PHYSICAL REVIEW D **80,** 083526 (2009)

# Thermodynamics in variable speed of light theories

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(Received 16 September 2008; published 26 October 2009)

The perfect fluid in the context of a covariant variable speed of light theory proposed by J. Magueijo is studied. On the one hand the modified first law of thermodynamics together with a recipe to obtain equations of state are obtained. On the other hand the Newtonian limit is performed to obtain the nonrelativistic hydrostatic equilibrium equation for the theory. The results obtained are used to determine the time variation of the radius of Mercury induced by the variability of the speed of light ($c$), and the scalar contribution to the luminosity of white dwarfs. Using a bound for the change of that radius and combining it with an upper limit for the variation of the fine structure constant, a bound on the time variation of $c$ is set. An independent bound is obtained from luminosity estimates for Stein 2015B.

DOI: 10.1103/PhysRevD.80.083526 PACS numbers: 98.80.Cq, 04.80.Cc

## I. INTRODUCTION

There are several very different motivations for studying the variation of fundamental constants. The coincidence of large dimensionless numbers arising from the combination of different physical constants led Dirac to propose the large number hypothesis and predict a time variation of them [1,2]. Theories with varying constants can also be a way of implementing Mach's principle. Extra-dimensional theories like superstring or Kaluza-Klein theories reduce in the low energy limit to effective theories in which the fundamental constants may vary in space and time.

Although no variation has been found in most experiments and observations performed up to date, the results from analysis of spectra from high redshift quasar absorption systems remain controversial. Some works have reported a variation in the fine structure constant [3–8], while other studies give null results [9–12]. Besides the motivations mentioned above, variable speed of light (VSL) theories are interesting because they could solve several cosmological puzzles [13–15].

In this work we study the thermodynamics and Newtonian limit of the varying speed of light theory developed by J. Magueijo [16]. This theory was an improvement of a previous version [14], in order to account more carefully of local Lorentz invariance and the dynamics of the scalar field behind $c$'s variation. The theory here considered has been strongly criticized in [17], responded in [18], criticized again in [19,20], and answered back in [21]. As we point out at the end of the paper, in Magueijo's theory the interaction between matter and the scalar field is not explicitly shown in full detail. This can be seen in the effective quantum creation of particles (see Sec. VI of Ref. [16]), even in the absence of an explicit interaction term in the matter Lagrangian ($b = 0$ case; see below). Nevertheless, being a theory that gathered a lot of attention in the recent past, we consider that it is very suitable as a first application of the general framework developed in this work, which we plan to apply to other (formally less controversial) theories such as bimetric theories, in future publications.

After a summary of this VSL theory (Sec. II) and of a Lagrangian approach to describe perfect fluids (Sec. III), the first law of thermodynamics and a recipe for obtaining equations of state are derived (Sec. IV). It is shown that the field associated with the variation of the speed of light can formally be considered as a new thermodynamic variable. Regarding this point we note that this field has two properties that are not common in thermodynamic variables: local universality and long scale variation. This has been remarked in a work in the context of another theory with variation of physical constants [22]. In Sec. V we perform the Newtonian limit and in Sec. VI we apply all the results to study the evolution of the radius of Mercury and derive a bound on the time variation of $c$. Section VII devotes to the estimation of the scalar contribution to the luminosity of a white dwarf, obtaining a stringent upper bound for the variation of $c$. In Sec. VIII we state our conclusions and we leave for the Appendix some results concerning the space and time dependence of the scalar field.

## II. BRIEF DESCRIPTION OF THE VSL THEORY

In the covariant and locally Lorentz invariant VSL theory proposed by Magueijo $c$ plays three different roles:

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(a) *c is a dynamical field*:
The spacetime variations of $c$ are represented by an adimensional scalar field $\psi$, so that

$$c = c_0 e^{\psi}, \tag{1}$$

where $c_0$ is a constant.
The general relativity (GR) action is modified and becomes

$$I = \int d^4x\sqrt{-g}\bigg(e^{a\psi}(R - 2\Lambda - \kappa\nabla_\mu\psi\nabla^\mu\psi) + \frac{16\pi G}{c_0^4}e^{b\psi}\mathcal{L}_m\bigg). \tag{2}$$

The metric is taken to be $\eta_{\mu\nu} = \mathrm{Diag}(-1, 1, 1, 1)$. $\mathcal{L}_m$ is the matter Lagrangian and $\kappa$, $a$, and $b$ are three parameters of the theory. We will take $a - b = 4$ as in [16]. The matter Lagrangian is required to have no explicit dependence on $c$ (*minimal coupling condition*), which leads to the second role played by $c$.

(b) *c parametrizes the variations of the other "constants"*: The minimal coupling condition fixes the scaling with $c$ of all the Lagrangian parameters up to the $\hbar(c)$ dependence, which is taken to be $\hbar \propto c^{q-1}$, where $q$ is the fourth parameter of the theory. For example, since the Compton wavelength appears in the Lagrangian of a quantum particle, the dependence on $c$ of the mass will be $m \propto \hbar/c \propto c^{q-2}$. In a similar fashion it can be determined that the charge of a quantum particle ($e$), the Bohr radius ($r_b$), and the fine structure constant ($\alpha$) are proportional to $c^q$, $c^{-q}$, and $c^q$, respectively.

(c) *c is an integrating and conversion factor*: The theory is covariant and locally Lorentz invariant in a generalized way explained in [16]. The key point is the use of an $x^0$ coordinate, instead of time, in all geometrical formulas. The main difference with constant $c$ theories is that local measurements of space and time ($dx$ and $dt$) are not generally differentials of a coordinate system (the partial derivatives do not commute). Nevertheless it is always possible to find integrating factors such that $dt\psi^\beta$ and $dx\psi^{\beta-1}$ are perfect differentials, with $\beta$ another parameter which, however, will not appear in the equations of our work, because the calculations will be done using the $x^0$ coordinate.

