

Dyons in $\mathcal{N}=4$ gauged supergravity

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We study monopole and dyon solutions to the equations of motion of the bosonic sector of $\mathcal{N}=4$ gauged supergravity in four dimensional space-time. A static, spherically symmetric ansatz for the metric, gauge fields, dilaton, and axion leads to soliton solutions which, in the electrically charged case, have compact spatial sections. Both analytical and numerical results for the solutions are presented.

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I. INTRODUCTION

The interest in gravitating monopole and other soliton solutions to the equations of motion of non-Abelian gauge theories coupled to gravity has recently increased in view of the role that these solutions may play as backgrounds of gauged supergravity models in the context of the anti-de Sitter (AdS) conformal field theory CFT correspondence.

Most of the known solutions are available only numerically. An exception is the self-gravitating monopole solution constructed by Chamseddine and Volkov (ChV) by solving the first order Bogomol'nyi-Prasad-Sommerfield (BPS) equations for an $\mathcal{N}=4$ gauged supergravity model in four dimensional space [1,2]. Different BPS solutions of the same model were also found in [3]. The ChV solution corresponds to a regular magnetic monopole and a geometry which is not asymptotically flat, the dilaton potential providing a position dependent negative cosmological term $\Lambda[\phi]<0$. This solution is purely magnetic and the axion field is set to zero. A one-parameter family of non-BPS monopole-like solutions to the equations of motion of this model was numerically constructed in [4]. In some finite range of the parameter $w^{(2)}$, which is related to the gauge field behavior at the origin, solutions are globally regular. Remarkably, when the parameter takes values in one of the regions outside this range, the corresponding solutions have compact spatial sections; i.e., the metric exhibits a singularity of the “bag of gold” type.

Among other related self-gravitating Yang-Mills solutions, particularly interesting are those found in [5–7] for the Einstein-Yang-Mills theory in asymptotically anti-de Sitter ($\Lambda<0$) space. In this case, monopole and dyon solutions are shown to exist (while for $\Lambda\geq 0$ electrically charged solutions are forbidden [5–8]). Interestingly enough, these dyon solutions, which have been shown to be stable in some cases, exhibit a noninteger magnetic charge, which shows their nontopological character.

It is the purpose of this work to construct dyon solutions to the equations of motion in the bosonic sector of $\mathcal{N}=4$ gauged supergravity in four dimensional space (the Freedman-Schwarz model [9]). We investigate static, spheri-

cally symmetric field configurations which, in view of the fact that they should carry electric charge, necessarily include a nontrivial axion field (which for simplicity was set to zero in the purely magnetic solutions of [1–4]). As we shall see, electrically charged BPS solutions which are regular at the origin do not exist. Then, one has to study the second order equations of motion which after the static, spherically symmetric ansatz reduce to a system of six coupled nonlinear radial equations that has to be solved numerically. Again, one finds a family of solutions now labeled by three parameters: the one already present for purely magnetic solutions and two new ones related to the axion and the electric field. The properties of these electrically charged solutions radically differ from those of the BPS monopole solution. First, the solutions exhibit a bag of gold behavior (singular geometry) in the whole range of parameters. After the study of geodesics for such configurations, one confirms that the geometry has a real singularity at some finite value ρ_* of the “radial” coordinate ρ . Moreover, in contrast with the BPS monopole solution and related to the singular geometry, the charges have a distinctive behavior and in particular the magnetic charge is finite but noninteger. Now, in the context of the AdS/CFT correspondence one should investigate whether the four dimensional singularity is acceptable or not, following either the criterion proposed in [10] or uplifting the solution to ten dimensions and making an analysis similar to that in [11]. This requires a thoughtful study on how the peculiar properties of dyon solutions manifest in $d=10$, an issue that is left for future investigations.

The paper is organized as follows. In Sec. II we introduce the model and derive the equations of motion with the spherically symmetric ansatz. We also define the conserved electric and magnetic charges. Purely magnetic and dyon solutions are discussed in Secs. III and IV, respectively. The analysis of the asymptotic behavior of the solutions leads to a precise characterization of bag of gold singularities and how they affect the resulting values of magnetic and electric charges. Finally, in Sec. V we give a summary of our results.

