Collision entropy and optimal uncertainty

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We propose an alternative measure of quantum uncertainty for pairs of arbitrary observables in the two-dimensional case, in terms of collision entropies. We derive the optimal lower bound for this entropic uncertainty relation, which results in an analytic function of the overlap of the corresponding eigenbases. Besides, we obtain the minimum uncertainty states. We compare our relation with other formulations of the uncertainty principle.

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I. INTRODUCTION

Quantum mechanics' uncertainty principle (UP) is a fundamental theoretical notion, being not just a side result of quantum mechanics but arguably one of its most important fundamental concepts. It establishes the existence of an irreducible lower bound for the uncertainty in preparing a system's state. The original statement made use of the dispersion in an observable's measurement. The concept of entropy [1] provided a totally new perspective on UPs. So-called entropic uncertainty relations (EURs) [2,3] are a relatively recent, related concept, that greatly improves the original one in the sense of allowing for nontrivial *state-independent* lower bounds. The formulation of the UP in terms of EURs, besides being more appropriate than the original statement from a theoretical point of view, acquires significant importance in the quantum information theory realm [2,3]. In particular, EURs provide entanglement criteria and foundations for the security of many quantum cryptographic protocols, among other applications [4–17].

In this paper we are concerned with the study of uncertainty relations between two quantum observables in the case of two-dimensional systems. We show that a construct called collision entropy, that is, the Rényi entropy of index 2, exhibits significant advantages as an uncertainty measure in this case. The paper is organized as follows. An abridged history of UP formulations can be found in Sec. II, focusing on the concepts that we develop further. In Sec. III, we consider the sum of collision entropies as a UP-indicator for a pair of arbitrary observables in two-dimensional Hilbert space. We obtain the optimal lower bound and the minimum uncertainty states for the proposed uncertainty measure, following the procedure by Ghirardi *et al.* [18], who exploit the Bloch representation to facilitate the associated minimization problem's tractability. Section IV is devoted to comparing our EUR with other inequalities found in the relevant literature. Finally, some conclusions are drawn in Sec. V, stressing that the use of the collision entropy, compared to the Shannon one, allows us to provide an analytical expression for the lower bound as well as the minimizing states.

II. SOME HISTORIC CONSIDERATIONS

A. Entropy as an alternative uncertainty measure

The original quantitative UP formulation was proposed in the famous paper by Heisenberg [19] and demonstrated by Kennard [20] in the case of position and momentum observables. Robertson [21] extended the relation for other cases. These formulations were based on products of variances for pairs of observables. A completely novel perspective for the UP, in the case of canonically conjugate variables, was introduced in the pioneering contributions by Hirschman [22], Mamojka [23], and Bialynicki-Birula and Mycielski [24], in the framework of information theory. Specifically, the new proposal was to employ the sum of Shannon entropies associated to the position and momentum distributions. In Ref. [24] it has been shown that the entropic relation is stronger than the Heisenberg one.

The introduction of the information-theory alternative was inspired by the power of entropy to describe properties such us uncertainty in connection to probability distributions. Indeed, information entropies become much more appropriate in the quantum, probabilistic world. Additionally, as discussed in [2], the use of standard deviations to express indeterminacy has some limitations. The original variance-formulation of UP has also been criticized [25,26] on the grounds that the associated bound, given by the expectation value of the commutator between the two observables, depends (if the commutator is not a c-number) on the state of the system and thus lacks a universal character. Moreover, it can be easily seen [18] that for bounded operators the lower bound is trivially zero, yielding no valuable information. An alternative that has been envisaged to circumvent this obstacle precisely consists in using as an UP measure the Shannon or other generalized entropies associated to the probability distributions of the two observables' outcomes.

De Vicente and Sanchez-Ruiz [27] provided the best lower bound for the sum of Shannon entropies in the case of an arbitrary pair of observables with discrete, N-level spectra (see also Ref. [28]). This was an improvement on the Maassen-Uffink uncertainty relation presented in Ref. [26], based on the Landau-Pollak inequality [41]. In the particular case of single qubit systems ($N = 2$), Sanchez-Ruiz [29] and Ghirardi *et al.* [18], independently extracted the *optimal* lower bound for the Shannon entropies' sum. For N up to 5, a very recent study by Jafarpour and Sabour [30] gave (numerically) a more stringent bound.

