Injective Envelopes and Local Multiplier Algebras of Some Spatial Continuous Trace C*-algebras*

Martín Argerami, Douglas Farenick, and Pedro Massey

Department of Mathematics and Statistics, University of Regina Regina, Saskatchewan S4S 0A2, Canada

Departamento de Matemática, Facultad de Ciencias Exactas Universidad Nacional de La Plata, Argentina

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Abstract

A precise description of the injective envelope of a spatial continuous trace C*-algebra A over a Stonean space Δ is given. The description is based on the notion of a weakly continuous Hilbert bundle, which we show herein to be a Kaplansky–Hilbert module over the abelian AW*-algebra $C(\Delta)$. We then use the description of the injective envelope of A to study the first- and second-order local multiplier algebras of A. In particular, we show that the second-order local multiplier algebra of A is precisely the injective envelope of A.

Introduction

A commonly used technique in the theory of operators algebras is to study a given C*-algebra A by one or more of its enveloping algebras. Well known examples of such enveloping algebras are the enveloping von Neumann algebra A^{**} and the multiplier algebra $M(A)$. In this paper we consider two others: the local multiplier algebra $M_{\mathrm{loc}}(A)$ and the injective envelope $I(A)$, both

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of which have received considerable study and application in recent years (see, for example, [1, 6, 7, 9, 11, 19, 21, 22]).

The C*-algebras $M_{\mathrm{loc}}(A)$ and $I(A)$ are difficult to determine precisely, even for fairly rudimentary types of C*-algebras A. For instance, if we denote by $C_0(T)$ an abelian C*-algebra and by $K(H)$ the ideal of compact operators over H, their local multiplier algebra and injective envelope have been readily computed; but the injective envelope of $C_0(T)\otimes K(H)$ is much more difficult to describe: see [15] for an abstract description and [3, 4] for a somewhat more concrete one.

Our first goal in the present paper is to make a further contribution to the issue of the determination of $I(A)$ and $M_{\mathrm{loc}}(A)$ from A by considering continuous trace C*-algebras studied by Fell [10] that arise from continuous Hilbert bundles. The class of such algebras contains in particular all C*-algebras of the form $C_0(T) \otimes K(H)$, which were studied in [4]. Because the centres of $I(A)$ and $M_{\mathrm{loc}}(A)$ are AW*-algebras, and thus have Stonean maximal ideal spaces, we restrict ourselves in this paper to locally compact Hausdorff spaces T that are Stonean. In so doing, we establish an important first step toward a complete analysis, in the case of non-Stonean T, of the C*-algebras $I(A)$, $M_{\mathrm{loc}}(A)$, and $M_{\mathrm{loc}}\left(M_{\mathrm{loc}}(A)\right)$ for spatial continuous trace C*-algebras A with spectrum T. As the passage from general T to Stonean T involves a number of technicalities, the application of the main results herein to the case of arbitrary locally compact Hausdorff spaces T will be deferred to a subsequent article.

Our second goal is to study and use the notion of a weakly continuous Hilbert bundle Ω_{wk} relative to a continuous Hilbert bundle Ω over a locally compact Hausdorff space T. Particular cases of this notion have been previously considered in [15, 23]. It is natural to consider Ω as a C*-module over the abelian C*-algebra $C_0(T)$; if, moreover, T is a Stonean space Δ, we then show Ω_{wk} carries the structure of a faithful AW*-module over $C(\Delta)$. In this latter situation, such C*-modules are called Kaplansky–Hilbert modules. We study the C*-modules Ω and Ω_{wk}, as well as certain C*-algebras of endomorphisms of these modules, using the beautiful machinery Kaplansky developed in his seminal work from the early 1950s [16]. In particular, we prove that the C*-algebra $B(\Omega_{\mathrm{wk}})$ of bounded adjointable endomorphisms of Ω_{wk} is the injective envelope and second-order local multiplier algebra of the C*-algebra $K(\Omega)$ of "compact" endomorphisms of Ω.

Assuming that $T = \Delta$, a Stonean space, and in postponing the precise definitions until the following section, we summarise in this paragraph the main results of the paper. In Section 2, we show that Ω_{wk} is a Kaplansky–Hilbert module that contains Ω as a C*-submodule such that $\Omega^{\perp} = \{0\}$. In

Section 3, we prove that $B(\Omega_{\mathrm{wk}})$ is the injective envelope of both $K(\Omega)$ and the Fell continuous trace C*-algebra A induced by the bundle Ω. Section 4 deals with local multipliers, and we show that $B(\Omega_{\mathrm{wk}})$ is the second-order local multiplier algebra of both $K(\Omega)$ and Fell algebra A. We also prove that the equality $M_{\mathrm{loc}}(M_{\mathrm{loc}}(A)) = I(A)$ holds for certain type I non-separable C*-algebras, generalising a result of Somerset [21]. Finally, in Section 5 we find that a direct-sum decomposition of Ω_{wk} leads to a corresponding decomposition of (the generally non-AW*) algebra $M_{\mathrm{loc}}(A)$ but not to a decomposition of A.

1 Preliminaries

If T is a locally compact Hausdorff space and $\{H_t\}_{t\in T}$ is family of Hilbert spaces, a vector field on T with fibres H_t is a function $\nu : T \to \bigsqcup_t H_t$ in which $\nu(t) \in H_t$, for every $t \in T$. Such a vector field ν is said to be bounded if the function $t \mapsto \|\nu(t)\|$ is bounded. From this point on, the notation $T \to \bigsqcup_t H_t$ will be taken to also imply that, for all t, the point t is mapped into the corresponding fibre H_t.

Definition 1.1. *A* continuous Hilbert bundle *[8] is a triple* $(T, \{H_t\}_{t\in T}, \Omega)$, *where* Ω *is a set of vector fields on* T *with fibres* H_t *such that:*

(I) Ω *is a* $C(T)$*-module with the action* $(f \cdot \omega)(t) = f(t)\omega(t)$*;*

(II) *for each* $t_0 \in T$, $\{\omega(t_0) : \omega \in \Omega\} = H_{t_0}$*;*

(III) *the map* $t \mapsto \|\omega(t)\|$ *is continuous, for all* $\omega \in \Omega$*;*

(IV) Ω *is closed under local uniform approximation—that is, if* $\xi : T \to \bigsqcup_t H_t$ *is any vector field such that for every* $t_0 \in T$ *and* $\varepsilon > 0$ *there is an open set* $U \subset T$ *containing* t_0 *and a* $\omega \in \Omega$ *with* $\|\omega(t) - \xi(t)\| < \varepsilon$ *for all* $t \in U$*, then necessarily* $\xi \in \Omega$*.*

Dixmier and Douady [8] show that (I), (II), and (IV) can be replaced by other axioms, such as those given by Fell [10], without altering the structure that arises. For example, in the presence of the other axioms, (II) is equivalent to "$\{\omega(t_0) : \omega \in \Omega\}$ is dense in H_{t_0}, for each $t_0 \in T$"; in the presence of (IV), axiom (I) can be replaced by "Ω is a complex vector space".

We turn next to the notion of a weakly continuous Hilbert bundle. If $(T, \{H_t\}_{t\in T}, \Omega)$ is a continuous Hilbert bundle then, by the polarisation identity, the function $t \mapsto \langle \omega_1(t), \omega_2(t)\rangle$ is continuous for all $\omega_1, \omega_2 \in \Omega$. In defining $\langle \omega_1, \omega_2\rangle$ to be the map $T \to \mathbb{C}$ given by $t \mapsto \langle \omega_1(t), \omega_2(t)\rangle$, one obtains a $C(T)$-valued inner product on Ω which gives Ω the structure of an inner product module over $C(T)$.

Definition 1.2. *A vector field $\nu : T \to \bigsqcup_t H_t$ is said to be* weakly continuous *with respect to the continuous Hilbert bundle $(T, \{H_t\}_{t\in T}, \Omega)$ if the function*

$$t \longmapsto \langle \nu(t), \omega(t)\rangle$$

is continuous for all $\omega \in \Omega$. The set of all bounded weakly continuous vector fields with respect to a given Ω will be denoted by $\Omega_{\rm wk}$, that is

$$\Omega_{\rm wk} = \{\nu : T \to \bigsqcup_t H_t : \sup_t \|\nu(t)\| < \infty \quad \textit{and } \nu \textit{ is weakly continuous}\}.$$

We will call the quadruple $(T, \{H_t\}_{t\in T}, \Omega, \Omega_{\rm wk})$ a weakly continuous Hilbert bundle *over T.*

We remark that when T is compact, $\Omega_{\rm wk}$ is a $C(T)$-module under the pointwise module action, and also $\Omega \subset \Omega_{\rm wk}$ (because then every continuous field on T is bounded). However, the function $t \mapsto \langle \nu_1(t), \nu_2(t)\rangle$ is generally not continuous for arbitrary $\nu_1, \nu_2 \in \Omega_{\rm wk}$. Thus, although $\Omega_{\rm wk}$ is, algebraically, a module over $C_b(T)$, it is not in general an inner product module over $C_b(T)$. Nevertheless, if T has the right topology—namely that of a Stonean space—then we show (Theorem 2.6) that it is possible to endow a weakly continuous Hilbert bundle with the structure of a C*-module over the C*-algebra of continuous complex-valued functions on T.

The continuous trace C*-algebras we consider herein were first studied by Fell [10]. We now recall their definition.

Assume that $\{A_t\}_{t\in T}$ is a family of C*-algebras indexed by the locally compact Hausdorff topological space T. An operator field is a map $a : T \to \bigsqcup_t A_t$ such that $a(t) \in A_t$, for each $t \in T$.

Definition 1.3. *Let $(T, \{H_t\}_{t\in T}, \Omega)$ be a continuous Hilbert bundle. An operator field $a : T \to \bigsqcup_{t\in T} K(H_t)$ is:*

i. almost finite-dimensional *(with respect to Ω) if for each $t_0 \in T$ and $\varepsilon > 0$ there exist an open set $U \subset T$ containing t_0 and $\omega_1, \dots, \omega_n \in \Omega$ such that*

   (a) $\omega_1(t), \dots, \omega_n(t)$ are linearly independent for every $t \in U$, and

   (b) $\|p_t a(t) p_t - a(t)\| < \varepsilon$ for all $t \in U$, where $p_t \in B(H_t)$ is the projection with range $\mathrm{Span}\,\{\omega_j(t) : 1 \le j \le n\}$*;*

ii. weakly continuous *(with respect to* Ω*) if the complex-valued function*

$$t \longmapsto \langle a(t)\omega_1(t), \omega_2(t)\rangle$$

is continuous for every $\omega_1, \omega_2 \in \Omega$.

Definition 1.4. ([10]) *Let* $(T, \{H_t\}_{t\in T}, \Omega)$ *be a continuous Hilbert bundle. The* Fell algebra of the Hilbert bundle $(T, \{H_t\}_{t\in T}, \Omega)$*, denoted by* $A = A(T, \{H_t\}_{t\in T}, \Omega)$*, is the set of all weakly continuous, almost finite-dimensional operator fields* $a : T \to \bigsqcup_{t\in T} K(H_t)$ *for which* $t \mapsto \|a(t)\|$ *is continuous and vanishes at infinity, endowed with pointwise operations and norm*

$$\|a\| = \max_{t\in T} \|a(t)\|, \qquad a \in A.$$

We shall make repeated use of the following fact about the Fell algebras of Hilbert bundles: if $A = A(T, \{H_t\}_{t\in T}, \Omega)$, for some continuous Hilbert bundle $(T, \{H_t\}_{t\in T}, \Omega)$, then A is a continuous trace C*-algebra with spectrum $\hat{A} \simeq T$ [10, Theorems 4.4, 4.5].

2 An AW*-module Structure for $\Omega_{\rm wk}$

Assume henceforth that $T = \Delta$ is a Stonean space; that is, Δ is Hausdorff, compact, and extremely disconnected. The abelian C*-algebra $C(\Delta)$ is an AW*-algebra and so one may ask whether the C*-modules Ω and $\Omega_{\rm wk}$ are AW*-modules in the sense of Kaplansky [16]. We shall show that this is indeed true for the module $\Omega_{\rm wk}$. As a consequence of this last fact we shall get that the C*-algebra $B(\Omega_{\rm wk})$ of bounded adjointable endomorphisms of $\Omega_{\rm wk}$ is an AW*-algebra of type I.

