} \to (-1)^j s_j^{x,y}$ (and that of b by $s_j^{y,z} \to -s_j^{y,z}$). The concurrence (6) will then exhibit the same properties, depending just on $|b|$ and for even n just on $|v|$ [17].

B. Exact partition function and concurrence

The partition function Z of the system is to be evaluated in the full grand-canonical ensemble of the fermionic representation. However, due to the parity dependence of the latter, this requires a number-parity-projected statistics [29]. Z can then be written as a sum of partition functions for each parity,

$$Z = \mathrm{Tr}\sum_{\sigma=\pm 1}\frac{1}{2}(1+\sigma P)e^{-\beta H_\sigma} = \frac{1}{2}\sum_{\sigma=\pm 1}(Z_0^\sigma + \sigma Z_1^\sigma), \tag{10}$$

where $\frac{1}{2}(1+\sigma P)$ is the projector onto parity σ and

$$Z_\nu^\sigma = \mathrm{Tr} P^\nu e^{-\beta H_\sigma} = e^{\beta b n/2}\prod_{k\in K_\sigma}[1 + (-1)^\nu e^{-\beta\lambda_k}], \tag{11}$$

for $\nu = 0, 1$. The expectation value of an operator O can then be similarly expressed as

$$\langle O \rangle = \frac{1}{2}Z^{-1}\sum_{\sigma=\pm 1}(Z_0^\sigma\langle O\rangle_0^\sigma + \sigma Z_1^\sigma\langle O\rangle_1^\sigma), \tag{12}$$

$$\langle O\rangle_\nu^\sigma = (Z_\nu^\sigma)^{-1}\mathrm{Tr}(P^\nu e^{-\beta H_\sigma}O), \quad \nu = 0, 1. \tag{13}$$

In the case of many-body fermion operators, the thermal version of Wick's theorem [30] cannot be applied in the final average (12), but it can be used for evaluating the partial averages (13) (as $P^\nu e^{-\beta H_\sigma} = e^{-\beta H_\sigma + i\nu\pi N}$ is still the exponential of a one-body operator), in terms of the contractions

$$g_L \equiv \langle c_i^\dagger c_j \rangle_\nu^\sigma = \frac{1}{n} \sum_{k \in K_\sigma} \langle c_k'^\dagger c_k' \rangle_\nu^\sigma \cos(L\omega_k), \tag{14}$$

where $L=i-j$ and $\langle c_k'^\dagger c_k' \rangle_\nu^\sigma = [1+(-1)^\nu e^{\beta\lambda_k}]^{-1}$ [Eq. (13)]. As $s_i^z = c_i^\dagger c_i - \frac{1}{2}$, this leads to

$$\langle s_i^z \rangle_\nu^\sigma = g_0 - \frac{1}{2}, \quad \left\langle \left(s_i^z + \frac{1}{2}\right)\left(s_j^z + \frac{1}{2}\right) \right\rangle_\nu^\sigma = g_0^2 - g_L^2, \tag{15}$$

for $i \neq j$. Using the identity $s_i^+ s_j^- = s_i^+ [\Pi_{k=i+1}^{j-1} (s_k^+ s_k^- + s_k^- s_k^+)] s_j^-$ for $i<j$, with $s_j^+ s_{j+1}^+ = c_j^\dagger c_{j+1}^\dagger$, $s_j^+ s_{j+1}^- = c_j^\dagger c_{j+1}$ [8], one also obtains

$$\langle s_i^+ s_j^- \rangle_\nu^\sigma = \frac{1}{2} \mathrm{Det}(A_L), \tag{16}$$

where A_L is the $L \times L$ matrix of elements

$$(A_L)_{ij} = 2g_{i-j+1} - \delta_{i,j-1}, \tag{17}$$

i.e., $\mathrm{Det}(A_1) = 2g_1$, $\mathrm{Det}(A_2) = 4[g_1^2 - g_2(g_0 - \frac{1}{2})]$. All terms in (4) and (6) can then be exactly evaluated.