Varying the action (2) with respect to the metric and $\psi$ leads to the equations:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} + \kappa\bigg(\nabla_\mu\psi\nabla_\nu\psi - \frac{1}{2}g_{\mu\nu}\nabla_\delta\psi\nabla^\delta\psi\bigg) + e^{-a\psi}(\nabla_\mu\nabla_\nu e^{a\psi} - g_{\mu\nu}\Box e^{a\psi}), \tag{3}$$

$$\Box\psi + a\nabla_\mu\psi\nabla^\mu\psi = \frac{8\pi G}{c^4(2\kappa + 3a^2)}(aT - 2a\rho_\Lambda c^2 - 2b\mathcal{L}_m) + \frac{1}{\kappa}\frac{d\bar{\Lambda}}{d\psi}, \tag{4}$$

where $T^{\mu\nu}$ is the matter stress energy tensor and $T$ is its trace. Besides, $\rho_\Lambda$ is the mass density corresponding to the cosmological constant, $\rho_\Lambda = \frac{\Lambda c^2}{8\pi G}$ and $\bar{\Lambda}$ is a linear combination of the matter and geometric cosmological constants, $\bar{\Lambda} = \Lambda + \frac{8\pi G}{c^4}\Lambda_m$.

After applying the Bianchi identities to Eq. (3) and using Eq. (4), an equation for the divergence of $T^{\mu\nu}$ is obtained:

$$T^\nu{}_{\mu;\nu} = -\psi_{;\nu}T^\nu{}_\mu b + \psi_{;\mu}b\mathcal{L}_m - \Lambda_{m;\mu}. \tag{5}$$

Note that matter energy is conserved only when $b = 0$, in all other cases there is exchange of energy between matter and the $\psi$ field.

The presence of the $\psi$ field can modify the law of conservation of the number of particles and the normalization condition for the four-velocity, e.g. it is found that for a classical particle $U^2 = U_0^2(c/c_0)^{-b} \neq -1$ [16]. Besides, the energy density and the total energy of a body in hydrostatic equilibrium will also vary if $c$ is not constant (we must distinguish between energy density and total energy because the size of a body can change in time as a result of the variation of $c$). In addition, the $c$ dependence of the mass is different for a classical and a quantum particle. Finally, the matter Lagrangian, which is not unique, appears in the equations of the theory. To take into account all these effects and ambiguities it is convenient to introduce four new parameters, $q_1$, $q_2$, $q_3$, and $q_4$:

$$c^{q_1}U^\mu U_\mu = c_0^{q_1}U_0^2 = \mathcal{C} \quad \text{Generalized normalization of the four-velocity.} \tag{6}$$

$$(nc^{q_2}U^\nu)_{;\nu} = 0 \quad \text{Generalized conservation of particle number.} \tag{7}$$

$$\rho = \rho_0 e^{q_3\psi} \quad c \text{ dependence of the energy density of a body in hydrostatic equilibrium.} \tag{8}$$

$$U = U_0 e^{q_4\psi} \quad c \text{ dependence of the total energy of a body in hydrostatic equilibrium.} \tag{9}$$

The "0" subscript denotes the value of these quantities when $\psi = 0$.

## III. THE PERFECT FLUID LAGRANGIAN

The task of obtaining a Lagrangian for the perfect fluid is not a trivial one due to the constraints imposed by the normalization of the velocity and the conservation of the number of particles. A. H. Taub gave one in 1954 [23] and another one was given by B. F. Schutz in 1970 [24], using a different but equivalent approach. Note that equivalent Lagrangians in usual theories (i.e. differing in a divergence) may not be equivalent in Magueijo's one, because of their explicit appearance in the evolution equations. In our work we have used Schutz's Lagrangian, hence we present in this section a brief summary of his approach and then we show how it can be used in the VSL theory under study.

### A. Schutz's Lagrangian

Schutz uses a formulation of relativistic hydrodynamics based on the utilization of 6 potentials to represent the velocity:

$$U_\nu = \mu^{-1}(\phi_{,\nu} + \xi\beta_{,\nu} + \theta s_{,\nu}), \tag{10}$$

where $\mu$ and $s$ are the specific enthalpy (enthalpy per unit mass) and the specific entropy, respectively. The physical meaning of the other potentials is also explored by Schutz.

The perfect fluid action is

$$I = \int\left(R + \frac{16\pi G p}{c_0^4}\right)(-g)^{1/2}d^4x,$$

where $p$ is the pressure of the fluid. Then the perfect fluid Lagrangian is $\mathcal{L}_m = p$. The following steps must be followed to vary the action:

(1) Choose an equation of state for the fluid and write it in terms of $\mu$ and $s$: $p = p(\mu, s)$.
(2) When varying the action make use of the thermodynamic relation $dp = \rho_m d\mu - \rho_m T ds$ (where $\rho_m$ is the rest mass density).
(3) Define the four-velocity vector field $U_\nu$ in terms of 6 scalar potentials: $U_\nu = \mu^{-1}(\phi_{,\nu} + \xi\beta_{,\nu} + \theta s_{,\nu})$.
(4) The normalization of $U$ is taken into account before starting the variations. This is done expressing $\mu$ in terms of $\phi$, $\xi$, $\beta$, $\theta$, $s$, $g^{\mu\nu}$: $\mu^2 = -g^{\mu\nu}(\phi_{,\mu} + \xi\beta_{,\mu} + \theta s_{,\mu})(\phi_{,\nu} + \xi\beta_{,\nu} + \theta s_{,\nu})$. So the independent variables are $\phi$, $\xi$, $\beta$, $\theta$, $s$, and $g^{\mu\nu}$. Any quantity appearing in the action must be considered a function of these variables.