B. Our entropic quantifier

The Rényi entropies [31] constitute a family of generalized information-theoretic measures that account for the

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uncertainty or lack of information associated to a probability distribution. In the finite-dimensional, discrete case the definition reads as

$$H_q(\{p_i\}) = \frac{1}{1-q} \ln\left(\sum_{i=1}^{N} p_i^q\right), \tag{1}$$

where $0 \leqslant p_i \leqslant 1$ and $\sum_{i=1}^{N} p_i = 1$. N is the number of levels, and the index $q > 0$ with $q \neq 1$. When $q \to 1$, H_q approaches the Shannon entropy $H_1 \equiv H = -\sum p_i \ln(p_i)$. In the particular case $q = 2$ the Rényi entropy is known as the collision entropy. This quantity is widely used in quantum information process and quantum cryptography. The collision entropy can be written in terms of the so-called purity of a given probability distribution; indeed, $H_2(\{p_i\})$ is the natural logarithm of the inverse of the purity, which is given by $\sum_{i=1}^{N} p_i^2$. A particularly interesting scenario arises when the entropic index q tends to infinity. Here the Rényi entropy, known as min-entropy, becomes $H_\infty(\{p_i\}) = -\ln P$, where $P = \max_i(p_i)$.

EURs using Rényi entropies as measures of uncertainty have been recently studied in the literature [32–36]. However, most of the concomitant EURs in these references deal just with (i) *complementary* observables (i.e., those whose eigenstates are linked by a Fourier transformation) and/or with (ii) *conjugated* Rényi indices q and q' (i.e., when $\frac{1}{q} + \frac{1}{q'} = 2$, which includes the Shannon case).

In the present contribution, we are not restricted in the way mentioned above. Instead, (i) arbitrary pairs of observables are considered, and (ii) we adopt the collision entropies' sum as the uncertainty quantifier. In other words, we propose and analyze an EUR which does not make use of the Riesz' theorem hypothesis of indices conjugation.

III. DERIVATION OF THE OPTIMAL BOUND FOR THE SUM OF COLLISION ENTROPIES

A. Our optimal relation in terms of collision entropies

The sum of the collision entropies for two observables $A, B \in \mathbb{C}^{2\times 2}$ for a system prepared in the quantum pure state $|\Psi\rangle \in \mathbb{C}^2$ is given by

$$\begin{aligned}\mathcal{U}(A,B;\Psi) &\equiv H_2(A) + H_2(B)\\ &= -\ln\left[p_1^2(A) + p_2^2(A)\right] - \ln\left[p_1^2(B) + p_2^2(B)\right],\end{aligned} \tag{2}$$

where $p_i(A) = |\langle a_i|\Psi\rangle|^2$ and $p_i(B) = |\langle b_i|\Psi\rangle|^2$ are the probabilities for the outcomes of observables A and B, respectively, whose eigenbases are denoted by $\{|a_1\rangle, |a_2\rangle\}$ and $\{|b_1\rangle, |b_2\rangle\}$, respectively.

The minimization problem is significantly ameliorated if one exploits the well-known Bloch representation, along lines similar to those of Ref. [18] that deals with Shannon UP. The most general normalized quantum pure state of a single qubit can be written (up to an unobservable phase factor) as $|\Psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$, with $0 \leqslant \theta \leqslant \pi$ and $0 \leqslant \phi \leqslant 2\pi$, where $\{|0\rangle, |1\rangle\}$ is the so-called computational basis. To each pure state $|\Psi\rangle$ a unique point on the Bloch sphere is assigned, represented by the unit vector $\vec{s} = (\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta) \in \mathbb{R}^3$. In terms of this Bloch vector one can write the density operator associated to the (pure) state of the system: $\rho = |\Psi\rangle\langle\Psi| = \frac{1}{2}(I + \vec{s}\cdot\vec{\sigma})$, with $\vec{\sigma} = (\sigma_X, \sigma_Y, \sigma_Z)$ denoting the Pauli matrices and I the 2×2 identity matrix. The observables in this representation acquire the form

$$A = \alpha_1 I + \alpha_2\, \vec{a}\cdot\vec{\sigma}, \tag{3}$$

$$B = \beta_1 I + \beta_2\, \vec{b}\cdot\vec{\sigma}, \tag{4}$$

where $\vec{a}, \vec{b} \in \mathbb{R}^3$ are unit vectors and α_1, α_2, β_1, and β_2 are real parameters.