In the thermodynamic limit $n \to \infty$, and for finite $L \ll n$, we can ignore parity effects and replace sums over k by integrals over $\omega \equiv \omega_k$. We can then directly employ Wick's theorem in terms of the elements

$$g_L = \langle c_i^\dagger c_j \rangle = \frac{1}{\pi} \int_0^\pi \frac{\cos(L\omega)}{1 + e^{\beta(b-v\cos\omega)}} d\omega. \tag{18}$$

This leads to [Eq. (5)]

$$p_L^+ = g_0^2 - g_L^2, \quad \alpha_L = \frac{1}{2} \mathrm{Det}(A_L), \tag{19}$$

and $p_L^- = p_L^+ + 1 - 2g_0$, where A_L is constructed with the elements (18). We then obtain the final expression

$$C_L = [|\mathrm{Det}(A_L)| - 2\sqrt{(g_0^2 - g_L^2)[(1-g_0)^2 - g_L^2]}]_+. \tag{20}$$

Note that for $T \to 0$, Eq. (18) yields $g_L = 0$ for $b > |v|$ and $g_L = \sin(L\omega)/(L\pi)$ (with $g_0 = \omega/\pi$) for $|b| < |v|$, where $\cos\omega = b/|v|$.

When the ground state is nondegenerate, Eqs. (19) and (20) are also exactly valid for finite n in the $T \to 0$ limit, using the exact contractions

$$g_L = \langle c_i^\dagger c_j \rangle_0 = \frac{1}{n} \sum_{k \text{ occ.}} \cos(L\omega_k), \tag{21}$$

where $\langle O \rangle_0$ denotes the ground-state average and the sum runs over the occupied levels (see next section).

III. RESULTS

A. Ground-state transitions and concurrence

Let us first describe the behavior in the $T \to 0$ limit. As $[H,N]=0$, the ground state of H can be characterized by the fermion number N, i.e., the total spin component $M=N-n/2$ in the spin representation. Since λ_k in (7) becomes negative for $b < v\cos\omega_k$, the ground state will exhibit n transitions $N \to N+1$ as b decreases from $|v|$ to $-|v|$, starting from $N=0$ (the aligned state $M=-n/2$) for $b>|v|$ ($\lambda_k > 0\ \forall\ k$) and ending with $N=n$ ($M=n/2$) for $b<-|v|$ ($\lambda_k < 0\ \forall\ k$).

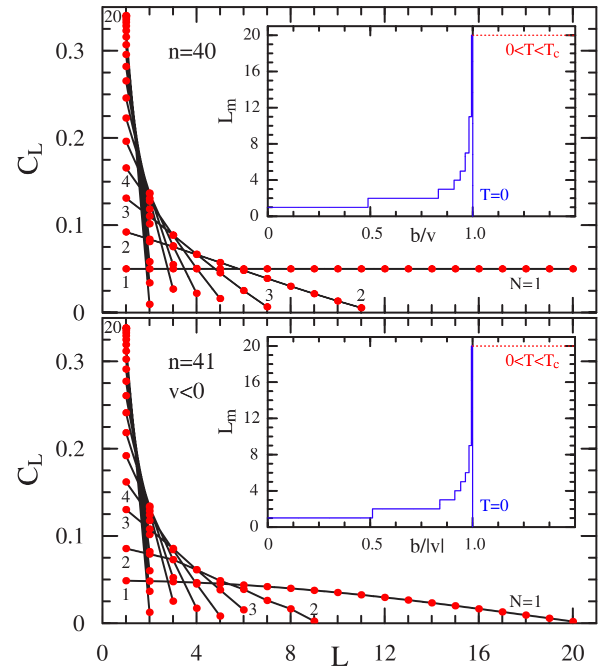

FIG. 1. (Color online) Top panel: Concurrence C_L as a function of site separation L in the 20 different entangled ground states of Hamiltonian (1) existing for $n=40$ qubits and $b>0$. Ground states are labeled by the effective fermion number $N=M+n/2 =1,\ldots,n/2$, which can be selected by adjusting the magnetic field b (N fermions for $b_{N+1} < b < b_N$). For $N=1$, C_L has the same value for all separations [Eq. (22)]. The inset depicts the entanglement range L_m vs the magnetic field in the same system, which for $b > v$ vanishes at $T=0$ but remains maximum if $0<T<T_c \equiv T_{n/2}$ (see next section). Bottom: Same details for an odd antiferromagnetic chain with $n=41$ qubits [results for the $T \to 0$ limit of $\rho(T)$]. For $N=1$ all pairs are still entangled but C_L decays for increasing L [Eq. (25)].

For $v>0$, the first transition $0 \to 1$ occurs at

$$b_1 = v,$$

i.e., when the lowest negative-parity level $\lambda_0 = b - v$ becomes negative. It represents, for $b>0$, the entangled-separable border at $T=0$. For b just below b_1 the ground state is the one-fermion state $c_0'^\dagger|0\rangle = (1/\sqrt{n})\Sigma_j c_j^\dagger|0\rangle$, i.e., the W state $(|\uparrow\downarrow\downarrow\ldots\downarrow\rangle + |\downarrow\uparrow\downarrow\ldots\downarrow\rangle + \ldots)/\sqrt{n}$, which exhibits *a constant concurrence*

$$C_L = 2/n \quad (N=1) \tag{22}$$

for any separation L (Fig. 1). Hence, the transition at $b=b_1$ is from a fully separable state for $b>v$ (aligned state) to a state where any pair is equally entangled.

