Light pseudoscalar mesons in a nonlocal $SU(3)$ chiral quark model

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We study the properties of the light pseudoscalar mesons in a three-flavor chiral quark model with nonlocal separable interactions. We concentrate on the evaluation of meson masses and decay constants, considering both the cases of Gaussian and Lorentzian nonlocal regulators. The results are found to be in quite good agreement with the empirical values, in particular in the case of the ratio f_K/f_π and the anomalous decay $\pi^0 \to \gamma\gamma$. In addition, the model leads to a reasonable description of the observed phenomenology in the η-η' sector, even though it implies the existence of two significantly different state mixing angles.

I. INTRODUCTION

The properties of the light pseudoscalar mesons (i.e., the pions, kaons, and etas) provide a suitable ground for the study of the basic nonperturbative features of quantum chromodynamics (QCD). As is well known, the QCD Lagrangian shows an approximate $U(3)_L \otimes U(3)_R$ chiral symmetry, which is spontaneously broken down to $U(3)_V$ in the low-momentum, nonperturbative regime. The fact that, instead of nine, only eight pseudoscalar quasi Goldstone bosons are observed in nature is usually explained in terms of the so-called $U(1)_A$ anomaly. This anomaly is again related to nonperturbative aspects of QCD, and it is believed to be mainly responsible for the rather large η' mass. Unfortunately, so far it has not been possible to obtain detailed information about the properties of the light pseudoscalar mesons directly from QCD, and most of the present theoretical work on the subject relies on low-energy effective theories. Among them the Nambu–Jona-Lasinio (NJL) model [1] and its three-flavor extensions [2–5] are some of the most popular ones. In the NJL model the quark fields interact via local effective vertices which are subject to chiral symmetry. If such an interaction is strong enough, chiral symmetry is spontaneously broken and pseudoscalar Goldstone bosons appear [6]. As an improvement of the local NJL scheme, some covariant nonlocal extensions have been studied in the last few years [7]. Nonlocality arises naturally in the context of several quite successful approaches to low-energy quark dynamics such as, for example, the instanton liquid model [8] and the Schwinger-Dyson resummation techniques [9]. It has been also argued that nonlocal covariant extensions of the NJL model have several advantages over the local scheme. Indeed, nonlocal interactions regularize the model in such a way that anomalies are preserved [10] and charges properly quantized, the effective interaction is finite to all orders in the loop expansion and there is no need to introduce extra cutoffs, soft regulators such as Gaussian functions lead to small next-to-leading order corrections [11], etc. In addition, it has been shown [12,13] that a proper choice of nonlocal regulator and model parameters can lead to some form of quark confinement, in the sense that the effective quark propagator has no poles at real energies.

Until now, most of the research work on nonlocal chiral models has been restricted to the flavor $SU(2)$ sector including applications to the baryonic sector [14,15] and to the study of phase transitions at finite temperature and densities [16]. The aim of the present paper is to extend this type of models so as to include strange degrees of freedom and to analyze the predictions for the masses and decay constants for the pions, kaons, and η-η' system.

This article is organized as follows. In Sec. II we present the general formalism and derive the expressions needed to evaluate the different meson properties. The numerical results for some specific nonlocal regulators together with the corresponding discussions are given in Sec. III, while in Sec. IV we present our conclusions. Finally, we include an appendix with some details concerning the evaluation of quark loop integrals.

II. FORMALISM

A. Effective action

We start with the Euclidean quark effective action

$$
S_E = \int d^4x \left\{ \overline{\psi}(x)[-i\gamma_\mu\partial_\mu + \hat{m}_c]\psi(x) - \frac{G}{2}[j_a^S(x)j_a^S(x) \right.
$$
$$
+ j_a^P(x)j_a^P(x)] - \frac{H}{4}A_{abc}[j_a^S(x)j_b^S(x)j_c^S(x)
$$
$$
\left. - 3j_a^S(x)j_b^P(x)j_c^P(x)] \right\}, \tag{1}
$$

where ψ is a chiral $U(3)$ vector that includes the light quark

fields, $\psi \equiv (u \; d \; s)^T$, while $\hat{m}_c = \text{diag}(m_u, m_d, m_s)$ is the current quark mass matrix. We will work from now on in the isospin symmetry limit, in which $m_u = m_d$. The currents $j_a^{S,P}(x)$ are given by

$$j_a^S(x) = \int d^4y \; d^4z \; r(y-x) r(x-z) \bar{\psi}(y) \lambda_a \psi(z), \quad (2)$$

$$j_a^P(x) = \int d^4y \; d^4z \; r(y-x) r(x-z) \bar{\psi}(y) i \gamma_5 \lambda_a \psi(z), \quad (3)$$

where the regulator $r(x-y)$ is local in momentum space, namely,

$$r(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{-i(x-y)p} r(p), \quad (4)$$

and the matrices λ_a, with $a = 0, \ldots, 8$, are the usual eight Gell-Mann 3×3 matrices—generators of $SU(3)$—plus $\lambda_0 = \sqrt{2/3} \mathbb{1}_{3\times3}$. Finally, the constants A_{abc} are defined by

$$A_{abc} = \frac{1}{3!} \epsilon_{ijk} \epsilon_{mnl} (\lambda_a)_{im} (\lambda_b)_{jn} (\lambda_c)_{kl}. \quad (5)$$

The corresponding partition function $Z = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \times \exp[-S_E]$ can be bosonized in the usual way introducing the scalar and pseudoscalar meson fields $\sigma_a(x)$ and $\pi_a(x)$, respectively, together with auxiliary fields $S_a(x)$ and $P_a(x)$. Integrating out the quark fields we get

$$Z = \int \mathcal{D}\sigma_a \; \mathcal{D}\pi_a \; \det A \int \mathcal{D}S_a \; \mathcal{D}P_a \; \exp\left[\int d^4x(\sigma_a S_a + \pi_a P_a)\right] \exp\left\{\int d^4x\left[\frac{G}{2}(S_a S_a + P_a P_a) + \frac{H}{4}A_{abc}(S_a S_b S_c - 3S_a P_b P_c)\right]\right\}, \quad (6)$$

where the operator A reads, in momentum space,

$$A(p,p') = (-\slashed{p} + \hat{m}_c)(2\pi)^4 \delta^{(4)}(p-p') + r(p)[\sigma_a(p-p') + i\gamma_5 \pi_a(p-p')]\lambda_a r(p'). \quad (7)$$

To perform the integration over the fields S_a and P_a we use the stationary phase approximation (SPA). This means to replace the integral over S_a and P_a by the integrand evaluated at its minimizing values $\tilde{S}_a(\sigma_b(x), \pi_c(x))$ and $\tilde{P}_a(\sigma_b(x), \pi_c(x))$. The latter are required to satisfy

$$\sigma_a + G\tilde{S}_a + \frac{3H}{4}A_{abc}[\tilde{S}_b\tilde{S}_c - \tilde{P}_b\tilde{P}_c] = 0,$$

$$\pi_a + G\tilde{P}_a - \frac{3H}{2}A_{abc}\tilde{S}_b\tilde{P}_c = 0. \quad (8)$$

Thus, within the SPA the bosonized effective action reads

$$S_E = -\ln\det A - \int d^4x\left[\sigma_a \tilde{S}_a + \pi_a \tilde{P}_a + \frac{G}{2}(\tilde{S}_a\tilde{S}_a + \tilde{P}_a\tilde{P}_a) + \frac{H}{4}A_{abc}(\tilde{S}_a\tilde{S}_b\tilde{S}_c - 3\tilde{S}_a\tilde{P}_b\tilde{P}_c)\right]. \quad (9)$$

At this stage we assume that the σ_a fields can have non-trivial translational invariant mean field values $\bar{\sigma}_a$ while the pseudoscalar field cannot. Thus, we write

$$\sigma_a(x) = \bar{\sigma}_a + \delta\sigma_a(x),$$

$$\pi_a(x) = \delta\pi_a(x). \quad (10)$$

Note that due to charge conservation only $\bar{\sigma}_{a=0,3,8}$ can be different from zero. Moreover, $\bar{\sigma}_3$ also vanishes in the isospin limit. After replacing Eqs. (10) in the bosonized effective action (9) and expanding up to second order in the fluctuations $\delta\sigma_a(x)$ and $\delta\pi_a(x)$ we get

$$S_E = S_E^{MFA} + S_E^{quad} + \cdots. \quad (11)$$

Here the mean field action reads

$$\frac{S_E^{MFA}}{V^{(4)}} = -2N_c \int \frac{d^4p}{(2\pi)^4} \text{Tr}\ln[p^2\mathbb{1}_{3\times3} + \Sigma^2(p)] - \frac{1}{2}\left[\sum_i\left(\bar{\sigma}_i\bar{S}_i + \frac{G}{2}\bar{S}_i\bar{S}_i\right) + \frac{H}{2}\bar{S}_u\bar{S}_d\bar{S}_s\right], \quad (12)$$

where for convenience we have changed to a new basis in which S_i, with $i = u, d, s$ (or equivalently $i = 1, 2, 3$), are given by

$$S_u = \sqrt{\frac{2}{3}}S_0 + S_3 + \frac{1}{\sqrt{3}}S_8, \quad S_d = \sqrt{\frac{2}{3}}S_0 - S_3 + \frac{1}{\sqrt{3}}S_8,$$

$$S_s = \sqrt{\frac{2}{3}}S_0 + \frac{2}{\sqrt{3}}S_8,$$

and similar definitions hold for $\bar{\sigma}_i$ in terms of $\bar{\sigma}_0$, $\bar{\sigma}_3$, and $\bar{\sigma}_8$. In Eq. (12) we have also defined $\Sigma(p) = \text{diag}(\Sigma_u(p), \Sigma_d(p), \Sigma_s(p))$, with $\Sigma_i(p) = m_i + \bar{\sigma}_i r^2(p)$, whereas the mean field values $\bar{S}_i$ are given by $\bar{S}_i = \tilde{S}_i(\bar{\sigma}_j, 0)$. Note that in the isospin limit $\bar{\sigma}_u = \bar{\sigma}_d$; thus, we have $\Sigma_u(p) = \Sigma_d(p)$.