The following lemmas are needed to describe the $C(\Delta)$-Hilbert module structure of $\Omega_{\rm wk}$.

Lemma 2.1. *Let* $f : \Delta \to \mathbb{R}$ *be a lower semicontinuous function such that there exist* $g \in C(\Delta)$ *and a meagre set* $M \subset \Delta$ *with* $f(s) = g(s)$ *for all* $s \in \Delta \setminus M$*. Then*

$$\sup_{s\in\Delta} g(s) = \sup_{s\in\Delta\setminus M} f(s) = \sup_{s\in\Delta} f(s).$$

Proof. Let $\rho = \sup_{s\in\Delta\setminus M} f(s) = \sup_{s\in\Delta\setminus M} g(s) \le \sup_{s\in\Delta} g(s)$; then $f(s) \le \rho$ for all $s \in \Delta \setminus M$. Because Δ is a Baire space, $\overline{\Delta \setminus M} = \Delta$; thus, by the lower semi-continuity, $f(s) \le \rho$ for every $s \in \Delta$. The same argument yields that $g(s) \le \rho$ for all $s \in \Delta$. □

Lemma 2.2. *Assume that* $(\Delta, \{H_s\}_{s\in\Delta}, \Omega)$ *is a continuous Hilbert bundle and* $\nu \in \Omega_{\rm wk}$*. Then*

i. *the function* $s \mapsto \|\nu(s)\|^2$ *is lower semicontinuous;*

ii. *there is a meagre subset* $M \subset \Delta$ *and a continuous function* $h : \Delta \to \mathbb{R}_+$ *such that*

(a) $h(s) = \|\nu(s)\|^2$ *for all* $s \in \Delta \setminus M$*, and*

(b) $\|h\| = \sup\limits_{s\in\Delta\setminus M} \|\nu(s)\|^2 = \sup\limits_{s\in\Delta} \|\nu(s)\|^2$.

Proof. Let $r \in \mathbb{R}$ be fixed and consider $U_r = \{s \in \Delta : r < \|\nu(s)\|^2\}$. We aim to show that U_r is open. Choose $s_0 \in U_r$. Thus, $r < \|\nu(s_0)\|^2$. By Parseval's formula, there are orthonormal vectors $\xi_1, \dots, \xi_n \in H_{s_0}$ such that $r < \sum_{j=1}^{n} |\langle \nu(s_0), \xi_j\rangle|^2 \leq \|\nu(s_0)\|^2$. Choose any $\mu_1, \dots, \mu_n \in \Omega$ such that $\mu_j(s_0) = \xi_j$, for each j. Because $\xi_1, \dots, \xi_n$ are orthogonal, $\mu_1(s), \dots, \mu_n(s)$ are linearly independent in an open neighbourhood of s_0. Hence, by [10, Lemma 4.2], there is an open set V containing s_0 and vector fields $\omega_1, \dots, \omega_n \in \Omega$ such that $\omega_1(s), \dots, \omega_n(s)$ are orthonormal for all $s \in V$, and $\omega_j(s_0) = \xi_j$ for each j. The function

$$g(s) = \sum_{j=1}^{n} |\langle \nu(s), \omega_j(s)\rangle|^2$$

on Δ is continuous and satisfies $g(s) \leq \|\nu(s)\|^2$, for every $s \in V$, and $r < g(s_0)$. Therefore, by the continuity of g, there is an open set $W \subset V$ containing s_0 such that $r < g(s) \leq \|\nu(s)\|^2$ for all $s \in W$. This proves that U_r contains an open set around each of its points. That is, U_r is open.

Because every bounded nonnegative lower semicontinuous function on a Stonean space Δ agrees with a nonnegative continuous function off a meagre set M [24, Proposition III.1.7], the function $h \in C(\Delta)$ as in (**ii**) exists and satisfies $h(s) = \|\nu(s)\|^2$ for $s \in \Delta \setminus M$.

The last statement follows from Lemma 2.1. □

Let $(\Delta, \{H_t\}_{t\in\Delta}, \Omega, \Omega_{\rm wk})$ be a weakly continuous Hilbert bundle over Δ. Given $\nu \in \Omega_{\rm wk}$, the function h that arises in Lemma 2.2 will be denoted by $\langle \nu, \nu\rangle$. There is no ambiguity in so doing because if $h_1, h_2 \in C(\Delta)$ and if $h_1(s) = h_2(s)$ for all $s \notin (M_1 \cup M_2)$ for some meagre subsets M_1 and M_2, then h_1 and h_2 agree on Δ. (If not, then by continuity, h_1 and h_2 would differ on an open set U; but $\emptyset \neq U \subset M_1 \cup M_2$ is in contradiction to the fact that no meagre set in a Baire space can contain a nonempty open set.)

Now use the polarisation identity to define $\langle \nu_1, \nu_2 \rangle \in C(\Delta)$ for any pair $\nu_1, \nu_2 \in \Omega_{\rm wk}$. This gives $\Omega_{\rm wk}$ the structure of pre-inner product module over $C(\Delta)$ whereby for each $\nu_1, \nu_2 \in \Omega_{\rm wk}$ there is a meagre subset $M_{\nu_1,\nu_2} \subset \Delta$ such that the continuous function $\langle \nu_1, \nu_2 \rangle$ satisfies

$$\langle \nu_1, \nu_2 \rangle (s) \;=\; \langle \nu_1(s), \nu_2(s) \rangle \,, \qquad \forall\, s \in \Delta \setminus M_{\nu_1,\nu_2} \,.$$

In particular, if $\nu \in \Omega_{\rm wk}$ and $\omega \in \Omega$, then

$$\langle \nu, \omega \rangle (s) \;=\; \langle \nu(s), \omega(s) \rangle \,, \qquad \forall\, s \in \Delta \,.$$

In fact, $\Omega_{\rm wk}$ is an inner product module over $C(\Delta)$, for if $\nu \in \Omega_{\rm wk}$ satisfies $\langle \nu, \nu \rangle = 0$, then Lemma 2.2 yields $\|\nu(s)\|^2 = 0$ for all $s \in \Delta$. Therefore,

$$\|\nu\| \;=\; \|\langle \nu, \nu \rangle\|^{1/2} \,, \quad \nu \in \Omega_{\rm wk} \,,$$

defines a norm on $\Omega_{\rm wk}$, where

$$\|\nu\|^2 \;=\; \sup_{s\in\Delta} \langle \nu(s), \nu(s) \rangle \;=\; \|\langle \nu, \nu \rangle\| \,. \tag{1}$$

Recall that given a C*-algebra B, a *Hilbert C*-module over B* is a left B-module E together with a B-valued definite sequilinear map $\langle \, , \, \rangle$ such that E is complete with the norm $\|\nu\| = \|\langle \nu, \nu \rangle\|^{1/2}$ (we refer to [17] for a detailed account on Hilbert modules).

Note that if $\nu \in \Omega_{\rm wk}$, then $|\nu|(s) := \langle \nu, \nu \rangle^{1/2}(s) \geq \|\nu(s)\|$ for $s \in \Delta$ and there exists a meagre set $M \subset \Delta$ with $|\nu|(s) = \|\nu(s)\|$ if $s \in (\Delta \setminus M)$ (Lemma 2.2). These facts will be used repeatedly from now on.

Proposition 2.3. *$\Omega_{\rm wk}$ is a C*-module over $C(\Delta)$ and Ω is a C*-submodule of $\Omega_{\rm wk}$.*

Proof. The only Hilbert C*-module axiom that is not obviously satisfied by $\Omega_{\rm wk}$ is the axiom of completeness. Let $\{\nu_i\}_{i\in\mathbb{N}}$ be a Cauchy sequence in $\Omega_{\rm wk}$. By the equality (1), $\{\nu_i(s)\}_{i\in\mathbb{N}}$ is a Cauchy sequence in H_s for every $s \in \Delta$. Let $\nu(s) \in H_s$ denote the limit of this sequence so that $\nu : \Delta \to \bigsqcup_{s\in\Delta} H_s$ is a vector field.

Choose $\omega \in \Omega$ and consider the function $g_{i,\omega} \in C(\Delta)$ given by $g_{i,\omega}(s) = \langle \omega(s), \nu_i(s) \rangle$. Let $\varepsilon > 0$. Then there is $N_\varepsilon \in \mathbb{N}$ such that $\|\nu_i - \nu_j\| < \varepsilon$, for all $i, j \geq N_\varepsilon$. Therefore, the Cauchy-Schwarz inequality yields

$$\sup_{s\in\Delta} |g_{i,\omega}(s) - g_{j,\omega}(s)| \;<\; \varepsilon \, \|\omega\| \,, \quad \forall\, i, j \geq N_\varepsilon \,.$$

Thus, the sequence $\{g_{i,\omega}\}_i$ is Cauchy in $C(\Delta)$; let $g_\omega \in C(\Delta)$ denote its limit. Observe that $g_\omega(s) = \lim_i \langle \nu_i(s), \omega(s) \rangle = \langle \nu(s), \omega(s) \rangle$, for all $s \in \Delta$. As the choice of $\omega \in \Omega$ is arbitrary, this shows that ν is weakly continuous. The Cauchy sequence $\{\nu_i\}_{i\in\mathbb{N}}$ is necessarily uniformly bounded by, say, $\rho > 0$, and then $\|\nu(s)\| \leq \rho$ for every $s \in \Delta$. That is, the function $s \to \|\nu(s)\|$ is bounded and so $\nu \in \Omega_{\text{wk}}$. Finally, if $i, j \geq N_\varepsilon$, then for any $s \in \Delta$ we have $\|\nu(s) - \nu_i(s)\| \leq \|\nu(s) - \nu_j(s)\| + \|\nu_j(s) - \nu_i(s)\| \leq \|\nu(s) - \nu_j(s)\| + \varepsilon$, and so letting $j \to \infty$ yields $\|\nu(s) - \nu_i(s)\| \leq \varepsilon$ for every $s \in \Delta$. That is, $\|\nu - \nu_i\| \to 0$, which proves that Ω_{wk} is complete.

For the case of Ω, let $\{\omega_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in Ω. For each $s \in \Delta$, $\{\omega_n(s)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in H_s; let $\omega(s)$ denote the limit. Since the limit is uniform, it is in particular locally uniform, and so $\omega \in \Omega$. Hence, Ω is complete. □

Definition 2.4. *A Hilbert C^*-module E over a C^*-algebra B is called a* Kaplansky–Hilbert module *if in addition B is an abelian AW^*-algebra and the following three properties hold [16, p. 842] (Kaplansky's original term for such a module was "faithful AW^*-module"):*

- ***i.*** *if $e_i \cdot \nu = 0$ for some family $\{e_i\}_i \subset B$ of pairwise-orthogonal projections and $\nu \in E$, then also $e \cdot \nu = 0$, where $e = \sup_i e_i$;*
- ***ii.*** *if $\{e_i\}_i \subset B$ is a family of pairwise-orthogonal projections such that $1 = \sup_i e_i$, and if $\{\nu_i\}_i \subset E$ is a bounded family, then there is a $\nu \in E$ such that $e_i \cdot \nu = e_i \cdot \nu_i$ for all i;*
- ***iii.*** *if $\nu \in E$, then $g \cdot \nu = 0$ for all $g \in B$ only if $\nu = 0$.*

Remark 2.5. *The element $\nu \in E$ obtained in the situation described in (ii) will sometimes be denoted as $\sum_i e_i \nu_i$. It should be emphasized that this is not a pointwise sum.*

Theorem 2.6. *Ω_{wk} is a Kaplansky–Hilbert module over $C(\Delta)$.*

Proof. For property (i), assume that $\nu \in \Omega_{\text{wk}}$ and $\{e_i\}_i \subset C(\Delta)$ is a family of pairwise-orthogonal projections with supremum $e \in C(\Delta)$ for which $e_i \cdot \nu = 0$ for all i. Because projections in $C(\Delta)$ are the characteristic functions of clopen sets, there are pairwise-disjoint clopen sets $U_i \subset \Delta$ such that $e_i = \chi_{U_i}$. Thus, for each i, using Lemma 2.2,

$$\begin{aligned} 0 = \|e_i \cdot \nu\|^2 &= \max_{s\in\Delta} \langle e_i \cdot \nu, e_i \cdot \nu \rangle(s) = \sup_{s\in\Delta} \langle e_i(s)\nu(s), e_i(s)\nu(s) \rangle \\ &= \max_{s\in\Delta} e_i(s) \; [\langle \nu, \nu \rangle(s)] = \max_{s\in U_i} \langle \nu, \nu \rangle(s) \,, \end{aligned}$$

and so $\langle\nu,\nu\rangle(s) = 0$ for every $s \in U_i$. Let $U = \bigcup_i U_i$. The set $\overline{U}$ is clopen and $\chi_{\overline{U}} = \sup_i e_i = e$ [5, §8]. As $\langle\nu,\nu\rangle$ is a continuous function that vanishes on U, it also vanishes on $\overline{U}$. Hence,

$$\|e \cdot \nu\|^2 = \max_{s\in\Delta} e(s)\,[\langle\nu,\nu\rangle(s)] = \max_{s\in\overline{U}} \langle\nu,\nu\rangle(s) = 0\,,$$

which yields property (i).