Due to the parity dependence of the energy levels, the next transition $1 \to 2$ does not take place when the next λ_k becomes negative ($b = v\cos\omega_k$) but rather when the lowest $\sigma=1$ level crosses the previous $\sigma=-1$ level, i.e., when $2\lambda_{\pm 1/2} = \lambda_0$, which leads to $b_2 = v[2\cos(\pi/n) - 1]$. In general, for $v>0$ the transitions $N-1 \to N$ occur at $b_N = 2v\cos\omega_k - b_{N-1}$, with $k=(N-1)/2$, which leads to the critical fields

$$b_N = v\frac{\cos\left[\left(N-\frac{1}{2}\right)\pi/n\right]}{\cos[\pi/(2n)]}, \quad 1 \leqslant N \leqslant n, \tag{23}$$

i.e., $b_N=v[\cos\omega_k - \sin\omega_k\tan(\pi/2n)]$. Thus, $b_N < b_{N-1}$, with $b_{n-N+1}=-b_N$ and $b_N \approx v\cos\omega_k$ for large n.

Equation (22) is valid for $b_2 < b < b_1$. The exact expression for C_L at the other N-fermion ground states is given by Eq. (20) with the elements (21), which become

$$g_L = \frac{\sin(NL\pi/n)}{n\sin(L\pi/n)}, \tag{24}$$

with $g_0=N/n=\lim_{L\to 0} g_L$. For $N=1$, $g_L=1/n \;\forall\; L$ and Eq. (20) leads to Eq. (22).

For $N\geqslant 2$, C_L will depend on the separation L, decreasing almost linearly with L for not too small n, as seen in Fig. 1. A series expansion of (20) yields the initial trend $C_L \approx (2N/n)[1-\pi L\sqrt{(N^2-1)/3/n}]$ for $NL \ll n$. The extent of pairwise entanglement decreases then rapidly as N increases (inset in Fig. 1), the separation between the most distant entangled qubits being $L_m \approx [(n+1.79)/3.57]$ for $N=2$ and, roughly, $L_m \approx [(n+4)/(2N)]$ for $2 < N \ll n/2$.

Just first and second neighbors ($L=1,2$) remain entangled for $|b| < 0.65v \;\forall\; n$ [and $|b| < 0.82v$ ($N \gtrsim n/5$) for $n\to\infty$] whereas only adjacent pairs ($L=1$) remain entangled for $|b| < 0.26v \;\forall\; n\neq 5$ [and $|b| < 0.5v$ ($N \gtrsim n/3$) for $n\to\infty$] (for $n=5$ second neighbors are entangled $\forall\; b>0$). The concurrence of adjacent pairs increases first linearly with N ($C_1 \approx 2N/n$ for $N \ll n$) and becomes maximum for $N=n/2$ ($n>4$), where $g_1=1/[n\sin(\pi/n)] \approx 1/\pi$ for large n and $C_1=2g_1(1+g_1)-1/2 \approx 0.339$.

For odd n, results for $v<0$ must be separately examined. The lowest negative-parity level is now $\lambda_{\pm[n/2]}=b-|v|\cos(\pi/n)$, so that the first transition occurs at

$$b_1^- = |v|\cos(\pi/n) \quad (v<0,\ n\ \text{odd}),$$

with the ground state twofold degenerate after the transition ($k=\pm[n/2]$). The concurrence of the mixture $\frac{1}{2}\sum_{k=\pm[n/2]}|k\rangle\langle k|$ of the two degenerate ground states $|k\rangle=(1/\sqrt{n})\sum_j e^{-i\omega_k j}c_j^\dagger|0\rangle$ [the $T\to 0$ limit of $\rho(T)$] is

$$C_L^- = 2\cos(L\pi/n)/n \quad (N=1) \tag{25}$$

which is again nonzero $\forall\; L$ (n is odd) although it now decays as L increases (bottom panel in Fig. 1). For $L \ll n$, $C_L^- \approx 2/n$, in agreement with (22), whereas for most distant qubits ($L=[n/2]$), $C_L^-=2\sin[\pi/(2n)]/n \approx \pi/n^2$. Hence, for large L a significant odd-even difference in C_L arises if $v<0$, even for large qubit number n, due to the ground-state degeneracy of the odd system.