In order to deal with the mesonic degrees of freedom, we also introduce a more convenient basis defined by

$$\phi_{ij} = \frac{1}{\sqrt{2}}(\lambda_a \phi_a)_{ij}, \quad (13)$$

where $\phi = \sigma, \pi$, and the indices i, j run from 1 to 3. For the pseudoscalar fields one has, in this way,

$$\delta\pi_{ij}=\begin{pmatrix} \dfrac{\pi^0}{\sqrt{2}}+\dfrac{\eta_8}{\sqrt{6}}+\dfrac{\eta_0}{\sqrt{3}} & \pi^+ & K^+ \\[2ex] \pi^- & -\dfrac{\pi^0}{\sqrt{2}}+\dfrac{\eta_8}{\sqrt{6}}+\dfrac{\eta_0}{\sqrt{3}} & K^0 \\[2ex] K^- & \overline{K}^0 & -\dfrac{2\eta_8}{\sqrt{6}}+\dfrac{\eta_0}{\sqrt{3}} \end{pmatrix}_{ij}. \tag{14}$$

The second piece of the effective action in Eq. (11)—quadratic in the meson fluctuations—can be written now as

$$S_E^{quad}=\frac{1}{2}\int\frac{d^4p}{(2\pi)^4}[G_{ij,kl}^+(p)\delta\sigma_{ij}(p)\delta\sigma_{kl}(-p)$$

$$+G_{ij,kl}^-(p)\delta\pi_{ij}(p)\delta\pi_{kl}(-p)], \tag{15}$$

where we have defined

$$G_{ij,kl}^\pm(p)=C_{ij}^\pm(p)\delta_{il}\delta_{jk}+((r^\pm)^{-1})_{ij,kl}, \tag{16}$$

with

$$C_{ij}^\pm(p^2)=-8N_c\int\frac{d^4q}{(2\pi)^4}$$

$$\times\frac{r^2(q^+)r^2(q^-)[(q^+\cdot q^-)\mp\Sigma_i(q^+)\Sigma_j(q^-)]}{[(q^+)^2+\Sigma_i^2(q^+)][(q^-)^2+\Sigma_j^2(q^-)]},$$

$$q^\pm=q\pm p/2, \tag{17}$$

$$r_{ij,kl}^\pm=G\,\delta_{il}\delta_{jk}\pm\frac{H}{2}\epsilon_{ikn}\epsilon_{jln}\overline{S}_n. \tag{18}$$

B. Mean field approximation and chiral condensates

The mean field values $\overline{\sigma}_u$ and $\overline{\sigma}_s$ can be found by minimizing the action S_E^{MFA}. Taking into account Eqs. (8), a straightforward exercise leads to the following set of coupled "gap equations":

$$\overline{\sigma}_u+G\overline{S}_u+\frac{H}{2}\overline{S}_u\overline{S}_s=0,$$

$$\overline{\sigma}_s+G\overline{S}_s+\frac{H}{2}\overline{S}_u^2=0, \tag{19}$$

where

$$\overline{S}_i=-8N_c\int\frac{d^4p}{(2\pi)^4}\frac{\Sigma_i(p)r^2(p)}{p^2+\Sigma_i^2(p)}. \tag{20}$$

The chiral condensates are given by the vacuum expectation values $\langle\overline{u}u\rangle=\langle\overline{d}d\rangle$ and $\langle\overline{s}s\rangle$. They can be easily ob-

tained by performing the variation of $Z^{MFA}=\exp[-S_E^{MFA}]$ with respect to the corresponding current quark masses. For $q=u,s$ one obtains

$$\langle\overline{q}q\rangle=-4N_c\int\frac{d^4p}{(2\pi)^4}\frac{\Sigma_q(p)}{p^2+\Sigma_q^2(p)}. \tag{21}$$

C. Meson masses and quark-meson coupling constants

From the quadratic effective action S_E^{quad} it is possible to derive the scalar and pseudoscalar meson masses as well as the quark-meson couplings. In what follows we will consider explicitly only the case of pseudoscalar mesons. The corresponding expressions for the scalar sector are completely equivalent, just replacing the upper indices "−" by "+." In terms of physical fields, the contribution of the pseudoscalar mesons to S_E^{quad} can be written as

$$S_E^{quad}|_P=\frac{1}{2}\int\frac{d^4p}{(2\pi)^4}\{G_\pi(p^2)[\pi^0(p)\pi^0(-p)$$

$$+2\pi^+(p)\pi^-(-p)]+G_K(p^2)[2K^0(p)\overline{K}^0(-p)$$

$$+2K^+(p)K^-(-p)]+G_\eta(p^2)\eta(p)\eta(-p)$$

$$+G_{\eta'}(p^2)\eta'(p)\eta'(-p)\}. \tag{22}$$

Here, the fields η and η' are related to the $U(3)_V$ states η_0 and η_8 according to

$$\eta=\cos\theta_\eta\,\eta_8-\sin\theta_\eta\,\eta_0, \tag{23}$$

$$\eta'=\sin\theta_{\eta'}\,\eta_8+\cos\theta_{\eta'}\,\eta_0, \tag{24}$$

where the mixing angles $\theta_{\eta,\eta'}$ are defined in such a way that there is no η-η' mixing at the level of the quadratic action. The functions $G_P(p^2)$ introduced in Eq. (22) are given by

$$G_\pi(p^2)=\left[\left(G+\frac{H}{2}\overline{S}_s\right)^{-1}+C_{uu}^-(p^2)\right], \tag{25}$$

$$G_K(p^2)=\left[\left(G+\frac{H}{2}\overline{S}_u\right)^{-1}+C_{us}^-(p^2)\right], \tag{26}$$

$$G_\eta(p^2) = \frac{G_{88}^-(p^2) + G_{00}^-(p^2)}{2}$$
$$- \sqrt{[G_{08}^-(p^2)]^2 + \left[\frac{G_{88}^-(p^2) - G_{00}^-(p^2)}{2}\right]^2}, \quad (27)$$

$$G_{\eta'}(p^2) = \frac{G_{88}^-(p^2) + G_{00}^-(p^2)}{2}$$
$$+ \sqrt{[G_{08}^-(p^2)]^2 + \left[\frac{G_{88}^-(p^2) - G_{00}^-(p^2)}{2}\right]^2}, \quad (28)$$

where

$$G_{88}^-(p^2) = \frac{1}{3}\left[\frac{6G - 4H\overline{S}_u - 2H\overline{S}_s}{2G^2 - GH\overline{S}_s - H^2\overline{S}_u^2} + C_{uu}^-(p^2) + 2C_{ss}^-(p^2)\right],$$

$$G_{08}^-(p^2) = \frac{\sqrt{2}}{3}\left[\frac{H(\overline{S}_s - \overline{S}_u)}{2G^2 - GH\overline{S}_s - H^2\overline{S}_u^2} + C_{uu}^-(p^2) - C_{ss}^-(p^2)\right],$$

$$G_{00}^-(p^2) = \frac{1}{3}\left[\frac{6G + 4H\overline{S}_u - H\overline{S}_s}{2G^2 - GH\overline{S}_s - H^2\overline{S}_u^2} + 2C_{uu}^-(p^2) + C_{ss}^-(p^2)\right].$$
$$(29)$$

The meson masses are obtained by solving the equations

$$G_P(-m_P^2) = 0, \quad (30)$$

with $P = \pi$, K, η, and η', while the η and η' mixing angles, which are in general different from each other, are given by

$$\tan 2\theta_{\eta,\eta'} = \left.\frac{2G_{08}^-(p^2)}{G_{00}^-(p^2) - G_{88}^-(p^2)}\right|_{p^2 = -m_{\eta,\eta'}^2}. \quad (31)$$