The Euler-Lagrange equations become

$$G_{\mu\nu} - \frac{8\pi G}{c_0^4}[(\rho + p)U_\mu U_\nu + p g_{\mu\nu}] = 0,$$
$$(\rho_m U^\nu)_{;\nu} = 0, \qquad U^\nu s_{,\nu} = 0, \qquad U^\nu \theta_{,\nu} = T,$$
$$U^\nu \beta_{,\nu} = 0, \qquad \text{and} \qquad U^\nu \xi_{,\nu} = 0.$$

The stress energy tensor is

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g_{\mu\nu}} = (\rho + p)U^\mu U^\nu + p g^{\mu\nu}. \tag{11}$$

It is important to note that $(\rho + p)$, $U_\mu$, $p$, $\rho_m$, and $T$ were defined in terms of $\phi$, $\xi$, $\beta$, $\theta$, $s$, and $g^{\mu\nu}$ after performing the variations, using the equations

$$p = p(\mu, s) \qquad \text{(equation of state)}, \qquad \rho_m = \left(\frac{\partial p}{\partial \mu}\right)_s,$$
$$T = \frac{-1}{\rho_m}\left(\frac{\partial p}{\partial s}\right)_\mu, \qquad (\rho + p) = \rho_m \mu,$$
$$U_\nu = \mu^{-1}(\phi_{,\nu} + \xi\beta_{,\nu} + \theta s_{,\nu}).$$

### B. Use of Schutz's Lagrangian in Magueijo's theory

Using Schutz's Lagrangian in the action (2) and varying it with respect to $\phi$, $\xi$, $\beta$, $\theta$, and $s$ we obtain

$$(e^{b\psi}\rho_m U^\nu)_{;\nu} = 0, \tag{12}$$

$$U^\nu s_{,\nu} = 0, \tag{13}$$

$$U^\nu \theta_{,\nu} = T, \tag{14}$$

$$U^\nu \beta_{,\nu} = 0, \tag{15}$$

$$U^\nu \xi_{,\nu} = 0. \tag{16}$$

Moreover, varying $\mathcal{L}_m$ with respect to $g^{\mu\nu}$ leads to

$$T^{\mu\nu} = (\rho + p)U^\mu U^\nu + p g^{\mu\nu}, \tag{17}$$

where $\rho$, $p$, $\rho_m$, $U^\mu$ are defined from $\phi$, $\xi$, $\beta$, $\theta$, and $s$ exactly in the same way as in Schutz's theory. In the VSL theory they can depend on $\psi$, but they will coincide with the usual quantities when $\psi = 0$. Besides, with these definitions of $\rho$, $p$, and $U^\mu$, the stress energy tensor in the VSL theory has the same form as the usual one for a perfect fluid with energy density $\rho$ and pressure $p$ (both quantities being measured in a momentarily comoving reference frame). These facts make it reasonable to consider $\rho$, $p$, $\rho_m$, and $U^\mu$ as the energy density, pressure, rest mass density, and four-velocity of the perfect fluid in the VSL theory that is being studied.

Note that by definition $U^\mu U_\mu = -1$, so $\mathbf{q_1} = \mathbf{0}$. This is different from Magueijo's result for a classical particle, $e^{b\psi}U^\mu U_\mu =$ constant, where the definition of the velocity is $U^\mu = \frac{dx^\mu}{d\lambda} = \frac{dx^\mu}{cd\tau}$. Although there is no contradiction with this, one has to be careful with the interpretations given to $U^\mu$.

## IV. THERMODYNAMICS

### A. First law

When energy is conserved, the divergence of $T^{\mu\nu}$ [Eq. (5)] is zero and the first law of thermodynamics is obtained projecting along $U^\mu$. We will do the same here but using $c^{q_1}U^\mu$ as the projector. Although using Schutz's Lagrangian leads to work with a four-velocity normalized to $-1(q_1 = 0)$, the calculations in this section will be done with an arbitrary $q_1$. The motivation is to obtain equations valid even when that condition is not satisfied.

The results of this section will be applied to systems whose scales are much smaller than cosmological scales, so we take $\Lambda_m = 0$.[1] Projecting $T^{\nu}_{\mu;\nu}$ along $c^{q_1}U^{\mu}$ and using Eqs. (5), (17), (6), and (7) together with Schutz's Lagrangian leads to[2]

$$d\rho + \left[1 + \frac{c^{q_1}}{\mathcal{C}}\right]dp - \frac{(\rho + p)}{n}dn = (\rho + p)\left(\frac{q_1}{2} + q_2 - b\right)d\psi. \quad (18)$$

It is convenient to express this relation in terms of the specific thermodynamic variables, $v = \frac{V}{N} = \frac{1}{n}$, $u = \rho v = \frac{U}{N}$, $s = \frac{S}{N}$ ($V$, $U$, and $S$ are the total volume, total energy, and total entropy of a system containing $N$ particles):

$$du + pdv - v(\rho + p)\left(\frac{q_1}{2} + q_2 - b\right)d\psi + v\left[1 + \frac{c^{q_1}}{\mathcal{C}}\right]dp = 0. \quad (19)$$

The first two terms are the only ones appearing in GR. The third term shows that $\psi$ formally plays the role of a new thermodynamic variable (changes in $\psi$ cause changes in the internal energy). Finally, the fourth term involves the pressure, which is not an independent variable (up to this point the independent variables are taken to be $v$ and $\psi$). The expression is valid only for thermodynamic processes which do not involve heat transfer, while in a more general situation, with an amount $dQ$ of heat being transferred, the equation becomes

$$du + pdv - v(\rho + p)\left(\frac{q_1}{2} + q_2 - b\right)d\psi + v\left[1 + \frac{c^{q_1}}{\mathcal{C}}\right]dp = dQ. \quad (20)$$

This is the modified first law of thermodynamics in the VSL theory.