The next transition for $v<0$ and n odd occurs when $\lambda_{n/2}+\lambda_{n/2-1}=\lambda_{[n/2]}$, i.e., at $b_2^-=|v|[1+\cos(2\pi/n)]-b_1^-$, and in general at $b_N^-=|v|\sum_{k=N/2-1}^{N/2}\cos\omega_k - b_{N-1}^-$, which leads to the smaller critical fields,

$$b_N^- = b_N\cos(\pi/n), \quad 1\leqslant N \leqslant n, \tag{26}$$

where b_N are the fields (23). Ground states remain twofold degenerate $\forall N\neq 0,n$, since there is just one fermion in the highest occupied level ($k=\pm(n-N)/2$).

Equation (25) holds for $b_2^- < b < b_1^-$. The expression of C_L^- for general N in the $T\to 0$ limit can be similarly obtained by using Eq. (21) for each of the degenerate ground states and then taking the average. The final result is

$$C_L^- = [|\mathrm{Re}[\mathrm{Det}(A_L^-)]| - 2\sqrt{(g_0^2-g_L^2)((1-g_0)^2-g_L^2)}]_+, \tag{27}$$

where A_L^- is constructed with the elements,

$$g_L^- = g_L e^{iL\pi/n}, \tag{28}$$

with g_L given again by Eq. (24). For $N=1$, Eq. (27) leads to Eq. (25). The behavior of C_L^- for $N\geqslant 2$ is similar to that of C_L [Eq. (20)], although it is smaller than C_L (due to the ground-state degeneracy) and its decay with L is less linear (see bottom panel). For instance, for $L=1$, $\mathrm{Re}[\mathrm{Det}(A_1^-)] = \mathrm{Det}(A_1)\cos(\pi/n)$, whence $C_1^- < C_1$, with $C_1^- \to C_1$ for large n.

Let us finally mention that for $n\to\infty$ and $\pi N/n \to \omega$, with L finite, Eqs. (24)–(28) both coincide exactly with the limit of Eq. (18) for $T\to 0$, where $b_N \to v\cos\omega$.

B. Results for finite temperatures

Illustrative exact results for $n=14-15$ and the thermodynamic limit $n\to\infty$ are depicted in Figs. 2 and 3. For T close to 0, the concurrence exhibits a stepwise behavior in finite chains, in agreement with the $T=0$ transitions previously described, presenting dips at the critical fields (23)–(26) due to the ground-state degeneracy at these points (level crossing). It is also verified that C_L is smaller in odd antiferromagnetic chains, particularly for large L close to $n/2$, in agreement with Eqs. (25)–(27).

While at $T=0$, there is no entanglement in the ground state for $b>b_1$, a fundamental result for $T>0$ is that $\rho(T)$ *remains entangled for all fields $b>b_1$ if T is sufficiently low*, leading to a small but *nonzero* concurrence C_L for *any* separation L if $0<T<T_L(b)$. Moreover, the limit temperature $T_L(b)$ approaches a *nonzero limit* T_L for $b\to\infty$ $\forall L$, being practically constant for $b\gtrsim|v|$ (and $b\gtrsim 0$ if $L=1$). This behavior applies for any n, including $n\to\infty$ as well as the special case $v<0$ and n odd, as seen in the lower panels of Figs. 2 and 3.

In order to rigorously prove the previous behavior, we note that for $b-|v| \gg kT$ ($e^{-\beta(b-|v|)} \ll 1$), we may keep just zero, one and two fermion states in $\exp(-\beta H)$, i.e.,

$$Z \approx e^{\beta bn/2}\left(1+\sum_{k\in K_-} e^{-\beta\lambda_k} + \sum_{k<k'\in K_+} e^{-\beta(\lambda_k+\lambda_{k'})}\right),$$

and similarly for $\rho(T)$. This leads to $\alpha_L \approx e^{-\beta b} I_L^-(\beta v)$ and $p_L^+ \approx e^{-2\beta b}[I_0^{+2}(\beta v) - I_L^{+2}(\beta v)]$ up to lowest order in $e^{-\beta b}$, where