II. THE MODEL

Let us start by writing bosonic part of the $d=4$ Freedman-Schwarz [9] supergravity model with Lagrangian

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$$\begin{aligned}
L = & \frac{R}{4} - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} \exp(-4\phi) \partial_\mu \mathbf{a} \partial^\mu \mathbf{a} \\
& - \frac{\exp(2\phi)}{4} \sum_{(\sigma)=1,2} \frac{1}{g_{(\sigma)}^2} F^{(\sigma)a}{}_{\mu\nu} F^{(\sigma)a}{}_{\mu\nu} \\
& - \frac{1}{2} \mathbf{a} \sum_{(\sigma)=1,2} \frac{1}{g_{(\sigma)}^2} *F^{(\sigma)a}{}_{\mu\nu} F^{(\sigma)a}{}_{\mu\nu} + \frac{1}{8} (g_{(1)}^2 + g_{(2)}^2) \\
& \times \exp(-2\phi) \quad (1)
\end{aligned}$$

We have chosen the signature of the space-time metric as $(-1,1,1,1)$ and the convention for the Riemann and Ricci tensors are $R^\alpha{}_{\beta\mu\nu} = \partial_\mu \Gamma^\alpha{}_{\beta\nu} - \dots$ and $R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}$. The gauge group is the direct product $SU(2) \times SU(2)$ and we call $A_\mu^{(\sigma)a}$ the corresponding gauge connections $[(\sigma)=1,2]$. The field strength is defined as

$$F_{\mu\nu}^{(\sigma)a} = \partial_\mu A_\nu^{(\sigma)a} - \partial_\nu A_\mu^{(\sigma)a} + \varepsilon_{abc} A_\mu^{(\sigma)b} A_\nu^{(\sigma)c}. \quad (2)$$

The dual tensor is defined as

$$*F_{\mu\nu}^{(\sigma)a} = \frac{1}{2} \sqrt{|g|} \varepsilon_{\mu\nu\alpha\beta} F^{(\sigma)a\alpha\beta}. \quad (3)$$

As usual, we denote by ϕ the dilaton field and by \mathbf{a} the axion.

We now consider Lagrangian (1) when the model is truncated so that $A_\mu^{(2)a} = 0$ (and $g_{(2)} = 0$) and write the corresponding equations of motion [omitting the $(\sigma) = 1$ index]

$$\begin{aligned}
\nabla_\mu \nabla^\mu \phi = & \frac{1}{2g^2} \exp(2\phi) F_{\mu\nu}^a F^{a\mu\nu} - 2 \exp(-4\phi) \partial_\mu \mathbf{a} \partial^\mu \mathbf{a} \\
& + \frac{1}{4} g^2 \exp(-2\phi) \nabla_\mu [\exp(2\phi) F^{a\mu\nu}] \\
& + \exp(2\phi) \varepsilon_{abc} A_\mu^b F^{c\mu\nu} = 2 *F^a{}_{\nu\mu} \partial_\mu \mathbf{a}, \\
R_{\mu\nu} = & 2 \partial_\mu \phi \partial_\nu \phi + 2 \exp(-4\phi) \partial_\mu \mathbf{a} \partial_\nu \mathbf{a} + \frac{2}{g^2} \exp(2\phi) \\
& \times \left(\frac{1}{4} F_{\mu\alpha}^a F_{\nu}^{a\alpha} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta}^a F^{a\alpha\beta} \right) - \frac{g^2}{4} g_{\mu\nu} \\
& \times \exp(-2\phi) \nabla_\mu \nabla^\mu [\exp(-4\phi) \mathbf{a}] \\
= & - \frac{1}{2g^2} *F_{\alpha\beta}^a F^{a\alpha\beta}. \quad (4)
\end{aligned}$$

As mentioned in the Introduction, Chamseddine and Volkov found exact static solutions for the bosonic sector of the truncated Freedman-Schwarz model for purely magnetic gauge fields, A_0^a and $\mathbf{a} = 0$ [1,2]. In fact, first order Bogomol'nyi equations [12] were obtained by analyzing the equations for Killing spinors and then an exact globally regular solution [also solving the second order equations (4)] was found. Contrary to the expectation that a neutral solution was

to be obtained since the model has no Higgs field, the behavior of the gauge field solution corresponds to a regular magnetic monopole. The nontrivial dilaton field provides a potential supporting such a solution, implying that the resulting geometry is not asymptotically flat. Although the absence of the Higgs scalar prevents the definition of a gauge invariant magnetic field, one can define a magnetic flux which for the Chamseddine-Volkov exact solution takes the unit value.

Finding dyon solutions (i.e., both magnetically and electrically charged) to Eqs. (4) is considerably more complicated, since the existence of nontrivial electric field strength components prevents to set the axion field to zero. Even the search of more tractable first order Bogomol'nyi equations becomes more involved since the compatibility condition for the Killing spinor equations, now including an axion field is highly nontrivial.