Incorporating Caratheodory's principle it can be shown that there exists an integrating factor for $dQ$ in Eq. (20) [25]. Defining $\frac{1}{T}$ as the integrating factor and $ds = \frac{dQ}{T}$, Eq. (20) becomes

$$du + v\left[1 + \frac{c^{q_1}}{\mathcal{C}}\right]dp + pdv - v(\rho + p) \times \left(\frac{q_1}{2} + q_2 - b\right)d\psi = Tds, \quad (21)$$

where $s$ (identified with the specific entropy of the system) and $T$ are two thermodynamic variables. To go on, it is convenient to introduce the function $f(\psi) = 1 + \frac{c^{q_1}}{\mathcal{C}} = 1 + \frac{c_0^{q_1}}{\mathcal{C}}e^{q_1\psi}$. We take as the time of reference the present epoch, so that $U_0^2 = -1$ (which is the usual normalization of the four-velocity) and $c_0^{q_1} = c_{\text{today}}^{q_1}$. Then from Eq. (6) it follows that $\mathcal{C} = -c_0^{q_1}$ and hence $f(\psi) = 1 - e^{q_1\psi}$. After choosing $s$, $v$, and $\psi$ as the independent variables and writing $p$ in terms of them, an expression for the first law of thermodynamics involving only state variables is finally obtained:

$$du = -\left(p + vf(\psi)\frac{\partial p}{\partial v}\right)dv + \left(T - vf(\psi)\frac{\partial p}{\partial s}\right)ds + \left((u + pv)\left(\frac{q_1}{2} + q_2 - b\right) - vf(\psi)\frac{\partial p}{\partial \psi}\right)d\psi. \quad (22)$$

### B. Equations of state

The first partial derivatives of the specific internal energy can be obtained directly from Eq. (22). There are three different equalities between the mixed partial derivatives which impose some restrictions on the functional dependence of $p$ and $T$ on $\psi$:

$$(f(\psi) - 1)\frac{\partial p}{\partial s} = \frac{\partial T}{\partial v}, \quad (23)$$

$$\frac{\partial p}{\partial \psi} = b_1 v\frac{\partial p}{\partial v}, \quad (24)$$

$$\frac{\partial T}{\partial \psi} = b_2 T + b_1 v\frac{\partial T}{\partial v}, \quad (25)$$

with $b_1 = \frac{q_1}{2} - q_2 + b$ and $b_2 = \frac{q_1}{2} + q_2 - b$.

Several observations corresponding to different epochs in the history of the Universe show that the $\alpha$ variation has been very small (or zero) [26], so the field $\psi$ must also be very small. Then, it makes sense to express the pressure as a power expansion in $\psi$[3]:

$$p = \sum_{k=0}^{\infty} p_k(v, s)\psi^k. \quad (26)$$

After replacing this series in Eq. (24) and equaling terms with the same power of $\psi$ a recurrent formula for the coefficients $p_k$ is obtained:

$$p_{k+1}(v, s) = \frac{b_1 v}{k + 1}\frac{\partial p_k(v, s)}{\partial v}, \quad (27)$$

where $p_0(v, s)$ is the pressure as a function of $v$ and $s$ for $\psi = 0$ and therefore it is obtained from the usual theories in which $c$ is constant.

Working to first order in $\psi$ we arrive at the following expression for the functional dependence of $p$ on $v$, $s$, and

[1] On the other hand, the cosmological constant is important for the evolution of $\psi$.

[2] If $\Lambda_m \neq 0$ but depends only on $\psi$, the equation will be valid after adding the term $-\frac{c^{q_1}}{\mathcal{C}}\frac{d\Lambda_m}{d\psi}d\psi$ to the right-hand side.

[3] The $\psi$ dependence of the temperature can be treated in a similar fashion.

$\psi$:

$$p \simeq p_0(v, s) + b_1 v \frac{\partial p_0}{\partial v} \psi. \tag{28}$$

## V. NEWTONIAN LIMIT

The Newtonian limit of this VSL theory can be obtained following the same steps as those used in GR. Although in the Newtonian limit time derivatives are negligible with respect to the spatial ones, special considerations are needed for the derivatives of $\psi$. For example, if the spatial extension of a system is small compared to the scales associated with the spatial variations of $\psi$ (which are cosmological scales) and one is interested in following its time evolution, time derivatives will be more interesting than the spatial ones. This will be the case in the following sections, where we will determine the evolution of the planetary radii and the luminosity of white dwarfs. We also remind that, as explained before, the cosmological constant is not included in our calculations. Finally, since the $\psi$ field must be very small, we will work to first order in $\psi$.

The condition of weak gravitational field allows to choose nearly Lorentz coordinates in which

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad \text{with} \quad h_{\mu\nu} \ll 1. \tag{29}$$

Using Eqs. (3) and (4), the Lagrangian $\mathcal{L}_m = p$ and the fact that $a\nabla_0\nabla^0\psi$ is negligible (see the Appendix), the following gravitational potential equation is obtained:

$$-\nabla^2 h_{00} = \frac{8\pi\tilde{G}}{c^4} T_{00}, \tag{30}$$

where $\tilde{G} = G(1 + \frac{a^2}{2\kappa+3a^2})$ is an effective gravitational constant. In the nonrelativistic limit of GR $h_{00}$ is identified with $-2\frac{\phi}{c^2}$ to arrive at Newton's gravitational potential equation, instead here we will go on working with $h_{00}$ due to the presence of the $c^2$ factor in that identification.

The Euler's equations for this theory are obtained projecting Eq. (5) perpendicularly to $U^\mu$ with the projector $g^{\mu\alpha} + U^\mu U^\alpha$ (here we have taken $q_1 = 0$ from the beginning):

$$-\frac{1}{2} T_{00} h_{00,i} + p_{,i} + \rho U^\nu U_{i,\nu} + \rho U_0 U^j g_{0j,i} = 0. \tag{31}$$

For quasistatical situations ($U_i \approx 0$) the equation becomes

$$\frac{1}{2} T_{00} \nabla h_{00} = \nabla p. \tag{32}$$

The hydrostatic equilibrium equation follows after combining Eqs. (30) and (32). For a system with spherical symmetry the result is

$$\frac{dp}{dr} = -\frac{\tilde{G}}{c^4}\frac{\rho(r)U(r)}{r^2}, \tag{33}$$

where $U(r)$ is the total energy inside the sphere of radius $r$ and $\frac{d}{dr}$ must be understood as a spatial derivative at constant time. To obtain this equation it is necessary to consider $c$ as a constant in the integral $\int_0^r \frac{4\pi r^2 \rho}{c^4} dr$. This can be done because the spatial variations of $\psi$ are negligible in noncosmological scales, as is demonstrated in the Appendix.