For the proof of property (ii), assume that $\{e_i\}_i \subset C(\Delta)$ is a family of pairwise-orthogonal projections such that $1 = \sup_i e_i$ and that $\{\nu_i\}_i \subset \Omega_{\rm wk}$ is a family such that $K = \sup \|\nu_i\| < \infty$; we aim to prove that there is a $\nu \in \Omega_{\rm wk}$ such that $e_i \cdot \nu = e_i \cdot \nu_i$ for all i. As before, assume that $e_i = \chi_{U_i}$ and $U = \bigcup_i U_i$. Then $1 = \sup_i e_i$ implies that $\overline{U} = \Delta$.

For each $\omega \in \Omega$, consider the unique function $f_\omega \in C(\Delta)$ such that $e_i\, f_\omega = e_i\,\langle\omega,\nu_i\rangle$ for all i (its existence guaranteed by the fact that Δ is the Stone–Čech compactification of U). Note that for $s \in U_i$ we have that $f_\omega(s) = \langle\omega(s),\nu_i(s)\rangle$. Hence, $|f_\omega(s)| \le K\,\|\omega(s)\|$ for $s \in U$; the same inequality holds for all $s \in \Delta$ because $\overline{U} = \Delta$ and both sides of the inequality are continuous functions of s. Moreover, if $\omega_1,\,\omega_2 \in \Omega$ and $\alpha \in \mathbb{C}$ then, for $s \in U$ we get that $f_{\alpha\,\omega_1+\omega_2}(s) = \alpha\, f_{\omega_1}(s) + f_{\omega_2}(s)$ and, therefore, that $f_{\alpha\,\omega_1+\omega_2} = \alpha\, f_{\omega_1} + f_{\omega_2}$. Thus, for each $s \in \Delta$ the function $\omega(s) \mapsto f_\omega(s)$ is a well-defined, bounded linear functional on H_s. Let $\nu(s) \in H_s$ be the representing vector for this functional, yielding a vector field $\nu : \Delta \to \bigsqcup_{s\in\Delta} H_s$. Since $\langle\nu(s),\omega(s)\rangle = \overline{f_\omega(s)}$, for every $\omega \in \Omega$, ν is weakly continuous. It remains to show that ν is a bounded vector field. If $s \in U$,

$$\|\nu(s)\| = \sup_{\omega\in\Omega,\|\omega(s)\|=1} |\langle\omega(s),\nu(s)\rangle| = \sup_{\omega\in\Omega,\|\omega(s)\|=1} |f_\omega(s)| \le \sup_i \|\nu_i\| = K\,,$$

which shows that $\|\nu(s)\|$ is uniformly bounded on U. Thus, since U is dense, the lower semicontinuous function $s \mapsto \|\nu(s)\|^2$ is bounded on Δ. Therefore, $\nu \in \Omega_{\rm wk}$.

Now we show that $e_i \cdot \nu = e_i \cdot \nu_i$, for all i. Fix i and $s \in U_i$ and consider $\omega \in \Omega$. Then,

$$\begin{aligned}\langle\omega(s),\, e_i(s)\,\nu(s)\rangle &= \langle\omega(s),\,\nu(s)\rangle = f_\omega(s)\\ &= e_i(s)\, f_\omega(s) = e_i(s)\langle\omega(s),\nu_i(s)\rangle\\ &= \langle\omega(s), e_i(s)\,\nu_i(s)\rangle\,.\end{aligned}$$

Since $(e_i\cdot\nu)(s) = 0 = (e_i\cdot\nu_i)(\mathrm{s})$ for $s \in \Delta\setminus U_i$ we conclude that $e_i\cdot\nu = e_i\cdot\nu_i$.

For the proof of property (iii), assume that $\nu \in \Omega_{\rm wk}$ satisfies $g\cdot\nu = 0$ for all $g \in C(\Delta)$. Then, in particular, $\langle\nu,\nu\rangle\cdot\nu = 0$, so $\langle\nu,\nu\rangle = 0$. Hence, from $\|\nu\| = \|\langle\nu,\nu\rangle\|^{1/2} = 0$ we conclude that $\nu = 0$. □

3 Endomorphisms of Ω and $\Omega_{\rm wk}$

Throughout this section A will denote the Fell C*-algebra of the continuous Hilbert bundle $(\Delta, \{H_s\}_{s\in\Delta}, \Omega)$, as described in Definition 1.4, with Δ Stonean. Let $B(\Omega)$ and $B(\Omega_{\rm wk})$ denote, respectively, the C*-algebras of adjointable $C(\Delta)$-endomorphisms of Ω and $\Omega_{\rm wk}$. Since, by Theorem 2.6, $\Omega_{\rm wk}$ is a Kaplansky–Hilbert AW*-module over $C(\Delta)$, $B(\Omega_{\rm wk})$ coincides with the set of all $C(\Delta)$-endomorphisms of $\Omega_{\rm wk}$ [16, Theorem 6] and is a type I AW*-algebra with centre $C(\Delta)$ [16, Theorem 7].

In the particular case where Ω is given by the trivial Hilbert bundle $(\Delta, \{H\}_{s\in\Delta}, C(\Delta, H))$ with H is a fixed Hilbert space, Hamana [15] proved that $B(\Omega_{\rm wk}) \cong C(\Delta)\overline{\otimes}B(H)$, the monotone complete tensor product of $C(\Delta)$ and $B(H)$.

For each $\nu_1, \nu_2 \in \Omega_{\rm wk}$, consider the endomorphism Θ_{ν_1,ν_2} on $\Omega_{\rm wk}$ defined by

$$\Theta_{\nu_1,\nu_2}(\nu) = \langle \nu, \nu_2\rangle \cdot \nu_1 , \quad \nu \in \Omega_{\rm wk} .$$

For a Hilbert bundle Ω_0, let

$$F(\Omega_0) = \left\{ \sum_{j=1}^{n} \Theta_{\omega_j,\omega'_j} \,:\, n \in \mathbb{N},\, \omega_j, \omega'_j \in \Omega \right\}.$$

We will consider both $F(\Omega)$ and $F(\Omega_{\rm wk})$.

If $\omega_1, \omega_2 \in \Omega$, then $\Theta_{\omega_1,\omega_2}(\omega) \in \Omega$ for all $\omega \in \Omega$, and so $F(\Omega) \subset B(\Omega)$. In fact, $F(\Omega)$ and $F(\Omega_{\rm wk})$ are algebraic ideals in $B(\Omega)$ and $B(\Omega_{\rm wk})$ respectively. The norm-closures of these algebraic ideals, namely $K(\Omega)$ and $K(\Omega_{\rm wk})$, are essential ideals in each of $B(\Omega)$ and $B(\Omega_{\rm wk})$—called the ideals of compact endomorphisms—and the multiplier algebras of $K(\Omega)$ and $K(\Omega_{\rm wk})$ are, respectively, $B(\Omega)$ and $B(\Omega_{\rm wk})$ [17].

When referring to rank-1 operators x acting on a Hilbert space H, we will use the notation $x = \xi \otimes \eta$ for such an operator—the action on $\gamma \in H$ given by $\gamma \mapsto \langle \gamma, \eta\rangle\, \xi$—and we reserve the notation $\Theta_{\xi,\eta}$ for "rank-1" operators acting on a Hilbert module.

The term "homomorphism" will be used to mean a $*$-homomorphism between C*-algebras.

For any C*-algebra B, we denote the injective envelope [13], [18, Chapter 15] of B by $I(B)$ (and we consider $I(B)$ as a C*-algebra rather than as an operator system).

The main result of the present section is the following.

Theorem 3.1. *There exist C*-algebra embeddings such that*

$$K(\Omega) \subset A \subset B(\Omega) \subset B(\Omega_{\rm wk}) = I(K(\Omega)). \qquad (2)$$

In particular, $I(K(\Omega)) = I(A) = I(B(\Omega)) = B(\Omega_{\rm wk})$.

The proof of Theorem 3.1 and a description of the inclusions in (2) begin with the following set of results.

Lemma 3.2. *For every $a \in A$ and $\omega \in \Omega$, the vector field $a \cdot \omega$ defined by $a \cdot \omega(s) = a(s)\omega(s)$ is an element of Ω.*

Proof. Let $a \in A$. Then $a^*a \in A_+$ and since all fields in A are weakly continuous, for every $\omega \in \Omega$ the map $s \mapsto \|a(s)\omega(s)\| = \langle a^*a \cdot \omega(s), \omega(s)\rangle^{1/2}$ is continuous.

Suppose $s_0 \in \Delta$ and $\varepsilon > 0$. Because $H_{s_0} = \{\mu(s_0) : \mu \in \Omega\}$, there is a $\mu \in \Omega$ such that $a(s_0)\omega(s_0) = \mu(s_0)$. Since

$$\|a \cdot \omega(s) - \mu(s)\|^2 = \|a(s)\omega(s)\|^2 + \|\mu(s)\|^2 - 2\mathrm{Re}\,\langle a(s)\omega(s), \mu(s)\rangle$$

is continuous on Δ and vanishes at s_0, there is an open set $U \subset \Delta$ containing s_0 such that $\|a \cdot \omega(s) - \mu(s)\| < \varepsilon$ for all $s \in U$. As Ω is closed under local uniform approximation, this proves that $a \cdot \omega \in \Omega$. □

Proposition 3.3. *The map $\varrho : A \to B(\Omega)$ given by $\varrho(a)\omega = a \cdot \omega$, for $a \in A$ and $\omega \in \Omega$ is an isometric homomorphism. Furthermore, $K(\Omega) \subset \varrho(A) \subset B(\Omega)$ as C*-algebras.*

Proof. It is clear that ϱ is a homomorphism, and so we only need to verify that it is one-to-one. To this end, assume that $\varrho(a) = 0$. Thus, $a(s)\omega(s) = 0$ for every $\omega \in \Omega$ and every $s \in \Delta$. Because $H_s = \{\omega(s) : \omega \in \Omega\}$, this implies that $a(s) = 0$ for all $s \in \Delta$, and so $a = 0$.