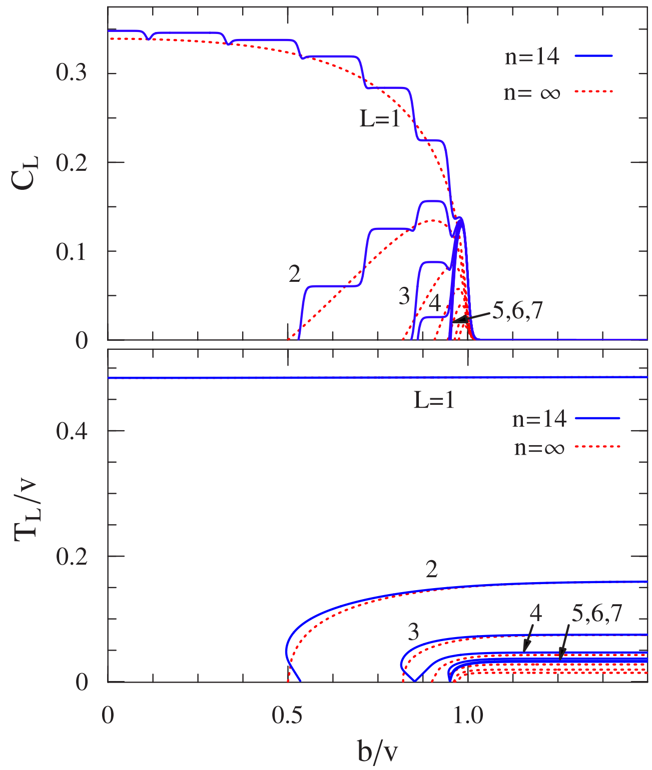

FIG. 2. (Color online) Concurrence (top) and limit temperatures for entanglement $T_L(b)$ (bottom) for pairs $i,i+L$ as a function of the magnetic field, for $n=14$ qubits and in the thermodynamic limit $n\to\infty$. The concurrence is plotted close to the $T\to 0$ limit ($T=0.005v$). All limit temperatures remain constant for $b/v\to\infty$ (see text). Results for even n lie mostly above those for $n\to\infty$, particularly for large L (where they saturate) and are independent of the sign of v.

$$I_L^{\pm}(\beta v)=\frac{1}{n}\sum_{k\in K_\pm} e^{\beta v\cos\omega_k}\cos(L\omega_k). \tag{29}$$

Hence, up to first order in $e^{-\beta b}$ we obtain

$$C_L\approx 2e^{-\beta b}[I_L^{-}(\beta v)-\sqrt{I_0^{+2}(\beta v)-I_L^{+2}(\beta v)}]_+. \tag{30}$$

Thus, as b increases the concurrence decreases exponentially when it is positive, but the limit temperature $T_L(b)$ becomes constant, as the entanglement condition $C_L>0$ becomes b independent, i.e.,

$$I_L^{-}(\beta v)>\sqrt{I_0^{+2}(\beta v)-I_L^{+2}(\beta v)}. \tag{31}$$

Equation (31) is always satisfied for sufficiently small but positive T, for any distance L, ensuring a nonzero concurrence and limit temperature $T_L(b)$ for any $b>|v|$. This is easy to prove for $v>0$, where for $T\to 0^+$, $I_L^{-}(\beta v)\approx e^{\beta v}/n > I_0^{+}(\beta v)\approx 2e^{\beta v\cos(\pi/n)}/n$. It also holds for $v<0$ (n odd), since in this case, for $T\to 0^+$, $I_L^{-}(\beta v)\approx e^{\beta|v|\cos(\pi/n)}\cos(L\pi/n)/n$, whereas the right-hand side of (31) becomes $\approx\sqrt{2}e^{\beta|v|\cos^2(\pi/n)}\sin(L\pi/n)/n<I_L^{-}(\beta v)$.

In the thermodynamic limit $n\to\infty$, and for finite $L\ll n$, we may neglect parity effects and just replace