## VI. EVOLUTION OF THE RADIUS OF MERCURY AND A BOUND FOR THE VARIATION OF *c*

The radius of a planet is determined with the hydrostatic equilibrium equation together with an equation of state and boundary conditions. The presence of $\psi$ in these equations causes in general time variations of the radius. On the other hand, the actual change in size of several bodies of the Solar System have been estimated using different topographical observations. For Mercury there is a stringent bound: its radius has not changed more than 1 km in the last $3.9 \times 10^9$ years [27]. This fact will allow us to obtain a bound for the temporal variation of $\psi$.

In Sec. VI A we will show that the hydrostatic equilibrium equation is equivalent to another equation in which the temporal dependence resides only in the gravitational constant. Then it will be possible to use the results of the work of McElhinny *et al.* and give a bound for the quantity $f(q_i)\dot{\psi}$, where $f(q_i)$ is a function of the parameters $b_1$, $q_3$, and $q_4$ (the dot denotes derivative with respect to time). In Sec. VI B those parameters will be expressed in terms of the parameters originally defined in [16]. Finally, in Sec. VI C the bound obtained for $f(q_i)\dot{\psi}$ will be combined with a bound for the variation of the fine structure constant to obtain an upper limit for $|\dot{\psi}|$.

### A. Transformation into a variable *G* theory

To solve the hydrostatic equilibrium equation it is necessary to have an equation of state relating the pressure and density. It can be obtained replacing the corresponding equation of state for the constant $c$ case in Eq. (28). For Mercury it is sufficient to work with a linear equation in $\rho$ [27]:

$$p_0 = K_{\text{sur}}\left(\frac{\rho_0}{\rho_{0\,\text{sur}}} - 1\right). \tag{34}$$

The quantities with a "0" subscript correspond to $\psi = 0$, sur indicates evaluation in the surface of Mercury, and $K_{\text{sur}}$ is the superficial compressibility.[4] Using Eqs. (28) and (34), and the nonrelativistic expression $\rho_0 \approx \frac{m_0 c_0^2}{v}$ (with $m_0$ the average mass of a particle), the equation of state to be used is obtained:

[4] In [27] a quotient between mass densities is used instead of $\frac{\rho_0}{\rho_{0\,\text{sur}}}$, nevertheless in the Newtonian limit they are equal since $\rho_{\text{energy}} \approx \rho_{\text{mass}} c^2$.

$$p = K_{\rm sur}\left(\frac{\rho_0}{\rho_{0\,\rm sur}} - 1\right) - b_1 K_{\rm sur}\psi\frac{\rho_0}{\rho_{0\,\rm sur}}. \qquad (35)$$

Replacing the equation of state (35) in Eq. (33), using the expression for $\frac{\partial\psi}{\partial r}$ given in the Appendix and taking into account the definitions of $q_3$ and $q_4$ we get

$$\frac{dp_0}{dr} = -\tilde{G}e^{(b_1+q_3+q_4-4)\psi}\frac{\rho_{m0}(r)M_0(r)}{r^2} - 2b_1\frac{K_{\rm sur}}{\rho_{0\,\rm sur}}\frac{a}{2\kappa+3a^2}e^{b_1\psi}G\frac{\rho_{m0}(r)M_0(r)}{r^2}, \qquad (36)$$

where $\rho_{m0}(r)$ is the mass density for $\psi = 0$ and $M_0(r)$ is the mass contained within radius $r$ (also for $\psi = 0$). The second term of the right-hand side is small compared to the first one,[5] so it can be multiplied by $e^{(q_2+q_4-4)\psi}$ [which is equal to $1 + O(\psi)$] and hence the equation becomes

$$\frac{dp_0}{dr} = -\bar{G}e^{(b_1+q_3+q_4-4)\psi}\frac{\rho_{m0}(r)M_0(r)}{r^2}, \qquad (37)$$

with $\bar{G}$ the final effective gravitational constant,

$$\bar{G} = G\left(1 + \frac{a^2}{2\kappa+3a^2} + 2b_1\frac{K_{\rm sur}}{\rho_{0\,\rm sur}}\frac{a}{2\kappa+3a^2}\right). \qquad (38)$$

The last term within the parenthesis depends on the particular properties of the body and could have, in principle, observational consequences. Nevertheless, it's value is too small relative to the second (and body independent) term for it's effect to be measurable.

The method used to set a limit on the actual time variation of the radius of Mercury is based on the observation of surface features [27]. A homologous change ($R \propto M^{1/3}$) cannot be detected through this procedure since it scales all linear dimensions equally. Neither can be observed a variation $R \propto r_b$ ($r_b$ is Bohr radius), since a change in $r_b$ implies a change of all macroscopic dimensions in the same proportion. For these reasons it is necessary to make a change of variables:

$$r^* = \frac{r}{r_b M^{1/3}}, \qquad M^* = \frac{M(r)}{M} = \frac{M_0(r)}{M_0}, \qquad (39)$$

where $M$ is the total mass of Mercury. Taking also into account that $m \propto c^{q-2}$ and $r_b \propto c^{-q}$, Eq. (37) becomes

$$\frac{dp_0}{dr^*} = -\bar{G}(t)\frac{M_0^{2/3}}{r_{b0}}\frac{\rho_{m0}(r^*)M_0^*(r^*)}{r^{*2}}, \qquad (40)$$

with

$$\bar{G}(t) = \bar{G}e^{(b_1+q_3+q_4-4+2/3q+2/3)\psi(t)} \qquad (41)$$

and $r_{b0}$ the Bohr radius for $\psi = 0$. This equation shows that the VSL theory that is being studied is equivalent, with respect to the hydrostatic equilibrium of Mercury, to a theory in which the only "constant" that varies is $G$.

Now we are in a position to use the results of [27]. The variation in the radius of Mercury ($R$) produced by the variation in $G$ can be parametrized as[6]

$$\frac{1}{R}\frac{dR}{dx_0} = -\frac{\delta}{\bar{G}(t)}\frac{d\bar{G}(t)}{dx_0}, \qquad (42)$$

or equivalently

$$\frac{1}{R}\frac{dR}{dt} = -\frac{\delta}{\bar{G}(t)}\frac{d\bar{G}(t)}{dt}, \qquad (43)$$

where $\delta$ is generally a function of $\bar{G}$ and $M$. Using models of Mercury, McElhinny *et al.* obtained the value $\delta = 0.02 \pm 0.005$ for that planet.