To show $K(\Omega) \subset \varrho(A) \subset B(\Omega)$ as C*-algebras, consider $\Theta_{\omega_1,\omega_2}$ with $\omega_1, \omega_2 \in \Omega$. The map $s \mapsto \|\Theta_{\omega_1(s),\omega_2(s)}\|$ is continuous because $\|\Theta_{\omega_1(s),\omega_2(s)}\| = \|\omega_1(s)\|\,\|\omega_2(s)\|$. For any $\eta_1, \eta_2 \in \Omega$, the map

$$\langle \Theta_{\omega_1,\omega_2} \cdot \eta_1, \eta_2\rangle(s) = \langle \eta_1, \omega_2\rangle(s)\,\langle \omega_1, \eta_2\rangle(s) = \langle \eta_1(s), \omega_2(s)\rangle\,\langle \omega_1(s), \eta_2(s)\rangle$$

is continuous. So $\Theta_{\omega_1,\omega_2}$ is also finite dimensional and weakly continuous, which shows that $\Theta_{\omega_1,\omega_2} \in A$ and $K(\Omega) \subset \varrho(A)$. □

Lemma 3.4. *With respect to the inclusion $\Omega \subset \Omega_{\rm wk}$, we have $\Omega^\perp = \{0\}$.*

Proof. Let $\nu \in \Omega_{\mathrm{wk}}$ be such that $\langle \nu, \omega\rangle = 0$, for every $\omega \in \Omega$. That is, for every $\omega \in \Omega$ and for every $s \in \Delta$, $\langle \nu(s), \omega(s)\rangle = 0$. If $\nu \neq 0$, there exists $s_0 \in \Delta$ such that $\nu(s_0) \neq 0$. By axiom (II) in Definition 1.1, there exists $\omega \in \Omega$ such that $\omega(s_0) = \nu(s_0)$, in contradiction to $\langle \nu(s_0), \omega(s_0)\rangle = 0$. $\square$

Lemma 3.5. *If $t_0 \in \Delta$ and $\xi \in H_{t_0}$, then there exists $\omega \in \Omega$ such that $\omega(t_0) = \xi$ and $\|\omega\| = \|\xi\|$.*

Proof. The case $\xi = 0$ is trivial. So assume that $\|\xi\| > 0$. Let $\omega' \in \Omega$ with $\omega'(t_0) = \xi$. Fix a clopen neighbourhood V of t_0 such that $V \subset \{t \in T : \|\omega'(t)\| \geq \|\omega'(t_0)\|/2\}$. Let $h'(\cdot) = \|\xi\| \cdot \|\omega'(\cdot)\|^{-1} \in C(V)$; then h' extends to a continuous function $h \in C(\Delta)$ with $h|_{\Delta\setminus V} = 0$. It is now straightforward to show that $\omega = h \cdot \omega' \in \Omega$ has the desired properties. $\square$

Proposition 3.6. *There exists an isometric homomorphism $\vartheta : B(\Omega) \to B(\Omega_{\mathrm{wk}})$ such that for $a \in A$, $\nu \in \Omega_{\mathrm{wk}}$,*

$$(\vartheta(\varrho(a))\nu)(s) = a(s)\nu(s), \quad s \in \Delta\,. \tag{3}$$

Proof. Assume that $b \in B(\Omega)$ and $\omega \in \Omega$, $s \in \Delta$. By Lemma 3.5,

$$\begin{aligned}
\|(b\,\omega)(s)\| &= \sup_{\xi\in H_s,\,\|\xi\|=1} |\langle (b\,\omega)(s), \xi\rangle| = \sup_{\eta\in\Omega,\,\|\eta\|=1} |\langle (b\,\omega)(s), \eta(s)\rangle| \\
&= \sup_{\eta\in\Omega,\,\|\eta\|=1} |\langle b\,\omega, \eta\rangle(s)| = \sup_{\eta\in\Omega,\,\|\eta\|=1} |\langle \omega(s), (b^*\eta)(s)\rangle| \\
&\leq \|\omega(s)\| \sup_{\eta\in\Omega,\,\|\eta\|=1} \|b^*\eta\| \leq \|\omega(s)\|\,\|b^*\| = \|\omega(s)\|\,\|b\|\,.
\end{aligned}$$

Therefore the function $\omega(s) \mapsto (b\,\omega)(s)$ is well defined and induces a bounded linear operator $b(s) \in B(H_s)$ such that $(b\,\omega)(s) = b(s)\,\omega(s)$, for $s \in \Delta$ and $\omega \in \Omega$, with $\sup_{s\in\Delta}\|b(s)\| \leq \|b\|$. Moreover,

$$\begin{aligned}
\|b\| &= \sup_{\|\omega\|=1} \|b\cdot\omega\| = \sup_{\|\omega\|=1}\sup_{s} \|b\cdot\omega(s)\| = \sup_{\|\omega\|=1}\sup_{s} \|b(s)\omega(s)\| \\
&\leq \sup_{\|\omega\|=1}\sup_{s} \|b(s)\|\,\|\omega(s)\| \leq \sup_{s}\|b(s)\| \leq \|b\|\,,
\end{aligned}$$

and so $\sup_{s\in\Delta}\|b(s)\| = \|b\|$. Suppose now that $\nu \in \Omega_{\mathrm{wk}}$ and $s \in \Delta$, and define a vector field $\vartheta b\nu$ by $(\vartheta b\,\nu)(s) = b(s)\,\nu(s)$. If $\eta \in \Omega$, then

$$\langle (\vartheta b\,\nu)(s), \eta(s)\rangle \;=\; \langle \nu(s), b(s)^*\eta(s)\rangle \;=\; \langle \nu(s), (b^*\eta)(s)\rangle$$

is continuous, which shows that $\vartheta b\,\nu$ is weakly continuous with respect to Ω. Since $\vartheta b\,\nu$ is also uniformly bounded, we conclude that $\vartheta b\,\nu \in \Omega_{\mathrm{wk}}$.

It is straightforward to show that the map $\nu \mapsto \vartheta b\,\nu$ is a bounded $C(\Delta)$-endomorphism of $\Omega_{\rm wk}$ and hence it gives rise to an element $\vartheta b \in B(\Omega_{\rm wk})$. It is clear that ϑ is a homomorphism. If $\vartheta b = 0$, then $b(s)\omega(s) = 0$ for all $\omega \in \Omega$, $s \in \Delta$ and so $b(s) = 0$ for all s; then $\|b\| = \sup_s \|b(s)\| = 0$, and $b = 0$. So ϑ is one-to-one, and thus isometric. Finally, it is clear that (3) holds by construction. □

One consequence of the proof of Proposition 3.6 is that for every $b \in B(\Omega)$ there exists an operator field $\{b(s)\}_{s\in\Delta}$ acting on the Hilbert bundle $\{H_s\}_{s\in\Delta}$ such that $(b\,\omega)(s) = b(s)\,\omega(s)$, for every $s \in \Delta$. This property, however, is not shared by all elements of $B(\Omega_{\rm wk})$.

Lemma 3.7. *If $z \in B(\Omega_{\rm wk})$ and $\Theta_{\omega,\omega} z \Theta_{\mu,\mu} = 0$ for all $\omega, \mu \in \Omega$, then $z = 0$.*

Proof. For any ξ, ω, $\mu \in \Omega$ we have that

$$0 = \Theta_{\omega,\omega}\, z\, \Theta_{\mu,\mu}\, \xi = \langle \xi, \mu\rangle\, \langle z\mu, \omega\rangle\, \omega.$$

Hence, we get that

$$0 = \langle \xi, \mu\rangle\, |\langle z\mu, \omega\rangle|^2 = \langle \xi, \mu\rangle\, |\langle \mu, z^*\omega\rangle|^2.$$

We are free to choose $\xi, \mu \in \Omega$. Fix s, and choose μ with $\mu(s) = z^*\omega(s)$; let $\xi = \mu$. Then, as $\mu \in \Omega$, we get $0 = \langle \mu, \mu\rangle(s) = \langle \mu(s), \mu(s)\rangle$, so $z^*\omega(s) = \mu(s) = 0$. As $s \in \Delta$ is arbitrary, $z^*\omega = 0$ for every $\omega \in \Omega$. For any $\nu \in \Omega_{\rm wk}$ and every $\omega \in \Omega$, $\langle z\nu, \omega\rangle = \langle \nu, z^*\omega\rangle = 0$. By Lemma 3.4 we conclude that $z\nu = 0$ for $\nu \in \Omega_{\rm wk}$ and hence $z = 0$. □

Proof of Theorem 3.1. We consider the embeddings $A \xrightarrow{\varrho} B(\Omega)$ and $B(\Omega) \xrightarrow{\vartheta} B(\Omega_{\rm wk})$ defined in Propositions 3.3 and 3.6. In this way, we get the inclusions in (2).

Because $B(\Omega_{\rm wk})$ is a type I AW*-algebra, it is injective [14, Proposition 5.2]. To show that $B(\Omega_{\rm wk})$ is the injective envelope $I(K(\Omega))$ of $K(\Omega)$, we need to show that the embedding $\vartheta \circ \varrho$ of $K(\Omega)$ into $B(\Omega_{\rm wk})$ is rigid [18, Theorem 15.8]: that is, we aim to prove that if $\phi : B(\Omega_{\rm wk}) \to B(\Omega_{\rm wk})$ is a unital completely positive linear map for which $\phi|_{K(\Omega)} = {\rm id}_{K(\Omega)}$, then $\phi = {\rm id}_{B(\Omega_{\rm wk})}$.

Let $\phi : B(\Omega_{\rm wk}) \to B(\Omega_{\rm wk})$ be such a ucp map with $\phi|_{K(\Omega)} = {\rm id}_{K(\Omega)}$. Suppose that $z \in B(\Omega_{\rm wk})$ and $\omega, \mu \in \Omega$. Then $\Theta_{\omega,\omega} z \Theta_{\mu,\mu} = \Theta_{\langle z\mu,\omega\rangle\omega,\mu} \in K(\Omega)$. Because $K(\Omega)$ is in the multiplicative domain of ϕ, we have that $\phi(axb) = a\phi(x)b$ for all $x \in B(\Omega_{\rm wk})$ and $a, b \in K(\Omega)$. This implies that

$$\Theta_{\omega,\omega}\phi(z)\Theta_{\mu,\mu} = \phi(\Theta_{\omega,\omega} z \Theta_{\mu,\mu}) = \phi(\Theta_{\langle z\mu,\omega\rangle\omega,\mu}) = \Theta_{\langle z\mu,\omega\rangle\omega,\mu} = \Theta_{\omega,\omega} z \Theta_{\mu,\mu},$$

and so $\Theta_{\omega,\omega}(z-\phi(z))\Theta_{\mu,\mu}=0$. Since ω,μ were arbitrary, Lemma 3.7 implies that $z-\phi(z)=0$ and so $\phi=\mathrm{id}_{B(\Omega_{\mathrm{wk}})}$.

We have shown above that the inclusion $K(\Omega)\subset B(\Omega_{\mathrm{wk}})$ is rigid. Moreover, $K(\Omega)$ is an essential ideal of $B(\Omega)$ and $K(\Omega)\subset A\subset B(\Omega)$. Hence, $I(K(\Omega))=I(A)=I(B(\Omega))=B(\Omega_{\mathrm{wk}})$. □

We conclude this section with a remark about the ideal $K(\Omega_{\mathrm{wk}})$ of $B(\Omega_{\mathrm{wk}})$. In type I AW*-algebras, the ideal generated by the abelian projections has a prominent role. As it happens, $K(\Omega_{\mathrm{wk}})$ is precisely this ideal.

Proposition 3.8. *The C*-algebra $K(\Omega_{\mathrm{wk}})$ coincides with the ideal $J\subset B(\Omega_{\mathrm{wk}})$ generated by the abelian projections of $B(\Omega_{\mathrm{wk}})$. So $K(\Omega_{\mathrm{wk}})$ is a liminal C*-algebra with Hausdorff spectrum.*

Proof. By [16, Lemma 13], a projection $e\in B(\Omega_{\mathrm{wk}})$ is abelian if and only if there exists $\nu\in\Omega_{\mathrm{wk}}$ such that $|\nu|$ is a projection in $C(\Delta)$ and $e=\Theta_{\nu,\nu}$. Hence, $J\subset K(\Omega_{\mathrm{wk}})$.