Now the meson fields have to be renormalized, so that the residues of the corresponding propagators at the meson poles are set equal to 1. This means that one should define renormalized fields $\tilde{\phi}(p) = Z_\phi^{-1/2}\phi(p)$ such that, close to the poles, the quadratic effective Lagrangian reads

$$\mathcal{L}_E^{quad}\big|_\phi = \frac{1}{2}(p^2 + m_\phi^2)\tilde{\phi}(p)\tilde{\phi}(-p). \quad (32)$$

In this way, the wave function renormalization constants Z_P are given by

$$Z_P^{-1} = \left.\frac{dG_P(p^2)}{dp^2}\right|_{p^2 = -m_P^2}, \quad (33)$$

with $P = \pi$, K, η, and η'. Finally, the meson-quark coupling constants G_{Pq} are given by the original residues of the meson propagators at the corresponding poles:

$$G_{Pq} = Z_P^{1/2}. \quad (34)$$

D. Weak decay constants of pseudoscalar mesons

By definition, the pseudoscalar meson weak decay constants are given by the matrix elements of the axial currents $A_\mu^a(x)$ between the vacuum and renormalized one-meson states at the meson pole:

$$\langle 0|A_\mu^a(0)|\tilde{\phi}_b(p)\rangle = if_{ab}p_\mu. \quad (35)$$

For $a, b = 1, \ldots, 7$, the constants f_{ab} can be written as $\delta_{ab}f_\phi$, with $\phi = \pi$ for $a = 1,2,3$ and $\phi = K$ for $a = 4-7$. In contrast, as occurs with the mass matrix, the decay constants become mixed in the $a = 0,8$ sector.

In order to obtain the expression for the axial current, one has to gauge the effective action S_E by introducing a set of axial gauge fields $\mathcal{A}_\mu^a$. For a local theory, this gauging procedure can be done simply by performing the replacement

$$\partial_\mu \to \partial_\mu + \frac{i}{2}\gamma_5\lambda_a\mathcal{A}_\mu^a(x). \quad (36)$$

However, since in the present case we are dealing with nonlocal fields, an extra replacement has to be performed in the regulator $r(x-y)$ [12,13,17]. One has

$$r(x-y) \to P\exp\left[\frac{i}{2}\int_x^y ds_\mu\gamma_5\lambda_a\mathcal{A}_\mu^a(s)\right]r(x-y), \quad (37)$$

where s represents an arbitrary path that connects x with y. In the present work we will use the so-called "straight line path," which means

$$s_\mu = x_\mu + \alpha(y_\mu - x_\mu), \quad (38)$$

with $0 \leq \alpha \leq 1$. Once the gauged effective action is obtained, it is straightforward to get the axial current as the derivative of such action with respect to $\mathcal{A}_\mu^a(x)$ evaluated at $\mathcal{A}_\mu^a(x) = 0$. Then, performing the derivative of the resulting expressions with respect to the renormalized meson fields, one can finally identify the corresponding meson weak decay constants. After a rather lengthy calculation, we find that the pion and kaon decay constants are given by

$$f_\pi = 4f_{uu}(-m_\pi^2)Z_\pi^{1/2}, \quad (39)$$

$$f_K = 2[f_{us}(-m_K^2) + f_{su}(-m_K^2)]Z_K^{1/2}, \quad (40)$$

where

$$f_{ij}(p^2) = 2N_c\int\frac{d^4q}{(2\pi)^4}\frac{(p\cdot q_\alpha^+)}{p^2}$$
$$\times\left.\frac{r(q_\alpha^+)r(q_\alpha^-)\Sigma_j(q_\alpha^-)}{[(q_\alpha^+)^2 + \Sigma_i^2(q_\alpha^+)][(q_\alpha^-)^2 + \Sigma_j^2(q_\alpha^-)]}\right|_{\alpha=1/2}$$
$$+ 4N_c\int\frac{d^4q}{(2\pi)^4}\frac{(p\cdot q)}{p^2}\frac{dr(q)}{dq^2}$$

$$\times \int_0^1 d\alpha \frac{1}{(q_\alpha^+)^2 + \Sigma_i^2(q_\alpha^+)}$$

$$\times \left\{ r(q_\alpha^-)[\Sigma_i(q_\alpha^+) - m_i] \right.$$

$$\times \frac{[(q_\alpha^+ \cdot q_\alpha^-) + \Sigma_i(q_\alpha^+)\Sigma_j(q_\alpha^-)]}{(q_\alpha^-)^2 + \Sigma_j^2(q_\alpha^-)}$$

$$\left. + r(q_\alpha^+)\Sigma_i(q_\alpha^+) \right\}, \qquad (41)$$

with

$$q_\alpha^+ = q + (1-\alpha)p,$$

$$q_\alpha^- = q - \alpha p. \qquad (42)$$

In the case of the η-η' system, two decay constants can be defined for each component $a = 0,8$ of the axial current [18]. They can be written in terms of the f_{ab} decay constants and the previously defined mixing angles $\theta_{\eta,\eta'}$ as

$$f_\eta^a = [f_{a8}(-m_\eta^2)\cos\theta_\eta - f_{a0}(-m_\eta^2)\sin\theta_\eta]Z_\eta^{1/2}, \quad (43)$$

$$f_{\eta'}^a = [f_{a8}(-m_{\eta'}^2)\sin\theta_{\eta'} + f_{a0}(-m_{\eta'}^2)\cos\theta_{\eta'}]Z_{\eta'}^{1/2}. \qquad (44)$$

Within our model, the decay constants f_{ab} for $a,b = 0,8$ are related to the f_{ij} defined in Eq. (41) by

$$f_{88}(p^2) = \frac{4}{3}[2f_{ss}(p^2) + f_{uu}(p^2)], \qquad (45)$$

$$f_{00}(p^2) = \frac{4}{3}[2f_{uu}(p^2) + f_{ss}(p^2)], \qquad (46)$$

$$f_{08}(p^2) = f_{80}(p^2) = \frac{4\sqrt{2}}{3}[f_{uu}(p^2) - f_{ss}(p^2)]. \qquad (47)$$

It is clear that both the nondiagonal decay constants f_{08}, f_{80} as well as the mixing angles θ_η and $\theta_{\eta'}$ vanish in the $SU(3)$ symmetry limit.

E. Anomalous $P \to \gamma\gamma$ decays

To go further with the analysis of light pseudoscalar meson decays, let us evaluate the anomalous decays of π^0, η, and η' into two photons. In general, the corresponding amplitudes can be written as

$$A(P \to \gamma\gamma) = e^2 g_{P\gamma\gamma}\epsilon_{\mu\nu\alpha\beta}\varepsilon_1^{*\mu}\varepsilon_2^{*\nu}k_1^\alpha k_2^\beta, \qquad (48)$$

where $P = \pi^0, \eta, \eta'$, and k_i and ε_i stand for the momenta and polarizations of the outgoing photons respectively.

In the nonlocal model under consideration the coefficients $g_{P\gamma\gamma}$ are given by quark loop integrals. Besides the usual "triangle" diagram, given by a closed quark loop with one meson and two photon vertices, in the present nonlocal scheme one has a second diagram [13] in which one of the quark-photon vertices arises from the gauge contribution to the regulator; see Eq. (37). The sums of both diagrams for π^0, η and η' decays yield

$$g_{\pi\gamma\gamma} = I_u(m_\pi^2)Z_\pi^{1/2},$$

$$g_{\eta\gamma\gamma} = \frac{1}{3\sqrt{3}}\Big[[5I_u(m_\eta^2) - 2I_s(m_\eta^2)]\cos\theta_\eta$$
$$- \sqrt{2}[5I_u(m_\eta^2) + I_s(m_\eta^2)]\sin\theta_\eta]Z_\eta^{1/2},$$

$$g_{\eta'\gamma\gamma} = \frac{1}{3\sqrt{3}}\Big[[5I_u(m_{\eta'}^2) - 2I_s(m_{\eta'}^2)]\sin\theta_{\eta'}$$
$$+ \sqrt{2}[5I_u(m_{\eta'}^2) + I_s(m_{\eta'}^2)]\cos\theta_{\eta'}]Z_{\eta'}^{1/2}, \qquad (49)$$

where the loop integrals $I_f(m_P^2)$ are given by

$$I_f(m_P^2) = \frac{8}{3}N_c \int \frac{d^4q}{(2\pi)^4} \frac{r(q-k_2)r(q+k_1)}{[q^2+\Sigma_f^2(q)][(q+k_1)^2+\Sigma_f^2(q+k_1)][(q-k_2)^2+\Sigma_f^2(q-k_2)]}$$

$$\times \left\{ \Sigma_f(q) + \frac{q^2}{2}\left[\frac{[\Sigma_f(q-k_2)-\Sigma_f(q)]}{(k_2\cdot q)} - \frac{[\Sigma_f(q+k_1)-\Sigma_f(q)]}{(k_1\cdot q)}\right]\right\} \qquad (50)$$

[notice that for on-shell photons these integrals are only functions of the scalar product $(k_1 \cdot k_2)$, which in Euclidean space is equal to $-m_P^2/2$]. In terms of the parameters $g_{P\gamma\gamma}$, the corresponding decay widths are simply given by

$$\Gamma(P \to \gamma\gamma) = \frac{\pi}{4}m_P^3\alpha^2 g_{P\gamma\gamma}^2, \qquad (51)$$

where α is the fine structure constant.