We want the tightest lower bound for the uncertainty measure (2) for given observables A and B over all possible states $|\Psi\rangle$. This is equivalent to searching for

$$\min_{\theta,\phi} \mathcal{U}(A,B;\Psi), \tag{5}$$

for α_i, β_i, $\vec{a}$, and $\vec{b}$ fixed. Without loss of generality, we consider $\vec{a}\cdot\vec{\sigma}$ instead of A (they have the same eigenbasis and then the same Rényi entropy). Analogously, we consider $\vec{b}\cdot\vec{\sigma}$ instead of B. Let us pass to the squared moduli of the inner products of eigenstates of A and B. In terms of the scalar product between the corresponding unit vectors $\vec{a}$ and $\vec{b}$ we have

$$(\,|\langle a_i|b_j\rangle|^2\,) = \begin{pmatrix} \frac{1+\vec{a}\cdot\vec{b}}{2} & \frac{1-\vec{a}\cdot\vec{b}}{2} \\ \frac{1-\vec{a}\cdot\vec{b}}{2} & \frac{1+\vec{a}\cdot\vec{b}}{2} \end{pmatrix}. \tag{6}$$

The greatest element is known as the overlap between eigenbases. Hence, denoting by γ the angle formed by the $\vec{a}$ and $\vec{b}$ directions, the overlap becomes

$$\begin{aligned}c \equiv \max_{i,j=1,2} |\langle a_i|b_j\rangle| &= \max_{\gamma\in(0,\pi)} \left\{\cos\frac{\gamma}{2}, \sin\frac{\gamma}{2}\right\}\\ &= \begin{cases} \cos\frac{\gamma}{2} & \text{if } 0 < \gamma \leqslant \pi/2, \\ \sin\frac{\gamma}{2} & \text{if } \pi/2 \leqslant \gamma < \pi, \end{cases}\end{aligned} \tag{7}$$

where we restrict the values of γ to the interval $(0,\pi)$. Due to symmetry arguments, results for $\gamma \in (\pi, 2\pi)$ can be obtained straightforwardly. Besides, $\gamma = 0$ and $\gamma = \pi$ (implying $c = 1$) are excluded since they correspond indeed to pairs of commuting observables. The particular case $\gamma = \pi/2$ gives $|\langle a_i|b_j\rangle| = 1/\sqrt{2}$ for all $i,j = 1,2$, corresponding to that special situation in which the observables are complementary. For the two-dimensional (2D) case, the range for the overlap c goes from $1/\sqrt{2}$ up to 1.

As our main result, we show that the uncertainty measure (2) has as a lower bound, a function depending only on the overlap $c \in [1/\sqrt{2}, 1)$, of the form

$$\mathcal{U}(A,B;\Psi) \equiv H_2(A) + H_2(B) \geqslant -2\ln\frac{1+c^2}{2}. \tag{8}$$

Moreover, the uncertainty measure (2) exhibits an upper bound, since $\mathcal{U}(A,B;\Psi) \leqslant 2\ln 2$. We stress that the EUR (8) is valid for arbitrary pairs of (two-dimensional) observables, not merely for those special ones that are complementary.

B. Derivation of the optimal bound and minimal uncertainty states

In order to demonstrate the above result, let us first write the uncertainty measure $\mathcal{U}(A,B;\Psi)$ in terms of the scalar products of $\vec{a}$ and $\vec{b}$ with the Bloch vector $\vec{s}$. A short calculation yields

$$\mathcal{U} = \mathcal{U}(\vec{a},\vec{b};\vec{s}) = -\ln\frac{1+(\vec{a}\cdot\vec{s})^2}{2} - \ln\frac{1+(\vec{b}\cdot\vec{s})^2}{2}. \quad (9)$$