To show that $K(\Omega_{\mathrm{wk}})\subset J$, assume $\nu\in\Omega_{\mathrm{wk}}$ is nonzero. Let $\varepsilon>0$. We will show that there is an $x_\varepsilon\in J$ such that $\|\Theta_{\nu,\nu}-x_\varepsilon\|<\varepsilon$. Let $V\subset\Delta$ be the (clopen) closure of $\{s\in\Delta:\ |\nu|(s)<\varepsilon^{1/2}\}$, $U=\Delta\setminus V$ (also clopen) and let $g=(1/|\nu|)\,\chi_U\in C(\Delta)_+$. Then $g|\nu|=\chi_U$ and $\|\,\chi_{\Delta\setminus U}|\nu|\,\|<\varepsilon^{1/2}$. Let $\nu'=g\cdot\nu$ so that $|\nu'|=\chi_U$. Hence, $\Theta_{\nu',\nu'}\in J$ and $\Theta_{\nu',\nu'}=g^2\cdot\Theta_{\nu,\nu}$. Let $x_\varepsilon=|\nu|^2\cdot\Theta_{\nu',\nu'}\in J$. Then

$$x_\varepsilon=|\nu|^2\cdot\Theta_{\nu',\nu'}=|\nu|^2\,g^2\,\Theta_{\nu,\nu}=\chi_U\,\Theta_{\nu,\nu},$$

and $x_\varepsilon-\Theta_{\nu,\nu}=\chi_{\Delta\setminus U}\cdot\Theta_{\nu,\nu}$. Then

$$\begin{aligned}\|x_\varepsilon-\Theta_{\nu,\nu}\| &= \sup_{\eta\in(\Omega_{\mathrm{wk}})_1}\|\chi_{\Delta\setminus U}\cdot\Theta_{\nu,\nu}\,\eta\| = \sup_{\eta\in(\Omega_{\mathrm{wk}})_1}\|\chi_{\Delta\setminus U}\cdot\langle\eta,\nu\rangle\,\nu\|\\ &= \sup_{\eta\in(\Omega_{\mathrm{wk}})_1}\max_{s\in\Delta\setminus U}|\langle\eta,\nu\rangle(s)|\,\|\nu(s)\|\\ &\le \sup_{\eta\in(\Omega_{\mathrm{wk}})_1}\max_{s\in\Delta\setminus U}|\eta|(s)\,|\nu|(s)|\,\|\nu(s)\|\le\max_{s\in\Delta\setminus U}|\nu|(s)^2<\varepsilon.\end{aligned}$$

As ε was arbitrary and J is closed, we conclude that $\Theta_{\nu,\nu}\in J$. The polarisation identity then shows that $\Theta_{\nu_1,\nu_2}\in J$ for all $\nu_1,\nu_2\in\Omega_{\mathrm{wk}}$. Hence, $F(\Omega_{\mathrm{wk}})\subset J$, and so $K(\Omega_{\mathrm{wk}})\subset J$.

It remains to justify the last assertion in the statement. By the main result of [12], the ideal generated by the abelian projections in a type I AW*-algebra is liminal and has Hausdorff spectrum. Hence, this is true of $K(\Omega_{\mathrm{wk}})$. □

4 Multiplier and Local Multiplier Algebras

In the previous section we established the inclusions $K(\Omega) \subset A \subset B(\Omega) \subset B(\Omega_{\rm wk})$, as C*-subalgebras, and we showed that $I(A) = B(\Omega_{\rm wk})$. The present section refines these inclusions to incorporate multiplier algebras and local multiplier algebras.

Given a C*-algebra C, we denote by $M(C)$ and $M_{\rm loc}(C)$ its multiplier and local multiplier algebra [2] respectively.

The second order local multiplier algebra of C is $M_{\rm loc}\left(M_{\rm loc}(C)\right)$, the local multiplier algebra of $M_{\rm loc}(C)$. By [11, Corollary 4.3], the local multiplier algebras (of all orders) of C are C*-subalgebras of the injective envelope $I(C)$ of C. In particular, $C \subset M_{\rm loc}(C) \subset M_{\rm loc}\left(M_{\rm loc}(C)\right) \subset I(C)$ as C*-subalgebras.

By a well known theorem of Kasparov [2, Theorem 1.2.33], [17, Theorem 2.4], $M(K(\Omega)) = B(\Omega)$. We remark that all the subalgebras we consider are essential in $B(\Omega_{\rm wk})$ (i.e. the annihilator is zero), and so whenever we write $M(C)$ for one of these subalgebras $C \subset B(\Omega_{\rm wk})$, we mean the concrete realization [20]

$$M(C) = \{x \in B(\Omega_{\rm wk}) : \; xC + Cx \subset C\}.$$

The following theorem is the main result of this section.

Theorem 4.1. *With the notations from the previous sections, we have the equality $M_{\rm loc}(A) = M_{\rm loc}(K(\Omega))$ and the following inclusions (as C*-subalgebras):*

$$\begin{aligned} M(A) &\subset M(K(\Omega)) = B(\Omega) \\ &\subset M_{\rm loc}(K(\Omega)) \subset M_{\rm loc}\left(M_{\rm loc}(K(\Omega))\right) = B(\Omega_{\rm wk}) . \end{aligned} \tag{4}$$

In particular, $M_{\rm loc}\left(M_{\rm loc}(A)\right) = I(A)$.

Ara and Mathieu have presented examples of Stonean spaces Δ and trivial Hilbert bundles Ω where the inclusion $M_{\rm loc}(K(\Omega)) \subset M_{\rm loc}\left(M_{\rm loc}(K(\Omega))\right)$ in (4) is proper [3, Theorem 6.13]. As a consequence of Theorem 4.1 and the fact that $B(\Omega_{\rm wk}) = I(K(\Omega))$, we see that this gap cannot occur for higher local multiplier algebras, i.e. for all $k \geq 2$, $M_{\rm loc}^{k+1}(K(\Omega)) = M_{\rm loc}^{k}(K(\Omega))$ — where $M_{\rm loc}^{k+1}(K(\Omega)) = M_{\rm loc}(M_{\rm loc}^{k}(K(\Omega)))$ for $k \geq 1$.

The proof of Theorem 4.1 is achieved through a number of lemmas.

Lemma 4.2. *The set*

$$F_+ = \{\sum_{j=1}^{n} \Theta_{\omega_j,\omega_j} \ : n \in \mathbb{N},\, \omega_j \in \Omega\}$$

is dense in the positive cone of $K(\Omega)$.

Proof. Assume that $h \in K(\Omega)_+$ and let $\varepsilon > 0$ be arbitrary. For each $s_0 \in \Delta$ consider the positive compact operator $h(s_0) \in K(H_{s_0})$. Then there are vectors $\xi_1, \ldots, \xi_{n_{s_0}} \in H_{s_0}$ such that

$$\|h(s_0) - \sum_{j=1}^{n_{s_0}} \xi_j \otimes \xi_j\| < \varepsilon\,.$$

Using (II) in Definition 1.1, choose $\omega_1, \ldots, \omega_{n_{s_0}} \in \Omega$ such that $\omega_j(s_0) = \xi_j$, $1 \le j \le n_{s_0}$, and let $\kappa_{s_0} = \sum_{j=1}^{n_{s_0}} \Theta_{\omega_j,\omega_j}$. By continuity of the operator fields in A, there is an open set $U_{s_0} \subset \Delta$ containing s_0 such that $\|h(s) - \kappa_{s_0}(s)\| < \varepsilon$ for all $s \in U_{s_0}$.

This procedure leads to an open cover $\{U_s\}_{s\in\Delta}$ of Δ, from which (by compactness) there exists a finite subcover $\{U_1, \ldots, U_m\}$ and corresponding fields $\kappa_i = \sum_{j=1}^{n_i} \Theta_{\omega_j^{[i]},\omega_j^{[i]}}$. Let $\{\psi_1, \ldots, \psi_m\} \subset C(\Delta)$ be a partition of unity subordinate to $\{U_1, \ldots, U_m\}$ and note that $\psi_i \cdot \Theta_{\omega_j^{[i]},\omega_j^{[i]}} = \Theta_{\psi_i^{1/2}\cdot\omega_j^{[i]},\psi_i^{1/2}\cdot\omega_j^{[i]}}$ for all j and i. Hence, the field $\kappa = \sum_{i=1}^{m} \psi_i \cdot \kappa_i$ is in F_+, and for each $s \in \Delta$,

$$\|h(s) - \kappa(s)\| = \|\sum_{i=1}^{m} \psi_i \cdot (h - \kappa_i)(s)\| \le \sum_{i=1}^{m} \psi_i(s)\|(h - \kappa_i)(s)\| < \varepsilon\,.$$

Hence, h is in the norm-closure of F_+. □

Lemma 4.3. *Let* $\{U_i\}_{i\in\Lambda}$ *be a family of pairwise disjoint clopen subsets of* Δ *whose union* U *is dense in* Δ, *and let* $c_i = \chi_{U_i} \in C(\Delta)$, *for each* $i \in \Lambda$. *Suppose that* $\{\omega_i\}_{i\in\Lambda}$ *is any bounded family in* Ω *and let* $\tilde{\omega} = \sum_{i\in\Lambda} c_i\,\omega_i \in \Omega_{\mathrm{wk}}$, *in the sense of Remark 2.5. If* $f \in C(\Delta)$ *is such that* $f(s) = 0$ *for* $s \in \Delta \setminus U$, *then* $f \cdot \tilde{\omega} \in \Omega$.

Proof. Fix $s_0 \in \Delta$ and let $\varepsilon > 0$. If $s_0 \in \Delta \setminus U$, then by the continuity of f and the fact that $f(s_0) = 0$ there exists an open subset $U_{s_0} \subset \Delta$ containing s_0 such that $|f(s)| < \varepsilon\|\tilde{\omega}\|^{-1}$ for all $s \in U_{s_0}$. Hence, the vector field $f \cdot \tilde{\omega}$ is within ε of the zero vector field $0 \in \Omega$ on the open set U_{s_0}.

On the other hand, if $s_0 \in U$, then there exists $j \in \Lambda$ such that $s_0 \in U_j$. By construction, $c_j \cdot \tilde{\omega} = c_j \cdot \omega_j$ and so $\tilde{\omega}(s) = \omega_j(s)$ for all $s \in U_j$. Because

$\|(f \cdot \tilde{\omega})(s) - (f \cdot \omega_j)(s)\| = 0$ for all $s \in U_j$, the vector field $f \cdot \tilde{\omega}$ is within ε of the vector field $f \cdot \omega_j \in \Omega$ on the open set U_j. Thus, by the local uniform approximation property (axiom (IV) in Definition 1.1), $f \cdot \tilde{\omega} \in \Omega$. □

The fact that $\Omega^{\perp} = \{0\}$ in Ω_{wk} (Lemma 3.4) suggests that Ω is somehow dense in Ω_{wk}. The next proposition makes this relation more explicit.

Proposition 4.4. *If $\nu \in \Omega_{\mathrm{wk}}$ and $\varepsilon > 0$, then there exist a family $\{c_i\}_{i\in\Lambda}$ of pairwise orthogonal projections in $C(\Delta)$ with supremum 1 and a bounded family $\{\omega_i\}_{i\in\Lambda} \subset \Omega$ such that $\|\nu - \sum_{i\in\Lambda} c_i \cdot \omega_i\| < \varepsilon$.*

Proof. By Lemma 2.2, the function $s \mapsto \|\nu(s)\|$ is lower semicontinuous; hence, there exists a meagre set M_ν such that the function $s \mapsto \|\nu(s)\|$ is continuous in the relative topology of $\Delta \setminus M_\nu$. Observe that $\overline{(\Delta \setminus M_\nu)} = \Delta$.

Fix $s_0 \in \Delta \setminus M_\nu$ and let $\omega \in \Omega$ be such that $\omega(s_0) = \nu(s_0)$. Since

$$\|\nu(s) - \omega(s)\|^2 \ = \ \|\nu(s)\|^2 + \|\omega(s)\|^2 - 2\mathrm{Re}\,\langle \nu, \omega\rangle(s)\,,$$

the continuity in the relative topology of $\Delta \setminus M_\nu$ guarantees the existence of an open subset U_{s_0} of Δ containing s_0 such that $\|\nu(s) - \omega(s)\| < \varepsilon/2$ for all $s \in (\Delta \setminus M_\nu) \cap U_{s_0}$. Hence, again by continuity we get that $\|\nu - \omega\|(s) < \varepsilon$ for all $s \in \overline{U}_{s_0}$. The set $\overline{U}_{s_0}$ is a clopen subset of Δ and $\Delta' = \Delta \setminus \overline{U}_{s_0}$ is also a Stonean space. Further, $M_\nu \cap \Delta' = M_\nu \cap (\Delta \setminus \overline{U}_{s_0})$ is a meagre set such that the function $s \mapsto \|\nu(s)\|$, for $s \in \Delta' \setminus (M_\nu \cap \Delta')$, is continuous in the relative topology.