F. Low-energy theorems

Chiral models are expected to satisfy some basic low-energy theorems. In this subsection we consider some important relations such as the Goldberger-Treiman (GT) and Gell-Mann–Oakes–Renner (GOR), showing explicitly that they are indeed verified by the present model.

We start with the GT relation. Taking the chiral limit $m_\pi^2 \to 0$ in $f_{uu}(-m_\pi^2)$ and $dG_\pi/dp^2|_{p^2=-m_\pi^2}$ appearing in Eqs. (39) and (33) we get

$$\lim_{m_\pi^2 \to 0} f_{uu}(-m_\pi^2) = \frac{1}{4}\overline{\sigma}_{u,0}Z_{\pi,0}^{-1},\quad (52)$$

where here, as in the rest of this subsection, the subindex 0 indicates that the corresponding quantity is evaluated in the chiral limit [notice that $\overline{\sigma}_{u,0}=\Sigma_{u,0}(0)$]. Replacing this expression in Eq. (39) and taking into account Eq. (34) we get

$$f_{\pi,0}G_{\pi q,0}=\overline{\sigma}_{u,0},\quad (53)$$

which is equivalent to the GT relation at the quark level in our model.

Let us turn to the GOR relation. Expanding $C_{uu}^-(p^2)$ in Eq. (17) to leading order in m_u and p^2 we get

$$C_{uu}^-(p^2)\simeq\frac{\overline{S}_{u,0}}{\overline{\sigma}_{u,0}}-m_u\frac{\langle\overline{u}u+\overline{d}d\rangle_0}{\overline{\sigma}_{u,0}^2}+p^2 Z_{\pi,0}^{-1}.\quad (54)$$

To obtain this result we have used the gap equations (19) and the expression for the chiral condensate given in Eq. (21). Now using Eq. (54) together with the equation for the pion mass,

$$G_\pi(-m_\pi^2)=0,\quad (55)$$

and taking into account the GT relation (53), one gets

$$m_u\langle\overline{u}u+\overline{d}d\rangle_0=-f_{\pi,0}^2 m_\pi^2,\quad (56)$$

which is the form taken by the well-known GOR relation in the isospin limit.

Next we discuss the validity of the Feynman-Hellman (FH) theorem for the case of the so-called pion sigma term. This theorem implies the relation [19]

$$\frac{dm_\pi^2}{dm_u}=\langle\pi|\overline{u}u+\overline{d}d|\pi\rangle,\quad (57)$$

where covariant normalization $\langle p'|p\rangle=2E_p(2\pi)^3\delta^3(\vec{p}'-\vec{p})$ has been used for the pion field. An expression for the left-hand side of Eq. (57) can be easily obtained by deriving Eq. (55) with respect to the u-quark mass. In fact,

$$0=\frac{dG_\pi(-m_\pi^2)}{dm_u}=\frac{\partial G_\pi(p^2)}{\partial m_u}\bigg|_{p^2=-m_\pi^2}$$
$$+\frac{\partial G_\pi(p^2)}{\partial p^2}\bigg|_{p^2=-m_\pi^2}\left(-\frac{dm_\pi^2}{dm_u}\right),\quad (58)$$

and thus

$$\frac{dm_\pi^2}{dm_u}=\frac{\dfrac{\partial G_\pi(p^2)}{\partial m_u}}{\dfrac{\partial G_\pi(p^2)}{\partial p^2}}\Bigg|_{p^2=-m_\pi^2}.\quad (59)$$

On the other hand, within the path integral formalism, one has

$$\langle\pi^a|\overline{u}u+\overline{d}d|\pi^b\rangle=\frac{\delta^3 S_E^j[j]}{\delta j(0)\,\delta\pi^a(p)\,\delta\pi^b(-p)}\bigg|_{\substack{\pi=j=0\\p^2=-m_\pi^2}}\times Z_\pi,\quad (60)$$

where

$$S_E^j[j]=S_E+\int d^4x[\overline{u}(x)u(x)+\overline{d}(x)d(x)]j(x),\quad (61)$$

S_E being the effective action of the model, given by Eq. (1). From the explicit form of S_E it is easy to see that

$$S_E^j[j]=S_E[m_u\to m_u+j(x)];\quad (62)$$

therefore, using the bosonized form of the effective action in Eq. (11), with S_E^{quad} given by Eq. (22), we get

$$\langle\pi|\overline{u}u+\overline{d}d|\pi\rangle=Z_\pi\frac{\partial G_\pi^j[j]}{\partial j(0)}\bigg|_{\substack{j=0\\p^2=-m_\pi^2}}$$

$$=\frac{1}{Z_\pi^{-1}}\frac{\partial G_\pi(p^2)}{\partial m_u}\bigg|_{p^2=-m_\pi^2}$$

$$=\frac{\dfrac{\partial G_\pi(p^2)}{\partial m_u}}{\dfrac{\partial G_\pi(p^2)}{\partial p^2}}\Bigg|_{p^2=-m_\pi^2}.\quad (63)$$

Comparing Eq. (63) with Eq. (59) we see that the FH theorem, as it should, holds in the present model. Moreover, using the GOR relation, Eq. (56), we obtain, up to leading order in m_u,

$$\langle\pi|\overline{u}u+\overline{d}d|\pi\rangle_0=-\frac{\langle\overline{u}u+\overline{d}d\rangle_0}{f_{\pi,0}^2}=\frac{m_\pi^2}{m_u}.\quad (64)$$

To conclude, let us analyze in the chiral limit the coupling $g_{\pi\gamma\gamma}$. Expanding the integrand of $I_u(m_\pi^2)$ in powers of k_1, k_2 and taking the limit $m_\pi^2 \to 0$ one obtains [20]

$$g_{\pi\gamma\gamma,0} = \frac{Z_{\pi,0}^{1/2}}{4\pi^2\sigma_{u,0}}. \tag{65}$$

Now, taking into account the GT relation, one finally has

$$g_{\pi\gamma\gamma,0} = \frac{1}{4\pi^2 f_{\pi,0}}, \tag{66}$$

which is the expected result according to low-energy theorems and chiral perturbation theory.

III. NUMERICAL RESULTS

In this section we discuss the numerical results obtained within the above-described nonlocal model. Our results include the values of meson masses, decay constants, and mixing angles, as well as the corresponding quark constituent masses, quark condensates, and quark-meson couplings. The numerical calculations have been carried out for two different regulators often used in the literature: the Gaussian regulator

$$r(p^2) = \exp(-p^2/2\Lambda^2) \tag{67}$$

and a Lorentzian regulator

$$r(p^2) = (1 + p^2/\Lambda^2)^{-1}, \tag{68}$$

where Λ is a free parameter of the model, playing the role of an ultraviolet cutoff momentum scale. Let us recall that these regulators are defined in Euclidean momentum space.

A. Fits to physical observables

The nonlocal model under consideration includes five free parameters. These are the current quark masses $\bar{m}$ and m_s ($\bar{m} = m_u = m_d$), the coupling constants G and H, and the cutoff scale Λ. In our numerical calculations we have chosen to fix the value of $\bar{m}$, whereas the remaining four parameters have been determined by requiring that the model reproduce correctly the measured values of four physical quantities. The observables we have used here are the pion and kaon masses, the pion decay constant f_π, and a fourth quantity, chosen to be alternatively the η' mass or the squared $\eta' \to \gamma\gamma$ decay constant $g_{\eta'\gamma\gamma}^2$. In the case of the Gaussian (Lorentzian) regulator, we find that for $\bar{m}$ above a critical value $m_{crit} \simeq 8.3$ MeV (3.9 MeV) the quark propagators have only complex poles in Minkowski space. This can be understood as a sort of quark confinement [12,13]. In contrast, for $\bar{m} \lesssim m_{crit}$ one finds that u- and d-quark Euclidean propagators do have at least two doublets of purely imaginary poles (i.e., real poles in Minkowski space).

In Table I we quote our numerical results for several quantities of interest. Besides the obtained values of meson masses and decay constants, we include in this table the cor-

responding results for quark condensates, constituent quark masses, and quark-meson coupling constants. We have taken into account both Gaussian and Lorentzian regulators, considering in each case values for the light quark mass $\bar{m}$ above and below m_{crit}. Sets GI, GIV, LI, and LIV have been determined by fitting the free parameters so as to reproduce the empirical value of $m_{\eta'}$, while for sets GII, GIII, LII, and LIII the η' mass is obtained as a theoretical prediction and the parameters have been determined by adjusting $g_{\eta'\gamma\gamma}^2$ to its present central experimental value. The last column of Table I shows the empirical values of masses and decay constants to be compared with our predictions.