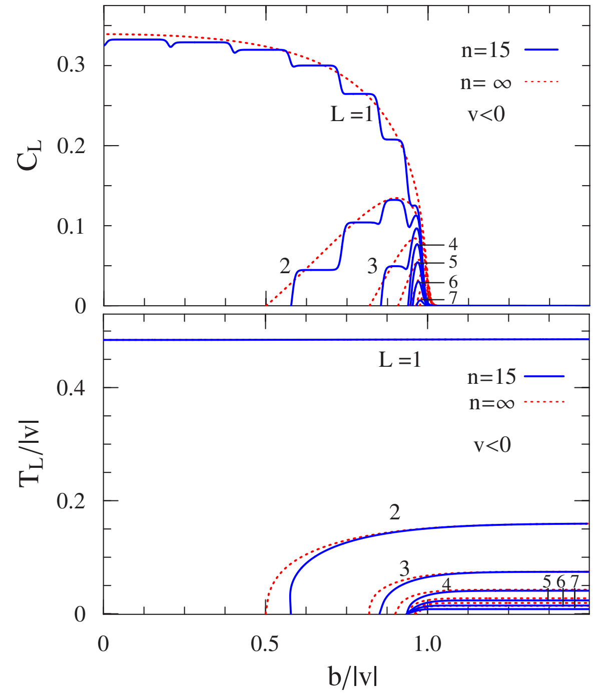

FIG. 3. (Color online) Same details as Fig. 2 for $n=15$ qubits and $v<0$. Results now lie mostly below those for $n\to\infty$ (which are the same as in Fig. 2) and do not saturate for large L. Results for $n=15$ and $v>0$ are similar to those of Fig. 2.

$$I_L^{\pm}(\beta v)\to\frac{1}{\pi}\int_0^{\pi} e^{\beta v\cos\omega}\cos(L\omega)d\omega=I_L(\beta v), \tag{32}$$

where $I_L(x)$ is the modified Bessel function of the first kind $\{I_L(x)\approx e^x[1+(1-4L^2)/8x]/\sqrt{2\pi x}$ for $x\to\infty$, with $I_L(-x)=(-)^L I_L(x)\}$. Equation (30) becomes then identical with the result obtained from Eqs. (18)–(20) [$g_L\to e^{-\beta b}I_L(\beta v)$ for $b-|v|\gg T$, with $\mathrm{Det}(A_L)\to 2g_L$]. Equation (31) then becomes

$$\sqrt{2}I_L(\beta|v|)>I_0(\beta v), \tag{33}$$

which is again *always* satisfied for sufficiently low $T\,\forall\,L$. The limit temperatures $T_L\equiv T_L(\infty)$ are then determined for $n\to\infty$ by the equation $\sqrt{2}I_L(\beta|v|)=I_0(\beta|v|)$, which leads to $T_1\approx 0.486|v|$, $T_2\approx 0.16|v|$, and

$$T_L\approx|v|\ln 2/L^2 \tag{34}$$

for large L [as $I_L(x)/I_0(x)\approx e^{-L^2/(2x)}$ for $x\gtrsim L^2$]. Thus, $T_L(b)$ *decreases as the inverse square of the pair distance* L for large b. The maximum value attained by C_L for $b>|v|$ nevertheless becomes small and decays exponentially with both b and L^2 ($C_L\approx e^{-(b/|v|-1)L^2/t}f(t)/L$ for $T=|v|t/L^2<T_L$, with $f(t)=\sqrt{2t/\pi}[e^{-t/2}-\sqrt{1-e^{-t}}]$). Equation (34) also indicates roughly the value of $T_L(b)$ at the critical region $b\approx|v|$, since it is almost constant for $b\geqslant|v|$ (Figs. 2 and 3).

On the other hand, for large $L\approx n/2$, the projected expression (30) is required *even for large* n. For instance, for even n and $L=n/2$, $\cos(L\omega_k)=0$ ($(-1)^k$) for k half-integer (integer). Hence, in this case $I_{n/2}^{+}(\beta v)=0$, while for $v>0$ and large n,