An application of Zorn's Lemma yields a maximal family $\{(\chi_{U_i}, \omega_i)\}_{i\in\Lambda}$ such that $U_i \cap U_j = \emptyset$ for $i \neq j$ and such that $\|\chi_{U_i}(\nu - \omega_i)\| < \varepsilon$. Maximality ensures that $\overline{(\cup_{i\in I} U_i)} = \Delta$, for otherwise we can enlarge this family by the previous procedure in the Stonean space $\Delta \setminus \overline{(\cup_{i\in\Lambda} U_i)}$. If we let $c_i = \chi_{U_i}$ for $i \in \Lambda$ then it is clear by Lemma 2.2 that $\|\nu - \sum_{i\in\Lambda} c_i \cdot \omega_i\| < \varepsilon$ as for every $j \in \Lambda$ we have that $\|c_j(\nu - \sum_{i\in\Lambda} c_i \cdot \omega_i)\| = \|c_j(\nu - \omega_j)\| < \varepsilon$ and $\bigvee_{i\in\Lambda} c_i = 1$. □

The next result is the key step in the proof of Theorem 4.1.

Proposition 4.5. *For every abelian projection $e \in B(\Omega_{\mathrm{wk}})$ and $\varepsilon > 0$ there is an essential ideal $I \subset K(\Omega)$ and $x \in M(I)$ such that $\|e - x\| < \varepsilon$.*

Proof. Assume that $e \in B(\Omega_{\mathrm{wk}})$ is an abelian projection and let $\varepsilon > 0$. Thus, by [16, Lemma 13], $e = \Theta_{\nu,\nu}$ for some $\nu \in \Omega_{\mathrm{wk}}$ for which $\langle \nu, \nu\rangle$ is a projection of $C(\Delta)$. By Proposition 4.4, there is a family $\{c_i\}_{i\in\Lambda}$ of pairwise orthogonal projections in $C(\Delta)$ with supremum 1 and a bounded family

$\{\omega_j\}_{j\in\Lambda} \subset \Omega$ such that $\|\nu - \tilde\omega\| < \varepsilon/(2\|\nu\|)$, where $\tilde\omega = \sum_{j\in\Lambda} c_j \cdot \omega_j \in \Omega_{\rm wk}$. Each c_j is the characteristic function of a clopen set U_j and the union U of these sets U_j is dense in Δ.

Let $I = \{a \in K(\Omega) : a(s) = 0, \forall\, s \in \Delta \setminus U\}$, which is an essential ideal of $K(\Omega)$. Define $F^I \subset F_+ \subset K(\Omega)_+$ to be the set

$$F^I = \{\sum_{i=1}^{n} \Theta_{\mu_i,\mu_i} : n \in \mathbb{N},\, \mu_i \in \Omega,\, \mu_i|_{\Delta\setminus U} = 0,\ i = 1,\ldots,n\}.$$

Suppose that $\eta \in \Omega$ satisfies $\|\eta(s)\| = 0$ for all $s \in \Delta \setminus U$, and consider $\Theta_{\eta,\eta} \in F^I$. Observe that $\Theta_{\tilde\omega,\tilde\omega}\,\Theta_{\eta,\eta} = \Theta_{\langle\eta,\tilde\omega\rangle\cdot\tilde\omega,\,\eta}$, which is an element of I because $\langle\eta,\tilde\omega\rangle(s) = \langle\eta(s),\tilde\omega(s)\rangle = 0$ for all $s \in \Delta\setminus U$ and $\langle\eta,\tilde\omega\rangle\cdot\tilde\omega \in \Omega$ by Lemma 4.3. Hence, $\Theta_{\tilde\omega,\tilde\omega}$ maps the set F^I back into I. Because F^I is dense in I_+, as we shall show below, $\Theta_{\tilde\omega,\tilde\omega} I \subset I$ and a similar computation shows that $I\Theta_{\tilde\omega,\tilde\omega} \subset I$. Furthermore, writing $x = \Theta_{\tilde\omega,\tilde\omega}$,

$$\|e - x\| = \|\Theta_{\nu,\nu} - \Theta_{\tilde\omega,\tilde\omega}\| \le (\|\nu\| + \|\tilde\omega\|)\,\|\nu - \tilde\omega\| < \varepsilon.$$

It remains to show that F^I is dense in I_+. To this end, assume $\varepsilon' > 0$ and $\kappa \in I_+$. Thus, $\kappa(s) = 0$ for all $s \in \Delta\setminus U$. Furthermore, by Lemma 4.2, there exists $h \in F_+$ such that $\|\kappa - h\| < \varepsilon'$. Let $\tilde h = \chi_{\Delta\setminus U}\cdot h$ and note that, as $\kappa \in I$, it is also true that $\|\kappa - \tilde h\| < \varepsilon'$. Now if h has the form $\sum_{j=1}^n \Theta_{\mu_j,\mu_j}$ for some $\mu_j \in \Omega$, then $\tilde h = \sum_{j=1}^n \Theta_{\chi_{\Delta\setminus U}\mu_j,\chi_{\Delta\setminus U}\mu_j} \in F^I$. □

Proof of Theorem 4.1. Because $K(\Omega)$ is an ideal of A, we have $M(A) \subset M(K(\Omega))$. Moreover, as $K(\Omega)$ is an essential ideal of A we conclude that $M_{\rm loc}(A) = M_{\rm loc}(K(\Omega))$ [2, Proposition 2.3.6]. On the other hand, the inclusions

$$B(\Omega) = M(K(\Omega)) \subset M_{\rm loc}(K(\Omega)) \subset M_{\rm loc}\left(M_{\rm loc}(K(\Omega))\right) \subset B(\Omega_{\rm wk})$$

hold by [11, Theorem 4.6].

Therefore, we are left to show that $M_{\rm loc}\left(M_{\rm loc}(K(\Omega))\right) = B(\Omega_{\rm wk})$. By [11, Corollary 4.3], an element $z \in I(K(\Omega)) = B(\Omega_{\rm wk})$ belongs to $M_{\rm loc}(K(\Omega))$ if and only if for every $\varepsilon > 0$ there is an essential ideal $I \subset K(\Omega)$ and a multiplier $x \in M(I)$ such that $\|z - x\| < \varepsilon$. By Proposition 3.8, $K(\Omega_{\rm wk})$ is the (essential) ideal of $B(\Omega_{\rm wk})$ generated by the abelian projections of $B(\Omega_{\rm wk})$; thus, by Proposition 4.5, $K(\Omega_{\rm wk}) \subset M_{\rm loc}(K(\Omega))$. Hence, $K(\Omega_{\rm wk})$ is an essential ideal of $M_{\rm loc}(K(\Omega))$ and so $M(K(\Omega_{\rm wk})) \subset M_{\rm loc}\left(M_{\rm loc}(K(\Omega))\right)$. However, $B(\Omega_{\rm wk}) = M(K(\Omega_{\rm wk}))$ by Kasparov's Theorem [17, Theorem 2.4] (or by a theorem of Pedersen [20]); hence,

$$B(\Omega_{\rm wk}) = M\left(K(\Omega_{\rm wk})\right) \subset M_{\rm loc}\left(M_{\rm loc}(K(\Omega))\right) \subset B(\Omega_{\rm wk}),$$

which yields $M_{\text{loc}}\left(M_{\text{loc}}(K(\Omega))\right) = B(\Omega_{\text{wk}})$. □

Somerset has shown that every separable postliminal (that is, type I) C*-algebra A has the property that $M_{\text{loc}}(M_{\text{loc}}(A)) = I(A)$ [22, Theorem 2.8]. Theorem 4.1 demonstrates that the same behavior occurs with (certain) nonseparable type I C*-algebras. Somerset's methods are different from ours in at least two ways: he employs the Baire $*$-envelope of a C*-algebra where we use the injective envelope and he uses properties of Polish spaces—spaces that arise from the separability of the algebras under study. It is reasonable to conjecture that $M_{\text{loc}}(M_{\text{loc}}(A)) = I(A)$ for all C*-algebras A that possess a postliminal essential ideal. To prove such a statement, it would be enough to prove it for any continuous trace C*-algebra A.

5 Direct Sum Decompositions

A Kaplansky–Hilbert module E over $C(\Delta)$ is said to be *homogeneous* [16] if there is a subset $\{\nu_j\}_{j\in\Lambda} \subset E$ – called an *orthonormal basis* – such that $\langle \nu_i, \nu_j\rangle = 0$ for all $j \neq i$, $|\nu_j| = 1$ for all j, and $\{\nu_j\}_{j\in\Lambda}^{\perp} = \{0\}$, where for any $\nu \in E$, $|\nu|$ is the continuous real-valued function $|\nu| = \langle \nu, \nu\rangle^{1/2} \in C(\Delta)$.

Kaplansky introduced the notion of homogeneous AW*-module with the aim of reducing the study of abstract AW*-modules to the slightly more concrete setting in which the modules have an orthonormal basis. This is justified by the following result:

Theorem 5.1 ([16])**.** *Let E be a Kaplansky-Hilbert module over $C(\Delta)$. Then there exist orthogonal projections $\{c_i\}_{i\in I} \subset C(\Delta)$ with supremum 1 such that $c_i\, E$ is a homogenous AW*-module over $c_i\, C(\Delta)$.*

Note that in the situation of Theorem 5.1, for each i there exists a clopen set $\Delta_i \subset \Delta$ with $c_i = \chi_{\Delta_i}$. The sets $\{\Delta_i\}$ are pairwise disjoint, and $\cup_i \Delta_i$ is dense in Δ.

In this section we consider the effect of a direct sum decomposition in the structures that have been studied in the previous sections, namely the Fell algebra A of the weakly continuous Hilbert bundle $(\Delta, \{H_s\}_{s\in\Delta}, \Omega, \Omega_{\text{wk}})$, and its local multiplier algebra $M_{\text{loc}}(A)$. We show that a decomposition of Ω_{wk} into a direct sum $\oplus_i c_i \Omega_{\text{wk}}$ given by a partition of the identity $\{c_i\}$ in $C(\Delta)$ leads one to consider two corresponding direct sum C*-algebras: $\oplus_i A_i$ and $\oplus_i M_{\text{loc}}(A_i)$, where A_i is a subalgebra of A for all i. We prove that A need not be isomorphic to $\oplus_i A_i$, yet $M_{\text{loc}}(A) \cong \oplus_i M_{\text{loc}}(A_i)$. The latter result is especially interesting if one recalls that $M_{\text{loc}}(A)$ is generally not an AW*-algebra [3, Theorem 6.13].

Theorem 5.2. *Let $(\Delta, \{H_s\}_{s\in\Delta}, \Omega)$ be a continuous Hilbert bundle over the Stonean space Δ. Assume that $\{\Delta_i\}_{i\in I}$ is a family of pairwise-disjoint clopen subsets of Δ whose union is dense in Δ, and for each $i \in I$ let $c_i = \chi_{\Delta_i} \in C(\Delta)$ and $\Omega_i = \{\omega_{|\Delta_i} : \omega \in \Omega\}$. Then:*

i. *$(\Delta_i, \{H_s\}_{s\in\Delta_i}, \Omega_i)$ is a continuous Hilbert bundle;*

ii. *$(\Omega_i)_{\rm wk} \cong c_i \cdot \Omega_{\rm wk}$ as C^*-modules;*

iii. *$\Omega_{\rm wk} \cong \bigoplus_i (\Omega_i)_{\rm wk}$ as C^*-modules;*

iv. *$B((\Omega_i)_{\rm wk}) \cong c_i \cdot B(\Omega_{\rm wk})$ as C^*-algebras;*

v. *$B(\Omega_{\rm wk}) \cong \bigoplus_i B((\Omega_i)_{\rm wk})$ as C^*-algebras.*

*In **ii** and **iii**, the isomorphism is considered together with the identification $C(\Delta_i) \simeq c_i\, C(\Delta)$.*

Proof. Being clopen in Δ, each Δ_i is itself a Stonean space, and it is easy to see that $C(\Delta_i) \cong c_i\, C(\Delta)$

i. For axiom (I) in Definition 1.1, we aim to show that Ω_i is a $C(\Delta_i)$ module. Let $\omega \in \Omega$ and consider $\omega_i = \omega|_{\Delta_i}$. Choose any $f_i \in C(\Delta_i)$. As Δ_i is clopen, f_i can be extended to $F_i \in C(\Delta)$ such that $f_i = F_i|_{\Delta_i}$, and $F_i|_{\Delta\setminus\Delta_i} = 0$. The action $f_i \cdot \omega_i = (F_i \cdot \omega)|_{\Delta_i}$ gives Ω_i the structure of a $C(\Delta_i)$ module. Axioms (II) and (III) of Definition 1.1 are trivially satisfied.