### B. Specification of the parameters

We will express the parameters $b_1$, $q_1$, $q_2$, $q_3$, and $q_4$ that have been introduced in this work in terms of the parameters $q$ and $b$ of the VSL theory. In our approximation $M = NM_N$ where $N$ is the total number of nucleons. Equation (7) requires that $N \propto \exp(-q_2\psi)$ while Eq. (12) implies that $M \propto \exp(-b\psi)$ and from the discussion in Sec. II, we see that $M \propto \exp[(q-2-q_2)\psi]$ and consequently $q_2 = b + q - 2$. From Eq. (8) $q_3 = q - 2 + 3q = 4q - 2$ while from Eq. (9) $q_4 = -q_2 + q - 2$. Also we have $b_1 \equiv q_1/2 - q_2 + b = -q + 2$ and as we explained before we consider the $q_1 = 0$ case.

Finally from Eq. (41) we conclude that

$$\bar{G}(t) = \bar{G}\exp\left[\left(b_1 + q_3 + q_4 - 4 + \frac{2}{3}q + \frac{2}{3}\right)\psi\right] = \bar{G}\exp\left[\left(\frac{11}{3}q - b - \frac{10}{3}\right)\psi\right]. \qquad (44)$$

### C. Bound for $\dot{c}/c$

Replacing the previous expression for $\bar{G}(t)$ in Eq. (43) one gets

$$\left(\frac{11}{3}q - b - \frac{10}{3}\right)\dot{\psi}(t) = -\frac{1}{\delta}\frac{\dot{R}}{R} \simeq 0 \pm 5\times10^{-12}\ {\rm y}^{-1}, \qquad (45)$$

with ${\rm y}^{-1} = 1/{\rm year}$. We have taken $\delta = 0.02 \pm 0.005$ and $\frac{\Delta R}{R} = 0 \pm 0.0004$ [27], where $\Delta R$ corresponds to a time interval approximately equal to $3.5 \times 10^9$ years.

This result can be combined with bounds for $\dot{\alpha}/\alpha$ that have been obtained using atomic clocks. We can use e.g. the one obtained in Ref. [29]:

[5] Assuming $b_1$, $a = O(1)$, the quotient between these terms is no larger than approximately $\frac{K_{\rm sur}}{\rho_{0\,\rm sur}}$. Since $K_{\rm sur} \approx 10^{12}\ \frac{\rm dynes}{\rm cm^2}$ and $\rho_{0\,\rm sur} = \rho_{m0\,\rm sur}c_0^2 \approx 4\times10^{21}\ \frac{\rm dynes}{\rm cm^2}$ [28], we see that $\frac{K_{\rm sur}}{\rho_{0\,\rm sur}} \approx 10^{-9}$.

[6] We call "$\delta$" the parameter called "$\alpha$" in [27] to avoid confusion with the fine structure constant.

$$\frac{\dot{\alpha}}{\alpha} = (4.2 \pm 6.9) \times 10^{-15}\ \mathrm{y}^{-1}. \tag{46}$$

In this paper we will consider the $b = 0$ case, which gives an upper bound for all non-negative values of $b$. Moreover, in the VSL theory $\frac{\dot{\alpha}}{\alpha} = q\dot{\psi}$, then Eq. (45) can be written as $-\frac{10}{3}\dot{\psi} = 0 \pm 5 \times 10^{-12}\ \mathrm{y}^{-1} - \frac{11}{3}\frac{\dot{\alpha}}{\alpha}$. Comparing this last equation with Eq. (46) we see that $-\frac{10}{3}\dot{\psi} \approx 0 \pm 5 \times 10^{-12}\ \mathrm{y}^{-1}$. The conclusive result is that

$$\frac{\dot{c}}{c} = \dot{\psi} = 0 \pm 2 \times 10^{-12}\ \mathrm{y}^{-1}. \tag{47}$$

This can be rewritten as a bound for the adimensional quantity $\psi' = H_0^{-1}\dot{\psi}$ after multiplying by the Hubble time $(H_0^{-1})$[7]:

$$\psi' = 0 \pm 3 \times 10^{-2}. \tag{48}$$

## VII. WHITE DWARFS LUMINOSITIES AND THE SCALAR FIELD

White dwarfs are excellent objects to test any energy injection from a scalar field [30,31]. This is due both to their low luminosity, as well as their extremely high heat conductivity, making all energy microscopically released to enhance the total luminosity. Most of them are adequately described by Newtonian physics and a zero temperature approximation, the latter hypothesis providing a polytrope type equation of state (EOS).

### A. The polytropic EOS

Using the subscript 0 to denote quantities without the presence of the scalar field, we write the polytrope equation as $p_0 = K_0\rho^\gamma$. Given Eq. (28), we have

$$p = p_0(\rho) - b_1\gamma\rho\frac{p_0}{\rho}\psi, \tag{49}$$

so we can write

$$p = K_0(1 - b_1\gamma\psi)\rho^\gamma, \tag{50}$$

and we recover a polytrope equation of state with a new constant $K_0 \to K = K_0(1 - b_1\psi\gamma)$. The Lane-Emden function with polytrope index $(\gamma - 1)^{-1}$ still applies, and consequently the expressions for the radius, mass, and internal energy of the star will be the same in terms of the effective constant $K$ [25]:

$$E = -\frac{3\gamma - 4}{5\gamma - 6}\frac{GM^2}{R}, \tag{51}$$

$$R = f^{1/2}\rho_c^{(\gamma-2)/2}\zeta_1, \tag{52}$$

$$M = 4\pi\rho_c^{(3\gamma-4)/2}f^{3/2}\zeta_1^2|\theta'(\zeta_1)|, \tag{53}$$

[7]We have taken $H_0^{-1} \approx 1.5 \times 10^{10}$ years.