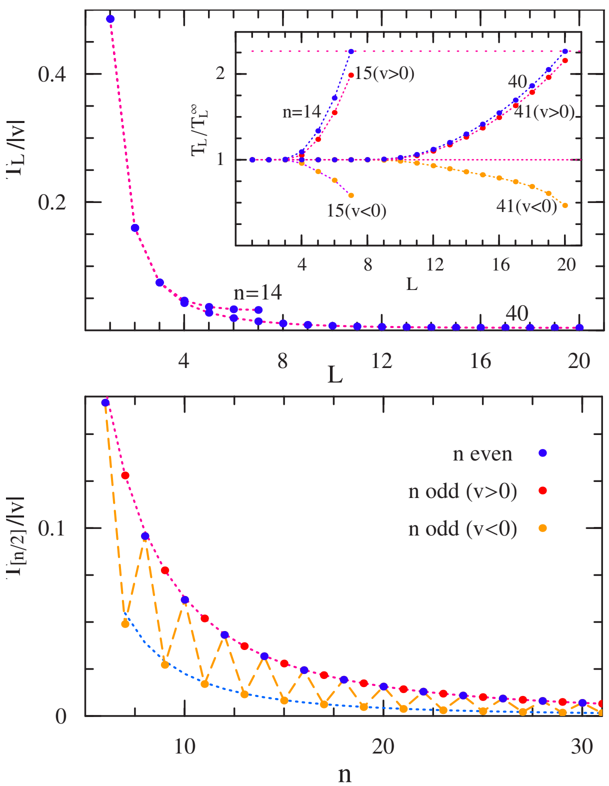

FIG. 4. (Color online) Top: Limit temperatures T_L for entanglement between pairs for large magnetic fields $b \gg v$ as a function of separation L, for $n=14$ and 40 qubits, determined by Eq. (31). Inset: Ratio between T_L and the value in the thermodynamic limit T_L^∞, determined by Eqs. (33) and (34). T_L deviates from T_L^∞ for $L \gtrsim n/4$, approaching, for $L \to n/2$, Eq. (36) for n even or n odd and $v>0$, and Eq. (37) for n odd and $v<0$. Bottom: The limit temperature for the most distant qubits ($L=[n/2]$), showing the odd-even staggering arising for $v<0$ (dashed line). The upper (lower) dotted line depicts the result of Eq. (36) [(37)].

$$I^{+(-)}_{0(n/2)}(\beta v) \approx e^{\beta v}\theta_{2(4)}(e^{-2\beta v \pi^2/n^2})/n, \tag{35}$$

after replacing $\cos\omega_k \approx 1-\omega_k^2/2$ [$\theta_2(u) \equiv 2\Sigma_{k=1/2}^{\infty} u^{k^2}$, $\theta_4(u) \equiv 1+2\Sigma_{k=1}^{\infty}(-1)^k u^{k^2}$, denote the elliptic theta functions]. These results also approximately hold for large odd n and $L=[n/2]$ if $v>0$. Equation (31) then becomes $I^-_{n/2}(\beta v) > I^+_0(\beta v)$, and since $\theta_2(u)=\theta_4(u)$ for $u=e^{-\pi}$, it leads to the limit temperature,

$$T_{[n/2]} \approx 2\pi v/n^2 \tag{36}$$

for the most distant pairs and $v>0$. It is greater than Eq. (34) for $L=n/2$ by a factor of $\pi/(2\ln 2) \approx 2.27$.

Equation (36) does not hold for $v<0$ if n is odd. In this case, we may directly employ the asymptotic expression of Eq. (31) for $T \to 0^+$, which for large n yields

$$T_{[n/2]} \approx \frac{|v|\pi^2}{2n^2 \ln[2\sqrt{2}n/\pi]}, \quad (v<0,\ n\ \text{odd}). \tag{37}$$

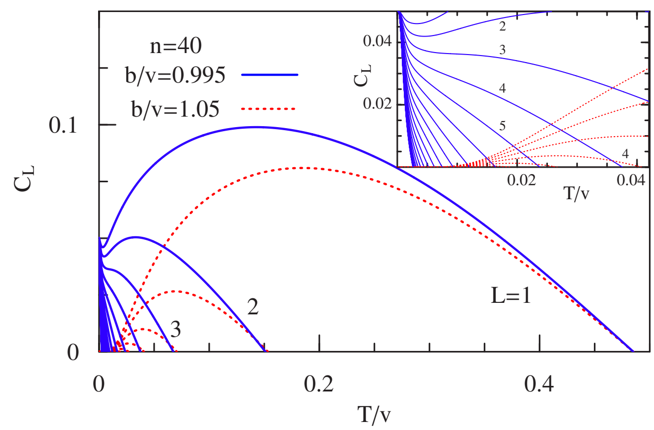

FIG. 5. (Color online) The thermal behavior of C_L for $n=40$ qubits near the transition at $b=v$. Limit temperatures remain stable at the transition, indicating the reentry of C_L for $T>0 \forall L$ for $b>v$. The inset is an enlargement of the low T region, showing the accumulation of the limit temperatures for large L at a value close to that given by Eq. (36).

Thus, in this case, there is an additional logarithmic factor in the denominator, which makes $T_{[n/2]}$ *lower* than Eq. (36) *and also Eq. (34) for* $L=n/2$, originating an odd-even staggering of $T_{n/2}$ if $v<0$.