For axiom (IV), let $\xi : \Delta_i \to \bigsqcup_{s\in\Delta_i} H_s$ be a vector field such that for every $s_0 \in \Delta_i$ and $\varepsilon > 0$ there is an open set $U_i \subset \Delta_i$ containing s_0 and a $\omega_i \in \Omega_i$ with $\|\omega_i(s) - \xi(s)\| < \varepsilon$ for all $s \in U_i$. Let $\Xi : \Delta_i \to \bigsqcup_{s\in\Delta} H_s$ be the vector field that coincides with ξ on Δ_i and is identically zero off Δ_i. By the definition of Ω_i, there is $\omega \in \Omega$ such that $\omega_i = \omega|_{\Delta_i}$. The set U_i is also open in Δ, and $\|\omega(s) - \Xi(s)\| < \varepsilon$ for all $s \in U_i$. If $s_0 \notin \Delta_i$ choose any open set V_i containing s_0 such that $V_i \cap U_i = \emptyset$ and let $\omega \in \Omega$ be arbitrary; then $0 = \|\chi_{\Delta_i}(s)\omega(s) - \Xi(s)\| < \varepsilon$ for all $s \in V_i$. Since $\chi_{\Delta_i} \cdot \omega \in \Omega$ and since Ω is closed under local uniform approximation, $\Xi \in \Omega$, whence $\xi \in \Omega_i$.

ii. Let $T_i : c_i\, \Omega_{\rm wk} \to (\Omega_i)_{\rm wk}$ be given by $T_i(c_i\nu) = \nu|_{\Delta_i}$. It is clear that T_i is well defined, linear, bounded, and has trivial kernel; to show that it is onto, note that if $\nu_i \in (\Omega_i)_{\rm wk}$, then—since Δ_i is clopen—the vector field $\nu : \Delta \to \bigsqcup_{s\in\Delta} H_s$ defined by $\nu(s) = 0$, for $s \notin \Delta_i$, and $\nu(s) = \nu_i(s)$, for $s \in \Delta_i$, has the property that $\langle \omega, \nu\rangle \in C(\Delta)$, for all $\omega \in \Omega$; so $\nu \in \Omega_{\rm wk}$ and $\nu_i = T_i(c_i\nu)$. It is also easy to check that T_i preserves inner products.

iii. Let $T : \Omega_{\rm wk} \to \bigoplus_i (\Omega_i)_{\rm wk}$, given by $T\nu = (T_i(c_i\nu))_{i\in I}$. The previous paragraph and Lemma 2.1 show that T is an isometry; we show now that T

is onto. Suppose that $\nu' = (\nu_i)_{i\in I} \in \bigoplus_i (\Omega_i)_{\rm wk}$. For each $i \in I$ let $\tilde{\nu}_i$ denote the vector field on Δ that coincides with ν_i on Δ_i and vanishes elsewhere. Then $\tilde{\nu}_i \in \Omega_{\rm wk}$ and $T_i(c_i\tilde{\nu}_i) = \nu_i$. Hence, if $\nu = \sum_i c_i\tilde{\nu}_i$ as in Remark 2.5, we have $\nu \in \Omega_{\rm wk}$ and $T\nu = \nu'$. Thus, $\Omega_{\rm wk}$ and $\bigoplus_i (\Omega_i)_{\rm wk}$ are isomorphic Banach spaces. Similar arguments show that $\bigoplus_i (\Omega_i)_{\rm wk}$ is a $C(\Delta)$-module and that T is module isomorphism. Hence, $\Omega_{\rm wk} \cong \bigoplus_i (\Omega_i)_{\rm wk}$ as C*-modules.
iv. Let $\rho_i : c_i\, B(\Omega_{\rm wk}) \to B((\Omega_i)_{\rm wk})$ be given by $\rho_i(c_ib)\, T_i(c_i\nu) = (b\nu)|_{\Delta_i}$. This map is well-defined because if $c_ib_1 = c_ib_2$ then for any $\nu \in \Omega_{\rm wk}$ we have $(b_1\nu)|_{\Delta_i} = (c_ib_1\nu)|_{\Delta_i} = (c_ib_2\nu)|_{\Delta_i} = (b_2\nu)|_{\Delta_i}$. A similar computation shows that ρ_i is one-to-one, and linearity is clear. To see that ρ_i is onto, let $b_i \in B((\Omega_i)_{\rm wk})$. Consider the injection $\tilde{}: (\Omega_i)_{\rm wk} \to \Omega_{\rm wk}$ where $\tilde{\nu}_i \in \Omega_{\rm wk}$ is the vector field that agrees with ν_i on Δ_i and is 0 elsewhere. Let $b \in B(\Omega_{\rm wk})$ be the operator given by $b\nu = \widetilde{b_i(\nu|_{\Delta_i})}$. Then $\rho_i(c_ib)(T_ic_i\nu) = (b\nu)|_{\Delta_i} = \widetilde{b_i(\nu|_{\Delta_i})}|_{\Delta_i} = b_i(\nu|_{\Delta_i}) = b_i\,(T_ic_i\nu)$, so $\rho_i(c_ib) = b_i$.
v. Let $\rho : B(\Omega_{\rm wk}) \to \bigoplus_i B((\Omega_i)_{\rm wk})$ be the map $\rho(b) = (\rho_i(c_ib))_{i\in I}$. It is clear that ρ is a homomorphism. If $\rho(b) = 0$ for some $b \in B(\Omega_{\rm wk})$, then – as each ρ_i is one-to-one – $c_ib = 0$ for all i; this implies that $b^*b = b^*(\sup_i(c_i \cdot I))b = \sup_i(b^*c_ib) = 0$ by [14, Corollary 4.10], so $b = 0$ and ρ is one-to-one. To show that ρ is onto, let $(b_i)_i \in \bigoplus_i B((\Omega_i)_{\rm wk})$; as each ρ_i is onto, there exist operators $b^i \in B(\Omega_{\rm wk})$ with $\rho_i(c_ib^i) = b_i$. Define $b \in B(\Omega_{\rm wk})$ by $b\nu = \sum_i c_ib^i\nu$ (in the sense of Remark 2.5; that is, $c_ib\nu = c_ib^i\nu$). Then $\rho_i(c_ib)\nu|_{\Delta_i} = (c_ib\nu)|_{\Delta_i} = (c_ib^i\nu)|_{\Delta_i} = \rho_i(c_ib^i)\nu|_{\Delta_i} = b_i\nu|_{\Delta_i}$. So $\rho(b) = (b_i)_i$. □

Proposition 5.3. *Assume the notation, hypotheses, and conclusions of Theorem 5.2. Then there exists an example where the canonical embedding $\Omega \hookrightarrow \bigoplus_i \Omega_i$ (via the isometry T from the proof of **iii** in Theorem 5.2) is not onto. In particular, Ω is properly contained in $\Omega_{\rm wk}$.*

Proof. Take Δ and the family of clopen subsets $\{\Delta_i\}_{i\in I}$ in Theorem 5.2 to be such that $\bigcup_{i\in I} \Delta_i \neq \Delta$. Thus, I is an infinite set. Let H be a Hilbert space with orthonormal basis $\{e_i\}_{i\in I}$ and consider the trivial Hilbert bundle $\Omega = C(\Delta, H)$ of all continuous functions $\omega : \Delta \to H$. As in Theorem 5.2, let $\Omega_i = C(\Delta_i, H)$.

For each $i \in I$, set $\omega_i \in \Omega$ with $\omega_i(s) = e_i$ for all s and consider $(\omega_i)_{i\in I} \in \bigoplus_i \Omega_i$. Under the isomorphism of Theorem 5.2, this element $(\omega_i)_{i\in I}$ is identified with $\omega = \sum_{i\in I} \chi_{\Delta_i} \cdot \tilde{\omega}_i \in \Omega_{\rm wk}$ (in the sense of Remark 2.5), where $\tilde{\omega}_i$ is any element of Ω that agrees with ω_i on Δ_i and vanishes off Δ_i. Under this identification, $\omega \notin \Omega$; that is, the function $s \mapsto \|\omega(s)\|$ fails to be continuous on Δ. We argue this by contradiction.

Assume that $s \mapsto \|\omega(s)\|$ is continuous on Δ. Because $\|\omega(s)\| = 1$ for all $s \in \cup_{i\in I}\Delta_i$, continuity implies that $\|\omega(s)\| = 1$ for $s \in \Delta$. Choose $s_0 \in \Delta \setminus (\cup_{i\in I}\Delta_i)$ and let $(s_\alpha)_{\alpha\in\Lambda} \subset \cup_{i\in I}\Delta_i$ be a net such that $s_\alpha \to s_0$. Let $\eta \in \Omega$ be the constant field $\eta(s) = \omega(s_0)$, for all $s \in \Delta$. Since $\omega \in \Omega_{\rm wk}$, we have

$$\lim_\alpha \langle \omega(s_\alpha), \eta(s_\alpha)\rangle \;=\; \langle \omega(s_0), \eta(s_0)\rangle \;=\; \langle \omega(s_0), \omega(s_0)\rangle \;=\; 1\,. \tag{5}$$

For each $\alpha \in \Lambda$ let $i(\alpha) \in I$ be such that $s_\alpha \in \Delta_{i(\alpha)}$. Thus, for every $\alpha \in \Lambda$, $I_\alpha = \{i(\beta) :\ \beta \in I,\ \beta \geq \alpha\}$ is an infinite set (for otherwise $s_0 \in \Delta_i$ for some $i \in I$). Therefore,

$$\lim_\alpha \langle \omega(s_\alpha), \eta(s_\alpha)\rangle \;=\; \lim_\alpha \langle e_{i(\alpha)}, \omega(s_0)\rangle \;=\; 0\,. \tag{6}$$

As (5) and (6) cannot be true simultaneously, we obtain a contradiction. Hence, $\omega \notin \Omega$. □

Our second reduction theorem below notes some consequences of Theorem 5.2 when applied to the injective envelope and local multiplier algebras of the Fell algebra A associated to a continuous Hilbert bundle.

Theorem 5.4. *Let $(\Delta, \{H_t\}_{t\in\Delta}, \Omega)$ be a continuous Hilbert bundle over the Stonean space Δ and let $A = (\Delta, \{K(H_t\}, \Gamma)$ denote the associated continuous trace C*-algebra of Fell. Assume that $\{\Delta_i\}_{i\in I}$ is a family of pairwise-disjoint clopen subsets of Δ whose union is dense in Δ, and for each $i \in I$ let $c_i = \chi_{\Delta_i} \in C(\Delta)$ and $\Omega_i = \{\omega_{|\Delta_i} : \omega \in \Omega\}$. Then:*

- ***i.*** *if A_i denotes the Fell algebra of $(\Delta_i, \{H_s\}_{s\in\Delta_i}, \Omega_i)$, then $A_i \cong c_i \cdot A$;*
- ***ii.*** $I(A_i) = B((\Omega_i)_{\rm wk})$;
- ***iii.*** $I(A) \cong \bigoplus_{i\in I} I(A_i)$;
- ***iv.*** $M_{\rm loc}(A) \cong \bigoplus_{i\in I} M_{\rm loc}(A_i)$.