where $\rho_c$ is the central density, $\zeta_1$ is the first root of the Lane-Emden function, and we have defined $f = K\gamma/(4\pi G(\gamma - 1))$. Unlike with the topographic analysis of planetary palaeoradii, the energy balance in the star is not invariant under the scaling (39). Thus, from Eq. (37) we see that the $G$ for the internal energy of the star is $G(t) = \bar{G}\exp[(b_1 + q_3 + q_4 - 4)\psi] = \bar{G}\exp[(3q - b - 4)\psi]$. On the other hand, the dependence of the radius $R$ on $\psi$ can be obtained solving Eq. (53) for the central density and replacing it in Eq. (52) [the $\psi$ dependences of the total mass and the compressibility were given in Sec. VI B and in Eq. (50), respectively]. Finally Eq. (51) leads to the following dependence of the internal energy:

$$E \propto \exp\psi f(q, b, \gamma), \tag{54}$$

where

$$f(q, b, \gamma) \equiv 3q - b - 4 + \frac{2\gamma - 4 + b(5 - 5\gamma) + q(3 - \gamma)}{3\gamma - 4}. \tag{55}$$

To go on we assume that the star is in equilibrium in the sense that all the energy injected by the field $\psi$ is radiated away. Therefore the equation for the luminosity induced by the $\psi$ field ($L_\psi$) becomes

$$L_\psi = -\dot{E} = -f(\gamma, q, b)E\dot{\psi}. \tag{56}$$

### B. Comparison with observation

We shall consider only white dwarfs well described by the nonrelativistic value $\gamma = 5/3$, in which case $f(q, b, \gamma) = \frac{13}{3}q - \frac{13}{3}b - \frac{14}{3}$. Unfortunately, only a handful of white dwarfs have well-measured masses, radii, and luminosities: indeed, these four or five stars are used to test the theory of white dwarfs, since the mass-radius relation requires exactly the same parameters we need to carry our comparison of theory and experiment. These stars and their properties have been reviewed in Ref. [32]. Table I shows the adopted values. We have excluded Sirius B from the sample, since relativistic effects are important in this case.

To obtain the bound for $\dot{\psi}$ we rewrite Eq. (56) as

$$\dot{\psi} = \frac{L_\psi}{E}\frac{3}{13b + 14} + \frac{13}{13b + 14}\frac{\dot{\alpha}}{\alpha}. \tag{57}$$

Using again the upper bound (46) for the present time variation of $\alpha$ and bounding $L_\psi$ by the observed luminosity of the white dwarfs ($L$) we obtain

$$\dot{\psi} \le \frac{L}{E}\left|\frac{3}{13b + 14}\right| + \left|\frac{13}{13b + 14}\frac{\dot{\alpha}}{\alpha}\right| \le \dot{\psi}_0\left|\frac{3}{13b + 14}\right| + \left|\frac{13}{13b + 14}\right|1.1 \times 10^{-14}\ \mathrm{y}^{-1}, \tag{58}$$

where

$$\dot{\psi}_0 \equiv \frac{(L/L_\odot)}{(E/E_0)} \frac{1}{\tau_\odot}, \tag{59}$$

and

$$E_0 = -\frac{3}{7} \frac{GM_\odot^2}{R_\odot} \tag{60}$$

is the would be internal energy of the Sun were it described by a Newtonian $\gamma = 5/3$ polytrope. $L_\odot$ is the solar luminosity and

$$\tau_\odot = \frac{E_0}{L_\odot} \simeq 1.32 \times 10^7 \text{ y} \tag{61}$$

is the Kelvin-Helmholtz solar contraction time scale.

The table shows upper bounds for $\dot{\psi}$ in $\text{y}^{-1}$ again for $b = 0$.

Stein 2015B provides the strongest bound. Using a value for the Hubble time $H_0^{-1} \propto 1.5 \times 10^{10}$ y we obtain

$$\frac{1}{H_0} \frac{\dot{c}}{c} = \frac{1}{H_0} \dot{\psi} = 0 \pm 2.1 \times 10^{-3}. \tag{62}$$

Comparing the expressions (45) and (58) we see that white dwarf physics provides the strongest constraints on the VSL theory near the present epoch for almost all values of the $b$ parameter, except for those near $b = -14/13$ which make Eq. (58) uninformative. It is also clear that combining both bounds (45) and (58) we can obtain a bound for $|\dot{\psi}|$ independent from the value of $b$ (although less strong than the bound given for $b \geq 0$): $|\dot{\psi}| \leq 2.2 \times 10^{-12}\ \text{y}^{-1}$.

## VIII. CONCLUSIONS

We have obtained the equations that describe a perfect fluid in the nonrelativistic limit and the first law of thermodynamics in the context of the covariant VSL theory proposed by J. Magueijo. We showed that the field $\psi$ can formally be considered as a new thermodynamic variable and we also showed how to obtain the equations of state in the VSL theory when the corresponding equations for constant $c$ are given.

The nonrelativistic hydrostatic equilibrium equation has the usual form with the gravitational constant $G$ replaced by an effective constant. The different variables (pressure, mass, mass density) depend on the $\psi$ field and so the radius of a planet should vary in time. Using bounds for the variation of the radius of Mercury and the fine structure constant we have set limits on $\dot{c}/c$: $\dot{c}/c = \dot{\psi} = 0 \pm 2 \times 10^{-12}\ \text{y}^{-1}$ (valid for positive values of the $b$ parameter). The most interesting thing about this result is that it gives a bound for $\dot{\psi}$, whereas the known limits for the variation of $\alpha$ and $e$ lead to bounds for the product $q\dot{\psi}$.

Under the same Newtonian approximation we obtained the dependence of the luminosity of a white dwarf on the time variation of the scalar field. The bound obtained is more stringent than the planetary radius bound by an order of magnitude.

The $b = 0$ assumption suggests a null coupling of the scalar field with matter. However, the $q \neq 0$ assumption implies a quantum coupling between $\psi$ and matter, not explicitly shown in the action (2). Of course this and other issues such as the microscopic origin of the energy exchange between the scalar field and ordinary matter as well as whether all the energy injected by $\psi$ on a star is radiated away or not, deserve further work. This we leave for future communications.