Therefore, the extremization of $\mathcal{U}$ becomes a geometric problem: For fixed directions $\vec{a}$ and $\vec{b}$, we need to find the unit vector $\vec{s}$ that bounds either from below or from above the quantity (9). Trivially, the maximum of this quantity corresponds to the case when $\vec{s}$ is just one of the two unit vectors along the direction perpendicular to both $\vec{a}$ and $\vec{b}$. Then $\mathcal{U}_{\rm Max} = 2\ln 2$. This happens indeed when all $p_i = 1/2$ and then the collision entropy for each observable is separately maximal. Let us now show that the minimum of the quantity (9) is given when $\vec{a}$, $\vec{b}$, and $\vec{s}$ are coplanar, a fact that reduces the number of variables in the minimization problem. Consider the function $U(x) = -\ln(\frac{1+x^2}{2})$ with $x \in [0,1]$. Straightforwardly, one sees that $U(x)$ is a strictly decreasing function in its domain. Thus, for a value $x_0 \in [0,1]$ one has $U(x) \geqslant U(x_0)$ for all $x \leqslant x_0$. Let Π be the plane determined by the fixed vectors $\vec{a}$ and $\vec{b}$, and let $\vec{d}$ be any unit vector belonging to an arbitrary orthogonal plane $\Pi^\perp$. If $\vec{d}_0$ is one of the two unit vectors that belong to Π and $\Pi^\perp$, then $|\vec{a}\cdot\vec{d}| \leqslant |\vec{a}\cdot\vec{d}_0|$ for all $\vec{d} \in \Pi^\perp$. Thus, $U(|\vec{a}\cdot\vec{d}|) \geqslant U(|\vec{a}\cdot\vec{d}_0|)$, the equality being satisfied when $\vec{d} = \pm\vec{d}_0$, that is, when $\vec{d} \in \Pi$ as well. An analogous result is obtained by changing $\vec{a}$ for $\vec{b}$. This justifies the fact that the minimum of $\mathcal{U}$ will be reached under the condition that $\vec{a}$, $\vec{b}$, and $\vec{s}$ all belong to the same plane. However, we still need to determine the direction of $\vec{s}$ relative to the fixed vectors $\vec{a}$ and $\vec{b}$.

Let us denote by χ the angle between $\vec{a}$ and $\vec{s}$. Accordingly, the uncertainty measure (9), expressed in terms of the angles χ and γ, becomes

$$\mathcal{U}_\gamma(\chi) = -\ln\left(\frac{1+\cos^2\chi}{2}\right) - \ln\left(\frac{1+\cos^2(\gamma-\chi)}{2}\right), \quad (10)$$

where χ can be restricted to the interval $[0,\pi]$ due to the periodicity of this function. Thus, the minimization problem (5) reduces to that of finding the minimum of $\mathcal{U}_\gamma(\chi)$ for γ fixed. Equating to zero the first derivative of $\mathcal{U}_\gamma(\chi)$ with respect to χ we arrive at the condition for a critical point, in the fashion

$$f(\chi) = f(\gamma-\chi), \quad (11)$$

where we have defined $f(x) = \frac{\sin 2x}{3+\cos 2x}$.

Let us solve now Eq. (11). First, for any fixed γ we have the trivial solutions $\chi_k = \frac{\gamma+k\pi}{2}$ with $k \in \mathbb{Z}$. Thus, in the interval $\chi \in [0,\pi]$ the two trivial solutions are

$$\chi_<(\gamma) \equiv \chi_1 = \frac{\gamma}{2}, \quad (12)$$

$$\chi_>(\gamma) \equiv \chi_2 = \frac{\gamma}{2} + \frac{\pi}{2}. \quad (13)$$

These solutions correspond to the straight lines plotted in Fig. 1. Geometrically, the solution $\chi_<$ corresponds to the vector $\vec{b}+\vec{a}$, pointing in the direction of the interior bisector of the angle determined by $\vec{a}$ and $\vec{b}$ directions. The solution $\chi_>$ corresponds to a different vector, pointing along the direction of $\vec{b}-\vec{a}$ and being perpendicular to the former. The norms of these two vectors can be simply expressed in terms of the overlap, since $\|\vec{b}\pm\vec{a}\| = 2c$. Therefore, the Bloch vectors corresponding to the solutions (12) and (13) become $\vec{s}_\lessgtr = \frac{\vec{b}\pm\vec{a}}{2c}$. Notably, these solutions have also been obtained by Ghirardi *et al.* in Ref. [18], where Shannon entropies have been employed instead.

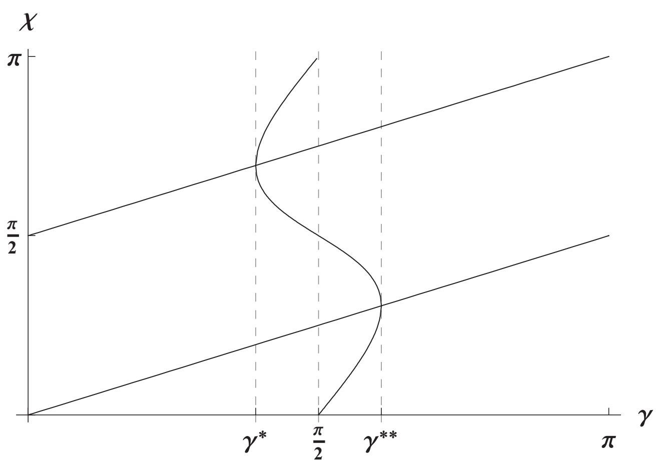

FIG. 1. Solutions of Eq. (11) for the angle χ between the Bloch vector $\vec{s}$ and the vector $\vec{a}$ corresponding to observable A. Operator B enters through the parameter γ which stands for the angle between $\vec{a}$ and $\vec{b}$.