As stated in Sec. II C, the meson masses are obtained by solving the equations $G_P(-m_P^2) = 0$ for $P = \pi$, K, η, and η'. Now, to perform the corresponding numerical calculations, one has to deal with the functions $C_{ij}(p^2)$ evaluated at Euclidean momentum $p^2 = -m_P^2$. These functions, defined by Eq. (17), include quark loop integrals that need to be treated with special care when the meson mass exceeds a given value $m_P > 2S_i$, where S_i is the imaginary part (in Euclidean space) of the first pole of the quark propagator. In practice this may happen in the case of the η' state, and physically it corresponds to a situation in which the meson mass is beyond a pseudothreshold of decay into a quark-antiquark state. A detailed discussion on this subject is given in the Appendix. In particular, it is seen that for $\bar{m} > m_{crit}$ (i.e., if the propagator has no purely imaginary poles) the quark loop integrals can be regularized in such a way that their imaginary parts vanish identically, and consequently the η' width corresponding to this unphysical decay is zero. In contrast, for $\bar{m} < m_{crit}$ the width is in general nonzero, and the presence of an imaginary part implies that the condition $G_{\eta'}(-m_{\eta'}^2) = 0$ cannot be satisfied. In any case, it is still possible to define the η' mass by looking at the minimum of $|G_{\eta'}(-p_0^2)|$. The situation is illustrated in Fig. 1, where we plot the absolute values of the functions $G_P(-p_0^2)$ for sets GI and GIV (upper and lower panels, respectively). For set GI ($\bar{m} > m_{crit}$) the $u\bar{u}$ pseudothreshold is reached at about 1 GeV, above the η' mass; thus, the quark loop integrals are well defined and $G_{\eta'}(-m_{\eta'}^2) = 0$. It can be seen that this is also the situation for the parameter sets GII, GIII, LII, and LIII. On the other hand, for set GIV ($\bar{m} < m_{crit}$, lower panel in Fig. 1) the $u\bar{u}$ pseudothreshold is reached at ~ 750 MeV, well below the η' mass. As stated, in this case one can define $m_{\eta'}$ by looking at the minimum of the function $|G_{\eta'}(-p_0^2)|$, which is represented by the dash-dotted curve. Though different from zero, the (unphysical) η' width is relatively small, and the minimum (which, as required, lies at $p_0 = 958$ MeV) is close to the horizontal axis in the chosen GeV^2 scale. Therefore we do not expect our results to be spoiled by confinement effects not included in the model. In any case, we believe that for $\bar{m} < m_{crit}$ the values of physical parameters related to η' decay may not be reliable, and they have not been included in Table I. A similar situation occurs for set LIV. Finally, in the case of set LI it turns out that while $\bar{m} > m_{crit}$ (no purely imaginary poles), the $u\bar{u}$ pseudo-

TABLE I. Numerical results for quark effective masses and condensates, and pseudoscalar meson masses, and decay parameters. The light quark mass $\overline{m}$ has been taken as input, and the parameters m_s, Λ, G, and H have been chosen so as to reproduce the empirical values of pion and kaon masses, the pion decay constant f_π, and, alternatively, the η' mass or the measured value of $g^2_{\eta'\gamma\gamma}$ (marked with an asterisk).

Set		Gaussian				Lorentzian				Empirical
		GI	GII	GIII	GIV	LI	LII	LIII	LIV	
$\overline{m}$	[MeV]	8.5	8.5	8.5	7.5	4.0	4.0	4.0	3.5	(3.4−7.4)
m_s	[MeV]	223	223	223	199	112	110	112	100	(108−209)
Λ	[MeV]	709	709	709	768	1013	1013	1013	1110	
$G\Lambda^2$		10.99	11.44	10.80	10.43	14.68	16.76	14.96	14.05	
$-H\Lambda^5$		295.3	275.4	303.7	305.1	743.4	573.8	720.4	821.0	
$-\langle\overline{u}u\rangle^{1/3}$	[MeV]	211	211	211	220	275	275	275	288	
$-\langle\overline{s}s\rangle^{1/3}$	[MeV]	186	187	185	204	297	307	299	314	
$\Sigma_u(0)$	[MeV]	313	313	313	295	299	299	300	281	
$\Sigma_s(0)$	[MeV]	650	662	645	607	562	615	569	518	
m_π	[MeV]	139*	139*	139*	139*	139*	139*	139*	139*	139
m_K	[MeV]	495*	495*	495*	495*	495*	495*	495*	495*	495
m_η	[MeV]	517	505	521	522	543	513	540	545	547
$m_{\eta'}$	[MeV]	958*	879	1007	958*	958*	778	908	958*	958
$G_{\pi q}$		3.28	3.28	3.28	3.09	3.13	3.13	3.13	2.94	
G_{Kq}		3.47	3.52	3.45	3.21	3.05	3.24	3.08	2.80	
$G_{\eta q}$		3.07	3.03	3.08	2.83	2.74	2.69	2.74	2.49	
$G_{\eta' q}$		1.62	2.01	1.21		1.36	2.21	1.13		
f_π	[MeV]	93.3*	93.3*	93.3*	93.3*	93.3*	93.3*	93.3*	93.3*	93.3
f_K/f_π		1.29	1.29	1.29	1.29	1.25	1.28	1.26	1.25	1.22
$g^2_{\pi\gamma\gamma}$	[GeV^{-2}]	0.073	0.073	0.073	0.074	0.074	0.074	0.074	0.074	0.075±0.005
$g^2_{\eta\gamma\gamma}$	[GeV^{-2}]	0.095	0.106	0.091	0.094	0.072	0.108	0.075	0.075	0.067±0.006
$g^2_{\eta'\gamma\gamma}$	[GeV^{-2}]	0.141	0.116*	0.116*		0.278	0.116*	0.116*		0.116±0.005

threshold occurs below the η' mass. In this case, in order to evaluate the loop integrals we have followed the regularization procedure described in the Appendix, in which the imaginary parts of quark loop integrals vanish, leading to a real η' pole.

By examining Table I it can be seen that, for the chosen values of $\overline{m}$, the results for the quark condensate $\langle\overline{u}u\rangle$, the ratio $\langle\overline{s}s\rangle/\langle\overline{u}u\rangle$, and the constituent quark masses $\Sigma_q(0)$ are similar to the values obtained within most theoretical studies [6,9]. The most remarkable differences between the results corresponding to both regulators are found in the values of the current quark masses and the quark condensates. For the Gaussian regulator the parameters $\overline{m}$ and m_s are found to be about 8 MeV and 200 MeV, respectively, while for the Lorentzian regulator the corresponding values are approximately 4 MeV and 100 MeV. Note, however, that the obtained values of the ratio $m_s/\overline{m}$ are very similar and somewhat above the present phenomenological range $(m_s/m_u)^{emp}=17-22$ [18]. On the contrary, as expected from the GT relation and its generalizations, quark condensates are found to be higher for the Lorentzian regulator parameter sets. In any case, the results for both regulators are in rea-

sonable agreement with standard phenomenological values [21] and the most recent lattice QCD estimates [22].

The predicted values for the kaon decay constant are also phenomenologically acceptable. Indeed, the prediction for the ratio f_K/f_π turns out to be significantly better than that obtained in the standard NJL model, where the kaon and pion decay constants are found to be approximately equal to each other [6] in contrast with experimental evidence. It is also worth noticing that we obtain a very good prediction for the $\pi^0\to\gamma\gamma$ decay rate. In this sense the nonlocal model shows a further degree of consistency in comparison with the standard local NJL model, where the quark momenta in the anomalous diagrams should go beyond the cutoff limit in order to get a good agreement with the experimental value [7].

Our results for η and η' masses and $\eta, \eta'\to\gamma\gamma$ anomalous decays require a more detailed discussion. In the Gaussian regulator case, it is seen that set GI, while fitting $m_{\eta'}$, leads to a rather large value for $g^2_{\eta'\gamma\gamma}$. On the contrary, for set GII, which fits the value of $g^2_{\eta'\gamma\gamma}$, the η' mass decreases up to a value of about 880 MeV (of course, one can also choose intermediate sets between these two). In addition, it is
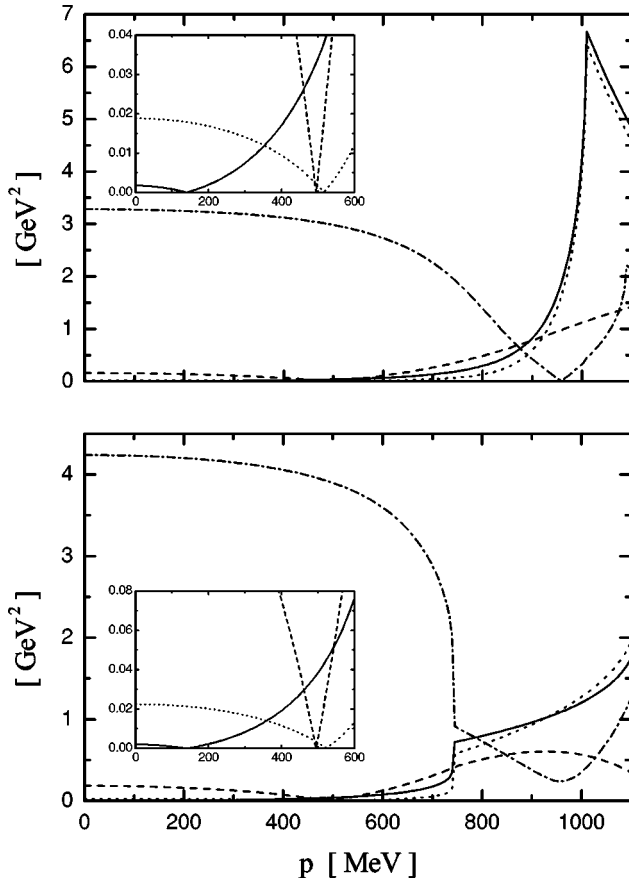
FIG. 1. Absolute values of the inverse meson propagators as functions of the momentum. The solid line corresponds to G_π, the dashed line to G_K, the dotted line to G_η, and the dash-dotted line to $G_{\eta'}$. The upper panel corresponds to set GI and the lower one to set GIV. In both cases the insets show a detail of the low-momentum region where the zeros corresponding to the ground-state pion, kaon, and eta mesons occur.