Proof. Let $A_i = (\Delta_i, \{K(H_s)\}_{s\in\Delta}, \Gamma_i)$ denote the Fell C*-algebra associated to the Hilbert bundle $(\Delta_i, \{H_s\}_{s\in\Delta_i}, \Omega_i)$. That is, Γ_i consists of all weakly continuous almost finite-dimensional operator fields $a_i : \Delta_i \to \bigsqcup_{s\in\Delta_i} K(H_s)$ such that $s \mapsto \|a_i(s)\|$ is continuous. We have that $B((\Omega_i)_{\rm wk})$ is a type I AW*-algebra with centre $C(\Delta_i)$.
i. For each $a_i \in \Gamma_i$ there is an $a \in \Gamma$ such that $a_i = a|_{\Delta_i}$. To verify this, let $a : \Delta_i \to \bigsqcup_{s\in\Delta} K(H_s)$ be the operator field defined by $a(s) = a_i(s)$, for $s \in \Delta_i$, and $a(s) = 0$, for $s \notin \Delta_i$. Since Δ_i is a clopen set, the maps

$s \to \|a(s)\|$ and $s \mapsto \langle a(s)\omega_1(s), \omega_2(s)\rangle$ are continuous for every $\omega_1, \omega_2 \in \Omega$. The operator field a is also locally finite-dimensional, again because Δ_i is clopen and a_i has the property on Δ_i. Hence, $a \in \Gamma$. Next, let $\pi_i : A_i \to c_i A$ be defined by $\pi_i(a_i) = c_i a$, where $a \in A$ is any operator field that restricts to a_i on Δ_i. This map is clearly well-defined, and a homomorphism.
ii. By Theorem 3.1, $B((\Omega_i)_{\mathrm{wk}}) = I(A_i) = I(c_i A)$.
iii. By [14, Lemma 6.2], $I(c_i A) = c_i I(A)$. Hence, $I(A_i) = B((\Omega_i)_{\mathrm{wk}})$ and Theorem 5.2 immediately yields $I(A) \cong \bigoplus_{i\in I} I(A_i)$.
iv. We take each $M_{\mathrm{loc}}(A_i)$ to be a C*-subalgebra of $B((\Omega_i)_{\mathrm{wk}})$. First we remark that the isomorphism ρ from Theorem 5.2 sends A into $\bigoplus_i A_i$. To see why, recall that $a\nu(s) = a(s)\nu(s)$, for all $a \in A$, $\nu \in \Omega_{\mathrm{wk}}$, and $s \in \Delta$ (Proposition 3.6). Since, for a given $i \in I$, the action of $\rho_i(a)$ on $\nu_i \in (\Omega_i)_{\mathrm{wk}}$ is defined by $\nu_i \mapsto (a\nu)|_{\Delta_i}$, where $\nu \in \Omega_{\mathrm{wk}}$ is any vector with $\nu|_{\Delta_i} = \nu_i$, it is easy to verify that $\rho_i(a)$ is a weakly continuous almost finite-dimensional operator field on Δ_i.

To show that $\rho\left(M_{\mathrm{loc}}(A)\right) \subset \bigoplus_i M_{\mathrm{loc}}(A_i)$, let $x \in M_{\mathrm{loc}}(A) \subset I(A)$ and suppose that $\varepsilon > 0$. Thus, there is an essential ideal $J \subset A$ and a multiplier $x \in M(J)$ such that $\|x - y\| < \varepsilon$. Further, there exists an open dense subset $U \subset \Delta$ such that

$$J \;=\; \{a \in A : a(s) = 0\,,\; s \in \Delta \setminus U\}\,. \tag{7}$$

For $i \in I$, let $U_i = \Delta_i \cap U$, which is an open dense set in Δ_i. Therefore,

$$J_i \;=\; \{a_i \in A_i :\; a(s) = 0,\; s \in \Delta_i \setminus U_i\} \tag{8}$$

is an essential ideal in A_i. We aim to show that $\rho_i(y) \in M(J_i)$. To this end, select $a_i \in J_i$. As $A_i \cong c_i \cdot A$, there is an $a \in A$ such that $a_i(s) = a(s)$ for all $s \in \Delta_i$. Moreover, $a \in A$ can be chosen so that $a(s) = 0$ for all $s \in \Delta \setminus \Delta_i$.

Because $a_i \in J_i$, we conclude that $a(s) = 0$ for all $s \in \Delta \setminus U$; that is, $a \in J$. Therefore, $ya \in J$, which implies that $ya(s) = 0$ for all $s \in \Delta \setminus U$. In particular, $ya(s) = 0$ for all $s \in \Delta_i \setminus U_i$. The element $\rho_i(y)a_i \in B((\Omega_i)_{\mathrm{wk}})$ is in fact an operator field since $\rho_i(y)a_i = \rho_i(y)\rho_i(c_i a) = \rho_i(c_i(ya)) \in A_i$. Then, for all $s \in \Delta_i \setminus U_i$ and $\nu \in \Omega_{\mathrm{wk}}$,

$$\begin{aligned}[\rho_i(y)a_i](s)(T_i c_i \nu)(s) &= \rho_i(y)a_i(T_i c_i \nu)(s) = \rho_i(c_i ya)(T_i c_i \nu)(s) \\ &= (ya)\nu|_{\Delta_i}(s) = (ya)(s)\nu|_{\Delta_i}(s) = 0.\end{aligned}$$

With ν being arbitrary, we conclude that $\rho_i(y)a_i(s) = 0$, that is $\rho_i(y)a_i \in J_i$, and so $\rho_i(y)$ is a left multiplier of J_i. By a similar argument, $\rho_i(y)$ is a right multiplier of J_i, and so $\rho_i(y) \in M(J_i)$. Thus, $\rho(y) \in \bigoplus_i M_{\mathrm{loc}}(A_i)$ and

$\|\rho(x)-\rho(y)\| = \|x-y\| < \varepsilon$. As $\varepsilon > 0$ was chosen arbitrarily, this proves that $\rho(x) \in \bigoplus_i M_{\mathrm{loc}}(A_i)$.

Conversely, let us show that $\bigoplus_i M_{\mathrm{loc}}(A_i) \subset \rho(M_{\mathrm{loc}}(A))$. Let $(x_i)_i \in \bigoplus_i M_{\mathrm{loc}}(A_i)$; thus for each $i \in I$, there exist an essential ideal $J_i \subset A_i$ and $y_i \in M(J_i)$ such that $\|x_i - y_i\| < \varepsilon$ for all $i \in I$. For each $i \in I$, there exists an open dense subset $U_i \subset \Delta_i$ such that J_i is given as in (8). Define $U = \bigcup_{i\in I} U_i$, which is an open dense subset of Δ and let J be the essential ideal of A defined as in (7) (for our present choice of U). Let $y \in B(\Omega_{\mathrm{wk}})$ be such that $\rho(y) = (y_i)_i$.

For each $\omega \in \Omega$, we have that $y\omega \in \Omega_{\mathrm{wk}}$.

Claim 1. If $\omega \in \Omega$ is such that $\omega(s) = 0$ for all $s \in \Delta \setminus U$, then $y\omega \in \Omega$ and $y\omega(s) = 0$ for $s \in \Delta \setminus U$.

Assuming Claim 1, consider the set $F_+ = \mathrm{span}\,\{\Theta_{\omega,\omega} :\ \omega \in \Omega,\ \omega(s) = 0 \ \text{ for } \ s \in \Delta \setminus U\}$, which by Lemma 4.2 is dense in K_+, where K is the essential ideal of $K(\Omega)$ defined by $K = K(\Omega) \cap J$. By the Claim, $y\Theta_{\omega,\omega} = \Theta_{y\omega,\omega} \in K$ for all $\omega \in \Omega$. Therefore, y is a left multiplier of K. Similarly, y is a right multiplier of K , which yields $y \in M(K)$. Hence, $(x_i)_{i\in I}$ is within ε of a multiplier—namely, $\rho(y)$—of an essential ideal of $\rho\,(K(\Omega))$. Thus, by the Frank–Paulsen description of local multiplier algebras [11], $(x_i)_{i\in I} \in \rho\,(M_{\mathrm{loc}}(K(\Omega)))$. By Theorem 4.1, $M_{\mathrm{loc}}(A) = M_{\mathrm{loc}}(K(\Omega))$, so $(x_i)_{i\in I} \in \rho(M_{\mathrm{loc}}(A))$.

We are now left with proving Claim 1. Assume that $\omega \in \Omega$ with $\omega(s) = 0$ for all $s \in \Delta \setminus U$. Let $i \in I$ and let $\omega_i = \omega|_{\Delta_i} \in \Omega_i$. Note that for every $\eta_i \in \Omega_i$, $\Theta_{\omega_i,\eta_i} \in J_i$, and hence $\Theta_{y_i\omega_i,\eta_i} = y_i\Theta_{\omega_i,\eta_i} \in J_i$. Also, $y_i\omega_i \in \Omega_i$. Indeed, suppose that $s_0 \in \Delta_i$ and let $\eta_i \in \Omega_i$ such that $\|\eta_i(s_0)\| = 1$. Choose a clopen subset $V_i \subset \Delta_i$ of s_0 for which $\|\eta_i(s)\| \geq 1/2$ for all $s \in V_i$ and define $f(s) = \chi_{V_i}(s)\|\eta_i(s)\|^{-2}$. Thus, $f \in C(\Delta_i)$ and so $f \cdot \eta_i \in \Omega_i$. Then, since $\Theta_{y_i\omega_i,\eta_i} \in J_i \subset A_i$, we have $\Theta_{y_i\omega_i,\eta_i}(f\cdot\eta_i) \in \Omega_i$. So $\chi_{V_i}\cdot y_i\omega_i = \Theta_{y_i\omega_i,\eta_i}(f\cdot\eta_i) \in \Omega_i$. Thus, $y_i\,\omega_i$ is a local uniform limit of vectors fields in Ω_i and hence, $y_i\,\omega_i \in \Omega_i$. Moreover, since $\Theta_{y_i\omega_i,\,\eta_i} \in J_i$ for any $\eta_i \in \Omega_i$, we have $y_i\omega_i(s) = 0$ for $s \in \Delta_i \setminus U_i$.

Since $(y\omega)(s) = (y_i\,\omega_i)(s)$ for $s \in \Delta_i$, the lower semicontinuous function $s \mapsto \|(y\omega)(s)\|$ is continuous on $\bigcup_i \Delta_i$ and vanishes on $(\bigcup_i \Delta_i) \setminus U$.

Claim 2. There exists $C > 0$ such that $\|y\omega(s)\| \leq C\,\|\omega(s)\|$, $s \in \Delta_i$, $i \in I$.

We will use Claim 2 to show that the function $s \mapsto \|(y\omega)(s)\|$ is continuous on Δ. Let $s \in \Delta \setminus (\bigcup_i \Delta_i)$ and let $(s_\alpha)_\alpha \subset \bigcup_i \Delta_i$ be a net such that $s_\alpha \to s$ in Δ. This implies that $\lim_\alpha \|\omega(s_\alpha)\| = 0$. By lower semicontinuity of the function $s \mapsto \|(y\omega)(s)\|$,

$$0 \leq \|y\omega(s)\| \leq \lim_{\alpha} \|y\omega(s_\alpha)\| \leq C \ \lim_{\alpha} \|\omega(s_\alpha)\| = 0\,,$$

and it follows that $s \mapsto \|(y\omega)(s)\|$ is continuous on Δ and vanishes in $\Delta \setminus U$. This establishes Claim 1.

We finish the proof by proving Claim 2. Fix $s \in \Delta_i$, and let $C = \sup_i \|y_i\|$. We already know that $y_i\omega_i \in \Omega_i$, and so

$$\begin{aligned}\|y\omega(s)\| = \|y_i\omega_i(s)\| = \|y_i\omega_i\|(s) \leq \|y_i\|\,\|\omega_i\|(s) \\ \leq C\,\|\omega_i\|(s) = C\,\|\omega_i(s)\| = C\,\|\omega(s)\|.\square\end{aligned}$$

Local multiplier algebras behave well under direct sums: $M_{\rm loc}(\oplus_i A_i) \cong \oplus_i M_{\rm loc}(A_i)$ [2, Proposition 2.3.6]. However, the isomorphism of local multiplier algebras in Theorem 5.4 cannot be established via that generic result:

Proposition 5.5. *Assume the notation, hypotheses, and conclusions of Theorem 5.4. Although ρ sends A into $\bigoplus_i A_i$, it need not be true that $A \cong \bigoplus_i A_i$.*

Proof. If Δ and Ω are as in Proposition 5.3, then $\rho(\Theta_{\omega,\omega}) = (\Theta_{\omega_i,\omega_i})_{i\in I} \in \oplus_{i\in I} A_i$, but $\rho(\Theta_{\omega,\omega}) \notin \rho(A)$. $\square$

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