## APPENDIX: SPATIAL AND TEMPORAL BEHAVIOR OF $\psi$

After some simplifications valid in the Newtonian limit, the equation for $\psi$ (in the case $a \neq 0$) becomes

$$\Box\psi = \frac{8\pi G}{c^4(2\kappa + 3a^2)} aT. \tag{A1}$$

Since $\psi$ is small, $c^4$ can be replaced by ${c_0}^4$ in the right-hand side of the equation (this is the first step of an iterative process of resolution):

$$\Box\psi = \frac{8\pi G}{c_0^4(2\kappa + 3a^2)} aT. \tag{A2}$$

$T$ can be written as the sum of a spatial part $T_s$ (corresponding to Mercury) and a temporal one $T_t$ (cosmological term). Then $\psi$ can also be separated into spatial ($\psi_s$) and temporal ($\psi_t$) components. The metrics to be used are the quasi-Minkowskian and the cosmological (Robertson-Walker) for the spatial and temporal parts, respectively.

### 1. Spatial behavior

Using that $T \simeq \rho =$ energy density of Mercury, the spatial part of Eq. (A2) can be written:

$$\nabla^2\psi_s = \frac{1}{r^2} \frac{d}{dr}\left(r^2 \frac{d\psi_s}{dr}\right) = \frac{8\pi G}{c_0^4} \frac{a}{2\kappa + 3a^2} \rho. \tag{A3}$$

After integrating one arrives at

TABLE I. Data and bounds for selected white dwarfs (data from Ref. [32]).

| Object | $M/M_\odot$ | $R/R_\odot$ | $L/L_\odot$ | $E/E_0$ | $\dot{\psi}_0$ | $\lvert\dot{\psi}\rvert \leq$ |
|---|---|---|---|---|---|---|
| Procyon B | 0.602 | 0.0123 | $5.8 \times 10^{-4}$ | 29.5 | $1.5 \times 10^{-12}$ | $3.3 \times 10^{-13}$ |
| 40 Eri B | 0.501 | 0.0136 | 0.014 | 18.5 | $5.7 \times 10^{-11}$ | $1.2 \times 10^{-11}$ |
| Stein 2015B | 0.66 | 0.011 | $3.1 \times 10^{-4}$ | 39.6 | $5.9 \times 10^{-13}$ | $1.4 \times 10^{-13}$ |

$$\frac{d\psi_s}{dr} = \frac{2G}{c_0^4}\frac{a}{2\kappa + 3a^2}\frac{U(r)}{r^2}. \tag{A4}$$

This is one of the formulas that has been used in our work. It can help us to obtain an estimation of the variation of $\psi$ inside Mercury:

$$\Delta\psi \approx \frac{2a}{2\kappa + 3a^2}\frac{4}{3}\pi\frac{1}{2}\left(\frac{G}{c_0^2}\bar{\rho}_m R^2\right), \tag{A5}$$

where $\bar{\rho}_m$ and $R$ are the average density and the radius of Mercury, respectively. The first factors are presumably $O(1)$ so

$$\Delta\psi \approx \frac{G}{c_0^2}\bar{\rho}_m R^2 \approx 10^{-11}. \tag{A6}$$

This number is small compared to the bound that is obtained for the variation of $\psi$ in the last $3.9 \times 10^9$ years (the relevant period of time for our study of Mercury) and besides the temporal and spatial behavior of this field can be separated. This justifies the steps of our analysis in which $\psi$ was considered constant inside Mercury.

### 2. Temporal behavior

The temporal part of Eq. (A2) is

$$\Box\psi_t = \frac{8\pi G}{c_0^2}\frac{a}{2\kappa + 3a^2}\rho_{mU}, \tag{A7}$$

where $\rho_{mU}$ is the average density of the Universe. Using the Robertson-Walker metric it becomes

$$\nabla_0\nabla^0\psi = \frac{8\pi G}{c_0^2}\frac{a}{2\kappa + 3a^2}\rho_{mU} + 3\frac{\dot{a}(t)}{a(t)}\dot{\psi}, \tag{A8}$$

where $a(t)$ is the scale factor of the metric and $a$ is one of the parameters of the VSL theory. In the way towards obtaining the gravitational potential equation in the Newtonian limit, the equation $G_0{}^0 = R_0{}^0 - \frac{1}{2}R = \frac{8\pi G}{c^4}T_0{}^0 + a(\nabla_0\nabla^0\psi - \Box\psi)$ appears. We want to show that $\nabla_0\nabla^0\psi$ is negligible compared to the order $\psi$ term in the Taylor's expansion of $\frac{8\pi G}{c^4}T_0{}^0$. To do this it will be demonstrated that each term appearing in Eq. (A8) can be neglected:

(i) First term of Eq. (A8): This term is negligible since $T_0{}^0 \approx \rho_{\rm Mercury}c^2$ and $\frac{\rho_{mU}}{\rho_{\rm Mercury}} \approx 10^{-30}$.

(ii) Second term of Eq. (A8):

$$3\frac{\dot{a}(t)}{a(t)}\dot{\psi} \approx 3\frac{H_0}{c_0^2}\frac{\Delta\psi}{\Delta t} = 3\frac{1}{c_0^2 H_0^{-1}\Delta t}\Delta\psi, \tag{A9}$$

where $\Delta\psi$ represents the change of $\psi$ in a time interval $\Delta t$. Taking $\Delta t \approx 4 \times 10^9$ years (this is the time interval for which we have a bound for the variation of the radius of Mercury), we get

$$3\frac{\dot{a}(t)}{a(t)}\dot{\psi} \approx 10^{-46}\Delta\psi \text{ cm}^{-2}. \tag{A10}$$

This quantity must be compared with

$$(-4)\frac{8\pi G}{c_0^4}\rho\psi \approx 10^2 \times \frac{G}{c_0^2} \times \rho_m \times \psi \approx 10^{-26}\psi \text{ cm}^{-2}, \tag{A11}$$

and hence we see that the second term of Eq. (A8) is also negligible.


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