found that the fit to $g^2_{\eta'\gamma\gamma}$ has a second solution—namely, the parameter set GIII—which leads to a η' mass of about 1 GeV. However, in this case $m_{\eta'}$ is found to be very close to the pseudothreshold point, and as a consequence both the values of the η'-quark coupling $G_{\eta'q}$ and the decay constant $g_{\eta'\gamma\gamma}$ are quite sensible to small changes in the parameters. On the other hand, for all four Gaussian regulator sets the results for m_η and $g_{\eta\gamma\gamma}$ do not change significantly. The values for the η mass are found to be in relatively good agreement with experiment, while $g^2_{\eta\gamma\gamma}$ turns out to be somewhat large. In the case of the Lorentzian regulator, the above-described situation becomes strengthened: set LI leads to an unacceptably large value for $g_{\eta'\gamma\gamma}$ (notice that, as stated, here $m_{\eta'}$ is above the pseudothreshold), while set LII leads to a too low η' mass. Set LIII seems to reproduce reasonably all measurable quantities, but as in the Gaussian regulator case, the result for $g_{\eta'\gamma\gamma}$ is highly unstable. In general, as a conclusion, one could say that the Gaussian regulator is preferred, leading to a reasonable global fit of all quantities considered here.

It is worth mentioning that the chosen value of $\bar{m}$ cannot be very far from the values considered in Table I. For higher

$\bar{m}$ one would obtain too low values for the quark condensates. On the other hand, if $\bar{m} \gg m_{crit}$, the η' mass turns out to be very far above the $\bar{u}u$ pseudothreshold, implying the existence of a large unphysical η' width.

B. η and η' mixing angles and decay constants

The problem of defining and (indirectly) fitting mixing angles and decay constants for the η-η' system has been reexamined several times in the literature (see Ref. [23], and references therein). On general grounds one has to deal with two different state mixing angles θ_P and four decay constants f_P^a, where $P = \eta, \eta'$ and $a = 0,8$. In standard analyses, however, it has been usual to parametrize the mixing between both meson states and decay constants using a single parameter (a mixing angle usually called θ) and introducing two decay constants f_8 and f_0 related to $g_{\eta\gamma\gamma}$ and $g_{\eta'\gamma\gamma}$ through low-energy equations analogous to Eq. (66). In the last few years, the analysis has been improved (mainly in the framework of next-to-leading order chiral perturbation theory), and several authors have considered the possible nonorthogonality of $(f_\eta^8, f_{\eta'}^8)$ and $(f_\eta^0, f_{\eta'}^0)$ [24,25], as well as that of the states η and η' [26]. For the sake of comparison, we follow here Ref. [24] and express the four decays constants f_P^a in terms of two decay constants f_a and two mixing angles θ_a, where $a = 0,8$: namely,

$$\begin{pmatrix} f_\eta^8 & f_\eta^0 \\ f_{\eta'}^8 & f_{\eta'}^0 \end{pmatrix} = \begin{pmatrix} f_8 \cos\theta_8 & -f_0 \sin\theta_0 \\ f_8 \sin\theta_8 & f_0 \cos\theta_0 \end{pmatrix}. \quad (69)$$

In our model, the decay constants f_P^a can be calculated from Eqs. (43)–(47). As shown below, it turns out that the angles θ_8 and θ_0 are in general different. This also happens with the mixing angles θ_η and $\theta_{\eta'}$, whose expressions are given in Eq. (31). Notice that these are consequences of the (rather strong) p^2 dependence of the functions $C_{ij}(p^2)$ and $f_{ij}(p^2)$ defined in Eqs. (17) and (41), respectively.

Our numerical results for the the parameters f_a, θ_a introduced in Eq. (69) and the mixing angles $\theta_{\eta,\eta'}$ are collected in Table II. In the last column of the table we quote the ranges in which the parameters f_a, θ_a fall within most popular theoretical approaches. We have taken these values from the analysis in Ref. [23], in which the results from different parametrizations have been translated to the four-parameter decay constant scheme given by Eq. (69). By looking at Table II it is seen that the results corresponding to Gaussian regulator sets lie within the range of values quoted in the literature, while in the case of the Lorentzian regulator the most remarkable difference corresponds to set LI, which leads to a large value of f_0 (this can be related with the unacceptably large value of $g_{\eta'\gamma\gamma}$ discussed above). The mixing angles θ_a can be compared to those obtained in a recent Bethe-Salpeter approach calculation [27], which leads to a somewhat larger (absolute) value of θ_0. As stated, for sets GIV and LIV the values for the η' decay constants have not been computed in view of the unphysical finite width acquired by the η' meson.

TABLE II. Numerical results for η and η' weak decay parameters and state mixing angles, according to the definitions given by Eqs. (31) and (69).

Set	GI	GII	GIII	GIV	LI	LII	LIII	LIV	Theory and phenomenology
f_8/f_π	1.33	1.34	1.32		1.32	1.36	1.31		(1.2–1.4)
f_0/f_π	1.28	1.18	1.18		1.64	1.18	1.05		(1.0–1.3)
θ_8	$-14.2°$	$-19.7°$	$-10.0°$		$-11.7°$	$-25.4°$	$-10.6°$		$-(22°-19°)$
θ_0	$-2.14°$	$-5.25°$	$-1.17°$		$1.66°$	$-6.52°$	$1.60°$		$-(10°-0°)$
θ_η	4.65	1.81	5.69	5.32	7.69	-0.72	6.90	8.00	
$\theta_{\eta'}$	-50.0	-47.1	-52.4		-48.4	-44.6	-50.6		

Finally, notice that for most parameter sets (both Gaussian and Lorentzian) the mixing angle θ_η gets a small positive value, whereas the angle $\theta_{\eta'}$ lies around $-45°$. That is to say, we find our mixing scheme to be very far from the approximation of a single mixing angle θ. For this reason we believe there is no reason to expect θ_η to be within the usually quoted range $\theta = -(10°-20°)$.

IV. CONCLUSIONS

In this work we have studied the properties of light pseudoscalar mesons in a three-flavor chiral quark model with nonlocal separable interactions, in which the $U(1)_A$ breaking is incorporated through a nonlocal dimension-9 operator of the type suggested by 't Hooft. We consider the situation in which the Minkowski quark propagator has poles at real energies as well as the case where only complex poles appear, which has been proposed in the literature as a realization of confinement. We concentrate on the evaluation of the masses and decay constants of the pseudoscalar mesons for two different nonlocal regulators: namely, Gaussian and Lorentzian.

As general conclusions, it is found that in this model the prediction for the ratio f_K/f_π turns out to be significantly better than that obtained in the standard NJL model, where the kaon and pion decay constants are found to be approximately equal to each other in contrast with the experimental evidence. In addition, the model overcomes the standard local NJL problem of treating the anomalous quark loop integrals in a consistent way. With respect to the η-η' system, according to our analysis the Gaussian regulator seems to be more adequate than the Lorentzian one to reproduce the observed phenomenology. In general, our results are in reasonable agreement with experimentally measured values, and the global fits can be still improved by considering regulators of more sophisticated shapes. Alternatively, one might consider adding degrees of freedom not explicitly included in the present calculations such as explicit vector and axial-vector interactions, two-gluon components for the η and η' mesons, etc. Finally, it is worth remarking that our fits lead to an η-η' system in which the $U(3)$ states η_8 and η_0 are mixed by two angles θ_η and $\theta_{\eta'}$ that appear to be significantly different from each other.