In addition to the trivial solutions, for a given range of $\gamma \in [\gamma^*,\gamma^{**}]$ it can be seen that there exist other solutions to Eq. (11). Unfortunately, they do not possess analytical expressions and have to be calculated numerically. The limiting (critical) values of γ^* and γ^{**} for the existence of two or more than two solutions satisfy

$$f'(\chi) = f'(\gamma-\chi),$$

coming from the condition that the maxima or, respectively, the minima of $f(\chi)$ and $f(\gamma-\chi)$ coincide. We can obtain in analytical fashion these critical values:

$$\gamma^* = \pi - \arccos(-1/3), \text{ and } \gamma^{**} = \arccos(-1/3). \quad (14)$$

We plot in Fig. 1 all solutions for χ in terms of the parameter γ. We see that γ^*, $\pi/2$, and γ^{**} are critical parameters, in the sense that the number of solutions of Eq. (11) changes. Let us now discuss in some detail the solutions pertaining to different regions of the parameter γ.

(1) $\gamma \in (0,\gamma^*]$. The only solutions of Eq. (11) are the trivial ones $\chi_<$ and $\chi_>$ in Eqs. (12) and (13). Note that $\chi_<$ corresponds to the minimum of $\mathcal{U}_\gamma$ and $\chi_>$ to the maximum.

(2) $\gamma \in (\gamma^*,\pi/2)$. $\chi_<$ and $\chi_>$ are still solutions of Eq. (11) but there exist other two solutions, whose values can be calculated numerically for each γ. The solution $\chi_<$ remains the absolute minimum of $\mathcal{U}_\gamma$ while $\chi_>$ is now a relative minimum and the other solutions yield the (same) maximum.

(3) $\gamma = \pi/2$. This is a special case because the observables A and B are *complementary*, that is, the overlap is precisely $c = 1/\sqrt{2}$. The solutions $\chi_< = \pi/4$ and $\chi_> = 3\pi/4$ yield the same minimum uncertainty measure, while the other three ones (0, $\pi/2$, and π) give the maximum.

(4) $\gamma \in (\pi/2,\gamma^{**})$. In this interval $\chi_<$ and $\chi_>$ reverse their behaviors with respect to those of item (2). That is, $\chi_<$ happens

to be a relative minimum while $\chi_>$ is the absolute minimum of $\mathcal{U}_\gamma$. The two other solutions correspond to maxima.

(5) $\gamma \in [\gamma^{**},\pi)$. Now $\chi_<$ corresponds to the maximum of $\mathcal{U}_\gamma$ and $\chi_>$ to the minimum. No other solutions exist in this interval.

Thus, depending on γ, the value of χ at which $\mathcal{U}_\gamma$ acquires its minimum can be given in concise fashion using the Heaviside function as

$$\chi_{\min}(\gamma) = \frac{\gamma}{2} + \frac{\pi}{2}\,\Theta\left(\gamma - \frac{\pi}{2}\right), \qquad \gamma \in \left(0,\frac{\pi}{2}\right) \cup \left(\frac{\pi}{2},\pi\right), \tag{15}$$

while $\chi_{\min}$ at $\gamma = \pi/2$ takes the values $\pi/4$ and $3\pi/4$ as discussed in item (3). Summing up, replacing the values of $\chi_<$ and $\chi_>$ in the uncertainty measure (10) we obtain its minimum as a function of γ, and finally

$$\mathcal{U}(A,B;\Psi) \geqslant \mathcal{U}_{\min}(A,B) = \begin{cases} -2\ln\left(\frac{1+\cos^2\frac{\gamma}{2}}{2}\right) & \text{if } 0 < \gamma \leqslant \pi/2, \\ -2\ln\left(\frac{1+\sin^2\frac{\gamma}{2}}{2}\right) & \text{if } \pi/2 \leqslant \gamma < \pi. \end{cases} \tag{16}$$

Recalling Eq. (7), we complete the proof of the uncertainty relation proposed in (8).