Let us start by proposing a static spherically symmetric ansatz for the space-time metric and the dilaton, axion, and gauge fields,

$$ds^2 = -N(r) \sigma^2(r) dt^2 + \frac{1}{N(r)} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2), \quad (5)$$

$$\begin{aligned}
A = & A_\mu^a t^a dx^\mu \\
= & u(r) t^3 dt + w(r) (-t^2 d\theta + t^1 \sin \theta d\varphi) + t^3 \cos \theta d\varphi \quad (6)
\end{aligned}$$

$$\phi = \phi(r), \quad \mathbf{a} = a(r), \quad (7)$$

where we have set $g=1$ without loss of generality. Let us note that the static, spherically symmetric ansatz (6) for the gauge field can be obtained from the general one introduced in [13,14] by a singular gauge transformation (see [6] for details) so that $w(r)$ can be associated with that introduced in the case of the 't Hooft-Polyakov monopole [15,16] through the ansatz $A_i^a = (1-w) \varepsilon_{aij} x^j / r^2$ and u with the scalar potential for the Julia-Zee dyon [17,18], $A_0^a = u(r) x^a / r$.

Inserting ansatz (5)–(7) into Eq. (4) one can write six independent radial equations of motion in the form

$$\begin{aligned}
r^2 \exp(-4\phi) \sigma^2 N^2 a'^2 + r^2 \sigma^2 N^2 \phi'^2 - r \sigma N^2 \sigma' \\
+ 2 \exp(2\phi) u^2 w^2 + 2 \exp(2\phi) \sigma^2 N^2 w'^2 = 0, \quad (8)
\end{aligned}$$

$$\begin{aligned}
4r^3 \sigma N \sigma' + 4r^2 \sigma^2 (-1 + N + rN') + 4 \exp(2\phi) \sigma^2 (w^2 - 1)^2 \\
+ 4 \exp(2\phi) r^4 u'^2 - r^4 \sigma^2 \exp(-2\phi) = 0, \quad (9)
\end{aligned}$$

$$\begin{aligned}
\sigma^2 N (2Nw' \phi' + w' N' + Nw'') + u^2 w + \sigma \sigma' N^2 w' \\
+ \sigma^2 N w \frac{1}{r^2} (1 - w^2) + 2 \sigma N u w a' \exp(-2\phi) = 0, \quad (10)
\end{aligned}$$

$$\begin{aligned}
& r^2 \phi'' N \sigma + \phi' (r^2 N' \sigma + 2r N \sigma + r^2 N \sigma') - \frac{r^2}{4} \sigma \exp(-2\phi) \\
& + 2r^2 \exp(-4\phi) \sigma N a'^2 + 2\sigma^{-1} N^{-1} u^2 w^2 \exp(2\phi) \\
& + r^2 \sigma^{-1} u'^2 \exp(2\phi) - \frac{1}{r^2} \sigma (w^2 - 1)^2 \exp(2\phi) \\
& - 2\sigma N w'^2 \exp(2\phi) = 0, \tag{11}
\end{aligned}$$

$$\begin{aligned}
& 2\sigma^2 N (w^2 - 1) a' \exp(-2\phi) + r^2 N (\sigma' u' + \sigma u'') + 2\sigma u w^2 \\
& - 2r N \sigma u' (1 + r\phi') = 0, \tag{12}
\end{aligned}$$

$$\begin{aligned}
& r^2 N \sigma a'' + r \sigma N [a' (2 - 8r\phi') + 4a (-2\phi' + 4r\phi'^2 - r\phi'')] \\
& + r^2 (\sigma N)' (a' - 4a\phi') + 2 \exp(4\phi) [(w^2 - 1)u' \\
& + 2uww'] = 0. \tag{13}
\end{aligned}$$

Defining magnetic and electric charges

As already noted in [1,2], defining a magnetic charge in the Freedman-Schwarz model is problematic since there is

no Higgs field breaking the gauge symmetry, thus providing a natural isospin direction to project the $SU(2)$ field strength on the direction of the residual Abelian symmetry [as one does for the original flat space 't Hooft-Polyakov (tHP) monopole configuration]. Note however that ansatz (6) is nothing but the gauge transformation of the original tHP ansatz (we call \bar{A} the gauge field ansatz in its original tHP form) entangling space and isospace indices,

$$A_i \rightarrow \bar{A}_i = S^{-1} A_i S + \frac{i}{e} S^{-1} \partial_i S = \frac{i}{e} (1-w) [\Omega, \partial_i \Omega],$$

$$A_0 = S^{-1} A_0 S = u(r) \Omega,$$

$$\Omega = t^a \frac{x^a}{r}. \tag{14}$$

Here S is given by

$$S = \begin{pmatrix} \exp(i(\phi + \eta)/2) \cos(\theta/2) & \exp(-i(\phi - \eta)/2) \sin(\theta/2) \\ -\exp(i(\phi - \eta)/2) \sin(\theta/2) & \exp(-i(\phi + \eta)/2) \cos(\theta/2) \end{pmatrix}, \tag{15}$$

where η is some function of (r, t) to be appropriately chosen [6].