APPENDIX: EVALUATION OF QUARK LOOP INTEGRALS

In this appendix we describe some details concerning the evaluation of the quark loop integrals $C_{ij}(p^2)$ defined in Eq. (17). Notice that in the nonlocal chiral model analyzed in this work all four-momenta are defined in Euclidean space. However, in order to determine the meson masses, the external momenta p in the loop integrals have to be extended to the spacelike region. Hence, without loss of generality, we choose $p = (\vec{0}, ip_0)$ and use three-momentum rotational invariance to write the quark loop integrals as

$$C_{ij}(-p_0^2) = \int dq_3\, dq_4\, q_3^2$$

$$\times \frac{F_{ij}(q_3, q_4, p_0)}{[(q^+)^2 + \Sigma_i^2(q^+)][(q^-)^2 + \Sigma_j^2(q^-)]},$$

(A1)

where $q_3 = |\vec{q}|$. The explicit form of $F_{ij}(q_3, q_4, p_0)$, which depends on the meson under consideration, can be easily obtained by comparing Eqs. (A1) and (17). In principle, the integration in Eq. (A1) has to be performed over the half-plane $q_3 \epsilon [0, \infty)$, $q_4 \epsilon (-\infty, \infty)$. For sufficiently small values of p_0 the denominator does not vanish at any point of this integration region. However, when p_0 increases it might happen that some of the poles of the integrand pinch the integration region, making the loop integral divergent. In such cases one has to find a way to redefine the integral in order to obtain a finite result. In practice, we need to extend the calculation of the loop integrals $C_{ij}(-p_0^2)$ to relatively large values of p_0 only when trying to determine the η' mass. For this particle, one only has to deal with integrals in which i

$=j$; therefore, we will restrict ourselves here to this case and the indices i,j will be dropped from now on.

Let us start by analyzing the zeros of the denominator in the integrand in Eq. (A1). This denominator can be written as $D=D^+D^-$, where

$$D^{\pm}=(q^{\pm})^2+\Sigma^2(q^{\pm}). \tag{A2}$$

It is clear that the zeros of D are closely related to the poles of the quark propagator $S(q)=[-\not{q}+\Sigma(q)]^{-1}$. In what follows we will assume that the regulator is such that the propagator has a numerable set of poles in the complex q plane and that there are no cuts. It is not hard to see that these poles appear in multiplets that can be characterized by two real numbers (S_r^{ν},S_i^{ν}), with $S_r^{\nu}\geq0$, $S_i^{\nu}>0$. The index $\nu\epsilon\mathbb{N}$ has been introduced to label the multiplets, with the convention that ν increases for increasing S_i^{ν}. It is convenient to distinguish between two different situations: (a) there are some purely imaginary poles—i.e., there are one or more ν for which $S_r^{\nu}=0$—and (b) no purely imaginary pole exists—i.e., $S_r^{\nu}>0$ for all ν. It can be shown that purely imaginary poles show up as doublets located at Euclidean momentum $(\sqrt{q^2})^{\nu}=\pm iS_i^{\nu}$, while complex poles [28] appear as quartets located at $(\sqrt{q^2})^{\nu}=S_r^{\nu}\pm iS_i^{\nu}$ and $(\sqrt{q^2})^{\nu}=-S_r^{\nu}\pm iS_i^{\nu}$. Clearly, the number and position of the poles depend on the specific shape of the regulator. For the Gaussian interaction, three different situations might occur. For values of $\bar{\sigma}$ below a certain critical value $\bar{\sigma}_c$, two pairs of purely imaginary simple poles and an infinite set of quartets of complex simple poles appear. It is possible to check that in this case one of the purely imaginary doublets is the multiplet which has the smallest imaginary part ($\nu=1$, according to our convention). At $\bar{\sigma}=\bar{\sigma}_c$, the two pairs of purely imaginary simple poles turn into a doublet of double poles with $S_r=0$, while for $\bar{\sigma}>\bar{\sigma}_c$ only an infinite set of quartets of complex simple poles is obtained. In the case of the Lorentzian interactions, there is also a critical value above which purely imaginary poles at

low momenta cease to exist. However, for this family of regulators the total number of poles is always finite.

As stated, for low enough external momentum p_0 the integrand in Eq. (A1) does not diverge along the integration region. As p_0 increases, the first set of poles to be met is that with the lowest value of S_i, namely $(\sqrt{q^2})^{\nu=1}$. In the calculation of the meson properties mentioned in the main text, we deal with relatively low external momenta, so that the effect of higher poles is never observed. Thus, in order to simplify the discussion we will only consider in what follows the first pole multiplet, dropping the upper index ν. The extension to the case in which other sets of poles become relevant will be briefly commented at the end of this appendix.

The denominator D vanishes when $D^+=0$ and/or $D^-=0$—i.e., when

$$(q^+)^2=q_3^2+q_4^2-\frac{p_0^2}{4}+iq_4p_0=S_r^2-S_i^2\pm2iS_rS_i \tag{A3}$$

and/or

$$(q^-)^2=q_3^2+q_4^2-\frac{p_0^2}{4}-iq_4p_0=S_r^2-S_i^2\pm2iS_rS_i. \tag{A4}$$

Solving these equations for q_4 we get in general eight different solutions. Four of them are given by

$$q_4^{(3,1)}=-\frac{S_iS_r}{\gamma(q_3,S_i,S_r)}+i\left(\pm\gamma(q_3,S_i,S_r)-\frac{p_0}{2}\right), \tag{A5}$$

$$q_4^{(4,2)}=-\frac{S_iS_r}{\gamma(q_3,S_i,S_r)}+i\left(\pm\gamma(q_3,S_i,S_r)+\frac{p_0}{2}\right), \tag{A6}$$

where

$$\gamma(q_3,S_i,S_r)=\sqrt{\frac{q_3^2+(S_i^2-S_r^2)+\sqrt{q_3^4+2q_3^2(S_i^2-S_r^2)+(S_i^2+S_r^2)^2}}{2}}, \tag{A7}$$

and the other four solutions are $q_4^{(i)\prime}=-\mathrm{Re}(q_4^{(i)})+i\,\mathrm{Im}(q_4^{(i)})$, with $i=1,\ldots,4$. In Eqs. (A5) and (A6), $q_4^{(3,1)}$ correspond to the zeros of D^+ and $q_4^{(4,2)}$ to those of D^-, while a similar correspondence holds for $q_4^{(i)\prime}$. For purely imaginary poles one has $S_r=0$; hence, only four independent solutions exist.

If p_0 is relatively small, the distribution of the poles in the complex q_4 plane is that represented in Fig. 2. Figure 2a holds for situation (a), in which the poles in the first multiplet are purely imaginary, while Fig. 2b corresponds to case (b), in which these poles are complex. In both figures the

dots indicate the zeros of D^+ and the squares those of D^-. As we see, for small values of p_0 half of the poles of D^+ [namely, $q_4^{(1)}$ for case (a) and $q_4^{(1)}$ and $q_4^{(1)\prime}$ for case (b)] are placed below the real axis, whereas the other half [$q_4^{(3)}$ for case (a) and $q_4^{(3)}$ and $q_4^{(3)\prime}$ for case (b)] lie above it. Something similar happens for the poles of D^-. Now, as p_0 increases, the poles move in the direction indicated by the arrows. For a certain value of p_0, the poles $q_4^{(2)}$ and $q_4^{(3)}$ meet on the real p_4 axis [the same obviously happens with $q_4^{(2)\prime}$ and $q_4^{(3)\prime}$ in case (b)], thus pinching the integration
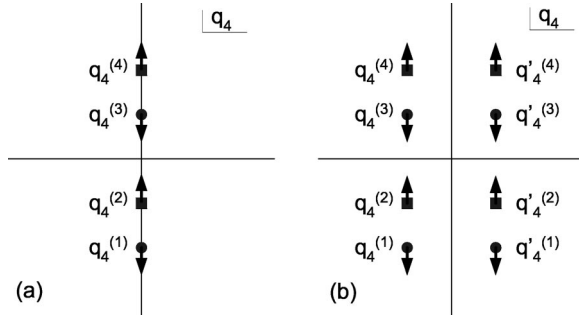
FIG. 2. Schematic distribution in the complex q_4 plane of the poles corresponding to the lowest S_i set for (a) $\bar{\sigma}<\bar{\sigma}_c$ and (b) $\bar{\sigma}>\bar{\sigma}_c$. In both cases the dots indicate the poles of D^+ and the squares those of D^-. These distributions correspond to a value of $p_0<2S_i$. The arrows indicate the movement of the poles as p_0 increases.

region of the (q_3,q_4) integral in Eq. (A1). The location of this so-called "pinch point" in the (q_3,q_4) plane, which we denote by (q_3^p,q_4^p), is given by the solution of $\text{Im}\,q_4^{(2)}=\text{Im}\,q_4^{(3)}=0$:

$$(q_3^p,q_4^p)=\left(\frac{\sqrt{(p_0^2-4S_i^2)(p_0^2+4S_r^2)}}{2p_0},\pm\frac{2S_iS_r}{p_0}\right). \quad (A8)$$

As we see, for $p_0<2S_i$ there is no pinch point [actually, it occurs for a complex value of q_3, outside the integration region in Eq. (A1)], while for $p_0\geqslant 2S_i$ one or two pinch points exist depending on whether $S_r=0$ [case (a)] or $S_r\neq 0$ [case (b)]. In this way, for $p_0>2S_i$ the integral in Eq. (A1) turns out to be ill defined.