For the sake of completeness *we also determine the states saturating our UP relation*. As already mentioned, these states have Bloch vectors $\vec{s}_<$ or $\vec{s}_>$ depending whether γ is, respectively, smaller or larger than $\pi/2$. Therefore, the corresponding minimum-uncertainty density operators become

$$\begin{aligned} \rho_< &= \frac{1}{2}\left(I + \frac{\vec{b}+\vec{a}}{2\cos\frac{\gamma}{2}}\cdot\vec{\sigma}\right) \qquad \text{if } 0 < \gamma \leqslant \pi/2, \\ \rho_> &= \frac{1}{2}\left(I + \frac{\vec{b}-\vec{a}}{2\sin\frac{\gamma}{2}}\cdot\vec{\sigma}\right) \qquad \text{if } \pi/2 \leqslant \gamma < \pi. \end{aligned} \tag{17}$$

Notice that for each γ there exists another minimum-UP state arising from the χ minimization of the uncertainty measure (10) in the interval $\chi \in [\pi,2\pi]$ that we did not consider due to the periodicity of this function. The solutions are $\tilde{\chi}_< = \frac{\gamma}{2} + \pi$ and $\tilde{\chi}_> = \frac{\gamma}{2} + \frac{3}{2}\pi$. The discussion of items (1) to (5) above apply also, replacing χ with $\tilde{\chi}$. It is not difficult to see that the optimum states $\tilde{\rho}_{\lessgtr}$ have Bloch vectors $\vec{\tilde{s}}_{\lessgtr} = -\frac{\vec{b}\pm\vec{a}}{2c}$ for $\gamma \lessgtr \pi/2$, respectively.

The particular case $\gamma = \pi/2$ (complementary observables) deserves some comment. In this case, there exist *four* states that minimize the uncertainty measure. To fix ideas, consider as an example the pair of observables $A = \sigma_x$ and $B = \sigma_y$. Their Bloch representations correspond to $\vec{a} = \check{\imath}$ and $\vec{b} = \check{\jmath}$, respectively, so that we speak of $\gamma = \pi/2$ and $c = 1/\sqrt{2}$. Our approach prescribes that any state of the system will have a collision-entropy uncertainty greater than or equal to $2\ln(4/3)$, with equality for the states $|\Psi_l\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i^{l+1/2}|1\rangle)$, for $l =$ 0, 1, 2, and 3, up to a global phase factor.

We can go beyond the case of complementary observables, since our inequality (16) allows us to quantitatively study the uncertainty related to the measurement of *any* pair of two-dimensional observables, that is, for *any* value of the overlap $c \in [1/\sqrt{2},1)$. For instance, among other interesting situations, we can deal with the Hadamard gate and the x (or z) spin projection. Assume $A = \frac{\sigma_x+\sigma_z}{\sqrt{2}}$ and $B = \sigma_z$. Then their corresponding vectors $\vec{a} = (1,0,1)/\sqrt{2}$ and $\vec{b} = (0,0,1)$ determine an angle $\gamma = \pi/4$ and, consequently, $c = \cos(\pi/8) \approx 0.924$. The following relation ensues: $2\ln 2 \geqslant H_2(\frac{\sigma_x+\sigma_z}{\sqrt{2}}) + H_2(\sigma_z) \geqslant 2\ln(\frac{8}{6+\sqrt{2}}) \approx 0.152$, with saturation of the last inequality for those qubit states that lie on the xz plane ($\phi = 0$) with $\theta = \pi/8$ or $\tilde{\theta} = 9\pi/8$.

IV. COMPARISON WITH OTHER UNCERTAINTY RELATIONS

We pass to discuss and compare our present results with other formulations of the UP.

A. Heisenberg-Robertson inequality in terms of standard deviations

We begin with the celebrated Heisenberg-Robertson (HR) inequality. For any pair of arbitrary observables A,B and a system described by the state $|\Psi\rangle$ one has

$$\Delta_\Psi A\, \Delta_\Psi B \geqslant \tfrac{1}{2}|\langle[A,B]\rangle_\Psi|, \tag{18}$$

where $\langle O\rangle_\Psi = \langle\Psi|O|\Psi\rangle$ is the mean value and $\Delta_\Psi O = \sqrt{\langle O^2\rangle_\Psi - \langle O\rangle_\Psi^2}$ the standard deviation of the observable O. In the 2D case, using the Bloch representation, the HR inequality (18) reads

$$|\alpha_2|\sqrt{1-(\vec{a}\cdot\vec{s})^2}\;|\beta_2|\sqrt{1-(\vec{b}\cdot\vec{s})^2} \geqslant |\alpha_2\beta_2||(\vec{a}\times\vec{b})\cdot\vec{s}|. \tag{19}$$