The behavior of T_L is depicted in Fig. 4. It is seen that for $L \gtrsim n/4$, it deviates from the $1/L^2$ law given by Eq. (34), approaching the values given by Eq. (36) or (37) for $L \approx n/2$. Figure 5 depicts the typical thermal behavior of C_L for $v>0$ near the transition at $b=b_1$. For $b<b_1$ there is entanglement between all pairs if T is lower than a certain

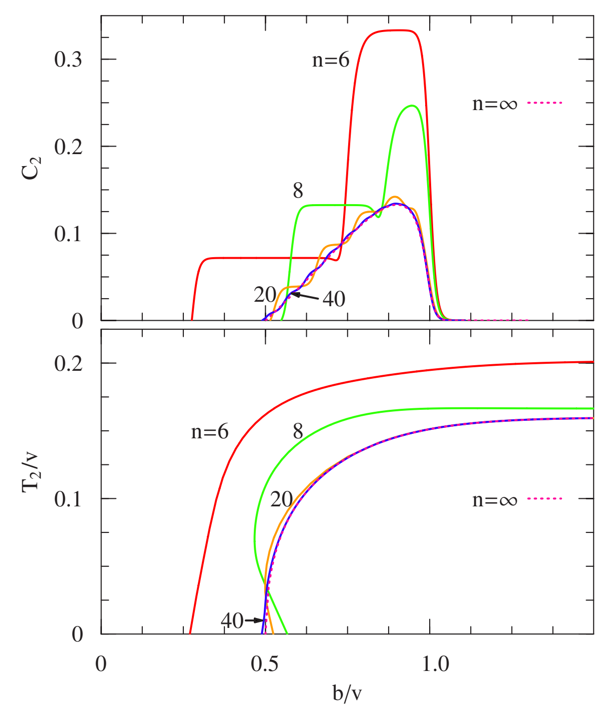

FIG. 6. (Color online) Concurrence (top) and limit temperatures for entanglement (bottom) for separation $L=2$ and different n, as a function of the magnetic field. The concurrence is plotted at $T=0.01v$. Dotted lines depict the thermodynamic limit.

temperature, given approximately by Eq. (36). It also shows the reentry of C_L for $T>0$ for $b>v$, which is quite prominent for low L.

Finally Fig. 6 depicts the typical behavior with the qubit number n of the concurrence and limit temperature. We have chosen a separation $L=2$. Although the thermodynamic limit is on the average rapidly approached, the stepwise behavior of C_L at low $T\approx 0.01v$ remains visible even for $n=40$, and deviations in the limit temperature can be significant at the onset. They are as well significant for small $n\lesssim 10$.

An interesting feature is that the slope of $T_L(b)$ can be *negative* in this region, a fact already seen in Fig. 2 for $n=14$, and visible here for $n=8$ and $n=20$. This occurs when the value of the onset field b_c for finite n (which for $L=2$ corresponds to b_2, b_3, b_7, and b_{14} for $n=6$, 8, 20, and 40) lies *above* the value for $n\to\infty$, as occurs for $n=8,20$. In these cases, there is a small field interval below b_c where entanglement between second neighbors exists only above a *threshold temperature* $T_L^i(b)>0$, up to the higher limit temperature $T_L(b)$.

A final comment is that we have checked all expressions by comparison with calculations for low $n\lesssim 10$ based on the direct diagonalization of H. In particular, for $n=2$, the entanglement condition (31) becomes exact $\forall$ b, as in this case there are just one- and two-fermion excited states, reducing ($L=1$) to $\sinh(\beta v)>1$ and leading to the known limit temperature $T_1=v/\ln(1+\sqrt{2})$ [19].

IV. CONCLUSIONS

We have provided an exact analytic treatment of the entanglement between arbitrary pairs in cyclic *XX* chains in the presence of a transverse magnetic field, valid at both zero and finite temperatures and for any qubit number n. We have shown that, in spite of its simplicity, this system exhibits very interesting features such as a discrete set of $[n/2]$ different entangled ground states at $T=0$ (and $b>0$), which can be easily selected by adjusting the magnetic field across the critical values (23) or (26), and which develop increasing entanglement ranges, reaching always full range (all pairs entangled) in an interval $b_2<b<b_1$, even for odd antiferromagnetic chains.

Moreover, while at $T=0$ the ground state is fully separable for $b>b_1$, we have rigorously proved that for $T>0$ there is a small but nonzero entanglement between any pair for all fields $b>b_1$ if T is sufficiently low, which decays exponentially with increasing field and with the square of the separation L. Limit temperatures T_L are roughly independent of b for $b\gtrsim v$ and decay as L^{-2} for $L\lesssim n/4$, but tend to saturate at $T_{[n/2]}$ [Eq. (36)] for most distant pairs ($L\approx n/2$) if n is even or $v>0$. We have also shown that, due to degeneracy of the ground state, pairwise entanglement in odd antiferromagnetic chains is weaker, particularly for distant pairs, where odd-even effects in the concurrence and T_L subsist for all n.

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