In order to find a proper regularization procedure, let us analyze a simpler situation in which the problem might be solved in Minkowski space through the usual "$i\epsilon$" prescription. We consider the loop integral that appears in the usual Nambu–Jona-Lasinio model with three-momentum cutoff:

$$I_2(-p_0^2)=2i\lim_{\epsilon\to 0^+}\int^{\Lambda_3}\frac{d^4q_M}{(2\pi)^4}$$

$$\times\frac{1}{[(q_M^+)^2-m^2+i\epsilon][(q_M^-)^2-m^2+i\epsilon]}, \quad (A9)$$

where we have added the subscript M to stress the fact that here the momenta are defined in Minkowski space. For sufficiently small values of p_0, even in the limit $\epsilon\to 0^+$, the integral is convergent and no regularization is needed. Thus one can simply perform the Wick rotation $p_4=ip_0$ and take the limit $\epsilon\to 0^+$ even before performing the integration. One gets, in this way,
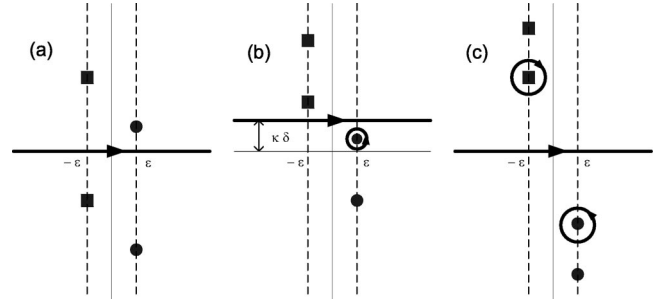

FIG. 3. Integration paths in the complex q_4 plane for (a) $q_3>\sqrt{p_0^2/4-m^2}+\delta$, (b) $\sqrt{p_0^2/4-m^2}-\delta<q_3<\sqrt{p_0^2/4-m^2}+\delta$, and (c) $q_3<\sqrt{p_0^2/4-m^2}-\delta$. The constant κ is an arbitrary real number satisfying $\kappa>1$.

$$I_2(p_4^2)=-2\int^{\Lambda_3}\frac{d^4q_E}{(2\pi)^4}\frac{1}{[(q_E^+)^2+m^2][(q_E^-)^2+m^2]}, \quad (A10)$$

which is an integral of the type given in Eq. (A1). Note that the poles of the propagators are such that this situation belongs to case (a), with $S_i=m$. For $p_0>2m$ the straightforward transformation from Minkowski to Euclidean space mentioned above cannot be done, since some poles go through the integration contours. The question is whether the result of the well-defined Miskowskian integral (A9) can be still recovered if one starts with the Euclidean integral (A10), which is ill defined for $p_0>2m$ due to the presence of a pinch point at $(q_3^p,q_4^p)=(\sqrt{p_0^2/4-m^2},0)$. It is not hard to prove that the answer is yes, once the q_4 integration contours and the pole positions are conveniently modified. The procedure requires one to introduce two small parameters ϵ and δ and take the limit $\delta\to 0^+$, $\epsilon\to 0^+$ at the end of the calculation. The parameter ϵ is used to shift the poles of D^+ and D^- (see Fig. 3), whereas δ is used to split the q_3 integration interval in three subintervals: the first region corresponds to $q_3>q_3^p+\delta$, the second to $q_3^p-\delta<q_3<q_3^p+\delta$, and the third to $q_3<q_3^p-\delta$. For each q_3 region we define a different q_4 integration contour, as represented in Fig. 3 (in the second region, Fig. 3b, also an arbitrary constant $\kappa>1$ is introduced). In fact, in the first and third regions the limit $\epsilon\to 0^+$ can be taken even before performing the integrations. These two q_3 regions give the full contribution to the real part of the result. However, more care has to be taken with the intermediate region, which is responsible for the full contribution to the imaginary part. For example—as it is well known from the "$i\epsilon$" Minkowskian formulation—changing the sign of ϵ does not affect the real part of the result but does change the sign of the imaginary part.

The prescription just described can be now applied to regularize any loop integral of the form given in Eq. (A1) in which $p_0>2S_i$. Let us consider first the case (a), for which the lowest set of poles has $S_r=0$ and the pinch point is located at $(q_3^p,q_4^p)=(\sqrt{p_0^2/4-S_i^2},0)$. Defining $C^{(a)}(-p_0^2)$ as an integral of the form given by Eq. (A1) for which only one set of purely imaginary poles contributes, we get

$$\mathrm{Re}[C^{(a)}(-p_0^2)] = \lim_{\delta \to 0^+} \left\{ R^{(a)}(-p_0^2, \delta) + \int_0^{q_3^p - \delta} dq_3 \int_{-\infty}^{\infty} dq_4 \frac{q_3^2 F(q_3, q_4, p_0)}{[(q^+)^2 + \Sigma^2(q^+)][(q^-)^2 + \Sigma^2(q^-)]} \right.$$

$$\left. + \int_{q_3^p + \delta}^{\infty} dq_3 \int_{-\infty}^{\infty} dq_4 \frac{q_3^2 F(q_3, q_4, p_0)}{[(q^+)^2 + \Sigma^2(q^+)][(q^-)^2 + \Sigma^2(q^-)]} \right\}, \tag{A11}$$

$$\mathrm{Im}[C^{(a)}(-p_0^2)] = -\frac{\pi^2}{2p_0} \frac{q_3^p F(q_3^p, 0, p_0)}{\left[1 + \dfrac{\partial \Sigma^2(q)}{\partial q^2} \Big|_{q^2 = -S_i^2} \right]^2}. \tag{A12}$$

Here $R^{(a)}(-p_0^2, \delta)$ is the so-called "residue contribution," responsible for the cancellation of the divergence appearing in the integrals in Eq. (A11) in the limit $\delta \to 0^+$. Its explicit expression reads

$$R^{(a)}(-p_0^2, \delta) = 2\pi \int_0^{q_3^p - \delta} \frac{dq_3}{\sqrt{q_3^2 + S_i^2}} \mathrm{Re} \left[\frac{q_3^2 F(q_3, q_4, p_0)}{[1 + \partial \Sigma^2(q^-)/\partial(q^-)^2][(q^+)^2 + \Sigma^2(q^+)]} \right]_{q_4 = q_4^{(2)} = i(p_0/2 - \sqrt{q_3^2 + S_i^2})}. \tag{A13}$$

For case (b) we have to extend the previous analysis to the situation in which the poles are complex even in the limit $\epsilon \to 0^+$. In this case there is an ambiguity on how to extend the "$i\epsilon$" prescription already in Minkowski space. Here we will follow the suggestion made in Ref. [29], in which opposite signs of ϵ are used for each pole and its Hermitian conjugate (both defined in Minkowski space). In our Euclidean framework, this corresponds to choose different signs of ϵ for sets $p_4^{(i)}$ and $p_4^{(i)'}$. It is not hard to see that with this prescription the contributions to the imaginary part of the quark loop integral coming from both sets of poles cancel each other. In this way, defining $C^{(b)}(-p_0^2)$ as an integral of the type given in Eq. (A1) for which only one set of complex poles contributes, we get

$$\mathrm{Im}[C^{(b)}(-p_0^2)] = 0. \tag{A14}$$

For the real part one obtains an expression similar to Eq. (A11), just replacing $R^{(a)}(-p_0^2, \delta)$ by $R^{(b)}(-p_0^2, \delta)$, with

$$R^{(b)}(-p_0^2, \delta) = 4\pi \int_0^{q_3^p - \delta} dq_3 \mathrm{Re} \left[\frac{q_3^2 F(q_3, q_4, p_0)}{[1 + \partial \Sigma^2(q^-)/\partial(q^-)^2][(q^+)^2 + \Sigma^2(q^+)](iq_4 + p_0/2)} \right]_{q_4 = q_4^{(2)}}. \tag{A15}$$

In principle, the extension of the present analysis to the situation in which further sets of poles are relevant is rather straightforward. However, some care has to be taken if one has more than one set of purely imaginary poles, since in that case double poles might show up for $p_0 > S_i^1 + S_i^2$.

One is faced with a similar problem in the calculation of the $\eta' \to \gamma\gamma$ decay constant $g_{\eta'\gamma\gamma}$, where the loop integral $I_u(m_{\eta'}^2)$ defined by Eq. (50) is divergent for $m_{\eta'} > 2S_i$. Though the situation is slightly more involved (one finds two pinch points instead of one), our regularization prescriptions can be trivially extended to include this case. In the same way, the method could be extended to more complicated situations, such as, e.g., those found in Schwinger-Dyson type of calculations [9,30]. Finally, we note that although our prescription has some similarities with that used in Ref. [13], the regularization procedure is not exactly the same. In the (q_3, q_4) integral which appears in Eq. (A11) the excluded region around the pinch point is a slide of infinite size in the q_4 direction and size 2δ in the q_3 direction, as opposed to the circular region of radius δ used in Ref. [13]. This leads to some minor differences in the numerical values of the regulated integrals.

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