Some questions regarding (19) may be cited here. If $|\Psi\rangle$ is an eigenstate of one of the two observables, $\vec{s}$ is parallel to the vector representing that observable and the HR-UP becomes trivial. No UP information is gained thereby (in terms of a standard deviation) for measuring the other observable. Moreover, for *any* state whose Bloch vector $\vec{s}$ belongs to the plane determined by $\vec{a}$ and $\vec{b}$, the right-hand side of (19) vanishes, thus representing trivial information about the bound for the standard deviations' product: $\Delta A\,\Delta B \geqslant 0$. Notice that in the derivation of our EUR (8) we showed that the minimum of $\mathcal{U}$ is attained for the case when $\vec{a}$, $\vec{b}$, and $\vec{s}$ lie in the same plane but only for those states satisfying Eqs. (12) or (13). Furthermore we stress that, unlike HR-UP, the bound we obtain is strictly greater than zero. Therefore, we conclude that, in 2D, *relevant* information about the uncertainty of the observables can be obtained by recourse to the collision entropy.

B. Luis relation in terms of purities

In the appendix of Ref. [37] Luis derived, for *complementary* observables with discrete spectrum of N states, an uncertainty relation following the work of Larsen [38]. In his notation,

$$\delta A\,\delta B \geqslant \left(\frac{2N}{N+1}\right)^2, \tag{20}$$

where $\delta O = 1/\sum_i p_i(O)^2$ is the inverse of the purity (or participation ratio) associated with observable O. The expression (20) is an improvement of the certainty relations that express the complementarity property, obtained by Luis in Refs. [39,40] for the case of 2D systems and general N-dimensional systems, respectively. Taking the natural logarithm in (20), this relation can be expressed in terms of the sum of collision entropies: $H_2(A) + H_2(B) \geqslant 2 \ln \frac{2N}{N+1}$. In the case $N = 2$ (one-qubit system) that we are here considering, the Luis bound is $2 \ln(4/3)$. This result coincides with our bound in (8) when $c = 1/\sqrt{2}$. While we extend the 2D formulation of the UP to arbitrary observables (i.e., for any overlap c), Luis obtains instead results that are valid only for complementary observables, although for arbitrary (finite) N dimensions.

C. Landau-Pollak relation in terms of maximum probabilities

The Landau-Pollak relation states that

$$\arccos \sqrt{P_A} + \arccos \sqrt{P_B} \geqslant \arccos c, \tag{21}$$

where $P_O = \max_i p_i(O)$ for $O = A,B$. This inequality is another alternative to the UP mathematical formulation, introduced for time-frequency analysis in [41] and adapted to physics two decades later [26]. We have in this regard an interesting result: The states that minimize $\mathcal{U}(A,B;\Psi)$ also saturate the inequality (21), which means that the lower bound in the Landau-Pollak uncertainty relation is optimal in 2D. Indeed, if we compute the maximum probabilities for the states that minimize $\mathcal{U}$, we find

$$P_A = P_B = \begin{cases} \cos^2 \frac{\gamma}{4} & \text{if } 0 < \gamma \leqslant \pi/2, \\ \sin^2 \frac{\gamma+\pi}{4} & \text{if } \pi/2 \leqslant \gamma < \pi, \end{cases} \tag{22}$$

in terms of the angle γ. Using Eq. (7), this can be simply reformulated as

$$P_A = P_B = \frac{1+c}{2}, \tag{23}$$

in terms of the overlap c. On the other hand, when (21) becomes an equality, the Landau-Pollak relation can be recast in the fashion

$$c = \sqrt{P_A P_B} - \sqrt{(1-P_A)(1-P_B)}. \tag{24}$$

It is easy to see that replacing (22) [or equivalently (23)], in (24), the right-hand side of this equation identically yields c for any γ. Therefore, the Landau-Pollak relation (21) and the EUR (8) (as well as its analogous Shannon-entropy expression [18,29]) are equivalent for one-qubit systems.

D. Maassen-Uffink relation in terms of min-entropies

In Ref. [26], Maassen and Uffink (MU) work out an improvement of the Shannon-entropy uncertainty relation and advance an entropic UP for the sum of the min-entropies $H_\infty(O) = -\ln P_O$ associated with two observables A and B characterized by finite, discrete spectra. The pertinent expression reads

$$H_\infty(A) + H_\infty(B) \geqslant -2 \ln \frac{1+c}{2}. \tag{25}$$

They encounter this relation by maximizing the product of the maximum probabilities, $P_A P_B$, subject to the Landau-Pollak inequality (21). It is straightforwardly seen that, replacing (22) [or equivalently (23)], in the left-hand side of (25), we obtain an equality. Therefore, we find that the states that saturate our EUR (8) given in terms of collision entropies, also saturate the MU-EUR (25) that uses min-entropies, with an equality in the Landau-Pollak relation. One may be tempted to conjecture that in the $N = 2$ case, for any entropic index $q > 0$, the minimum of the sum of the q-Rényi entropies is reached when $P_A = P_B = \frac{1+c}{2}$. Let us define the function $\mathcal{F}_q(c) \equiv \frac{2}{1-q} \ln[(\frac{1+c}{2})^q + (\frac{1-c}{2})^q]$. The question is whether one can assure that $H_q(A) + H_q(B) \geqslant \mathcal{F}_q(c)$ for any positive q. In the particular cases $q = 2$ and $q \to \infty$, respectively, we do obtain the lower bounds $\mathcal{F}_2(c) = -2 \ln \frac{1+c^2}{2}$ and $\mathcal{F}_\infty(c) = -2 \ln \frac{1+c}{2}$, that in turn correspond to the right-hand sides in the EURs (8) and (25). However, we cannot prove at this point that the claim remains valid neither for any q nor for any arbitrary pair of 2D observables. As a counterexample, consider, for instance, the $q \to 1$ Shannon case. It has been proved that the function $\mathcal{F}_1(c) = -(1+c) \ln \frac{1+c}{2} - (1-c) \ln \frac{1-c}{2}$ gives the absolute minimum of $H_1(A) + H_1(B)$ *only* when the overlap c belongs to the interval $[c^*,1)$, with $c^* \simeq 0.834$ (we refer the reader to [28], where a detailed analytical study of this point is provided). In other words, for those pairs of 2D observables with overlap between $1/\sqrt{2}$ and c^*, Eq. (23) does not correspond to the optimal solution regarding the minimization problem for the sum of Shannon entropies.

V. CONCLUDING REMARKS

In the one-qubit scenario we have derived an optimal lower bound for the collision entropies' sum associated with an arbitrary pair of observables. Although we have dealt with the simplest conceivable system, the relevance of our EUR given in (8) is that

(i) we obtain a lower bound that is optimal,
(ii) we find indeed the family of states that saturate the inequality,
(iii) we consider arbitrary pairs of observables, and
(iv) we take into account pairs of Rényi entropies where the corresponding indices are not conjugate ones.

We emphasize that the conjunction of the last two points has not received much attention in the literature. Previous works were based on the Riesz theorem, which imposes the conjugacy restriction for the entropic indices.

Another advantage of using collision entropies, as compared with results given in terms of Shannon ones (see Refs. [18,29]), is that the lower bound in our case is analytical. This could be useful for future applications, for instance in connection with entanglement criteria, state discrimination, quantum cryptographic protocols, etc.

Moreover, we have shown that the states that minimize the collision entropy UP measure defined by Eq. (2) also saturate the EUR (25) given by Maassen and Uffink and, additionally, saturate the Landau-Pollak relation (21). They yield no relevant information concerning the Heisenberg-Robertson standard-deviation formulation, which turns out to be trivial in our scenario.

Furthermore, it can be proved that the existence of relation (25) guarantees a nontrivial entropic uncertainty inequality for Rényi entropies of arbitrary (positive) indices. This is done making use of the monotonicity property of the family of Rényi entropies H_q with respect to the index q. The present study has allowed us to advance entropic UPs of the form

$$H_q(A) + H_{q'}(B) \geqslant -2 \ln \frac{1+c^2}{2} \tag{26}$$

for *any* couple $(q;q') \in \mathcal{R} = \{0 < q \leqslant 2, 0 < q' \leqslant 2\}$, where A and B are *any* arbitrary 2D observables. Within the region $\mathcal{R}$ of the q-q' plane, the relation (26) is more stringent than the one derived following Maassen-Uffink's prescription (25). In order to prove the assertion (26) we just need the fact that Rényi entropy is strictly decreasing with the entropic index. Thus, the left-hand side in (26) becomes greater than or equal to $H_2(A) + H_2(B)$, which in turn is lower bounded as in (8). The uncertainty relation (26) is in general nonoptimal. We claim that at least it is optimal at the vertex $(q;q') = (2;2)$ of the rectangular region $\mathcal{R}$.

Note that the extension of our EUR to *mixed states* can be easily made due to the fact that the collision entropy is a concave function for one-qubit systems [42]. Generalizations to N-level systems are the subject of active current research.

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