Then, as for the pure Yang-Mills pioneering monopole ansatz of Yang and Wu [19], one can define a projected field strength in the form

$$\mathcal{F}_{\mu\nu} = \text{Tr}(\bar{F}_{\mu\nu} \Omega) \tag{16}$$

which, written in terms of fields in the original gauge, becomes

$$\mathcal{F}_{\mu\nu} = \text{Tr}(F_{\mu\nu} S \Omega S^{-1}), \tag{17}$$

leading to a magnetic field $\mathcal{B}^i = (1/2\sqrt{-g}) \epsilon^{ijk} \mathcal{F}_{ij}$ which takes the form

$$\mathcal{B}^r = \frac{1}{\sqrt{-g}} (1 - w^2). \tag{18}$$

and an electric field $\mathcal{E}^i = \mathcal{F}^{0i}$ of the form

$$\mathcal{E}^r = \frac{u'(r)}{\sigma^2}. \tag{19}$$

Now, we define the electric and magnetic charges in the form

$$Q_M = \frac{1}{4\pi} \int dS_k \sqrt{-g} \mathcal{B}^k,$$

$$Q_E = \frac{1}{4\pi} \int dS_k \sqrt{-g} \mathcal{E}^k, \tag{20}$$

so that for ansatz (5), (6) one has

$$Q_M = 1 - w(\infty)^2, \tag{21}$$

$$Q_E = \frac{u' r^2}{\sigma} \Big|_{\infty}. \tag{22}$$

As stated in the Introduction, exact solutions of the first order Bogomol'nyi equations also solving the second order equations of motion of the truncated Freedman-Schwarz model are known [1,2]. They correspond to monopole-like gauge field configurations (with magnetic charge 1 and zero electric charge) and a globally hyperbolic regular geometry. Apart from these exact solutions, other magnetically charged non-BPS solutions were found in [4]. Let us then analyze, for completeness, this well-studied purely magnetic case and then discuss dyon solutions.

III. NON-BPS MONOPOLE SOLUTIONS

We then start from Eqs. (8)–(13) with $u = a = 0$, thus reducing the system to the one with four coupled nonlinear equations. Now, as pointed in [1,2], the bosonic action admits a global symmetry which in our notation corresponds to

$$\phi \rightarrow \phi - \epsilon, \quad N \rightarrow \exp(2\epsilon)N. \quad (23)$$

Following the Noether theorem, there is a conserved current which leads to the following relation between N , σ , and ϕ :

$$N\sigma^2 = \exp[2(\phi - \phi_0)], \quad (24)$$

where ϕ_0 is an integration constant. Using this relation, one can reduce the coupled system of four equations of motion to three independent equations for three unknown functions,

$$r \rightarrow \rho,$$

$$(w, \phi, N, \sigma) \rightarrow (w, \phi, R), \quad (25)$$

where we have chosen R as the third unknown function, implicitly defined while introducing a new variable ρ replacing the original radial variable r through

$$r^2 = \exp[2(\phi - \phi_0)]R^2(\rho). \quad (26)$$

This change, inspired in the one that yields to the exact ChV solution [1,2], leads to a metric of the form

$$ds^2 = \exp(2\phi - 2\phi_0)[-dt^2 + d\rho^2 + R^2(\rho)d\Omega^2]. \quad (27)$$

After some algebra, the resulting system can be written as

$$w'' + 2\phi'w' - \frac{1}{R^2}w(w^2 - 1) = 0,$$

$$\phi'' + 4\phi'^2 + \frac{1}{R^2}(R'^2 - 2w'^2 - 1) + 6\frac{R'}{R}\phi' - 1 = 0,$$

$$R'' + \frac{1}{R}(3w'^2 - R'^2 + 1) - 4R\phi'^2 + R - 6\phi'R' = 0, \quad (28)$$

where $w' = dw/d\rho$.

These equations together with the regularity condition at the origin imply

$$\begin{aligned} w &= 1 - w^{(2)}\rho^2 + O(\rho^4), \\ \phi &= \phi^{(0)} + \phi^{(2)}\rho^2 + O(\rho^4), \\ R &= \rho - R^{(3)}\rho^3 + O(\rho^5). \end{aligned} \quad (29)$$

Here $w^{(2)}$ and $\phi^{(0)}$ are free parameters. Note that the equations of motion are invariant under the transformation

$$\begin{aligned} \rho &\rightarrow \lambda\rho, & R &\rightarrow \lambda R, \\ w &\rightarrow w, & \phi &\rightarrow \phi + \log \lambda, \end{aligned} \quad (30)$$

and hence the value of $\phi^{(0)}$ can be arbitrarily fixed. In order to compare with the BPS solution in [1,2], we choose $\phi^{(0)} = -\log 2/2$. Then Eqs. (28) imply the following relation between coefficients $\phi^{(2)}$, $R^{(3)}$ and $w^{(2)}$,

$$\phi^{(2)} = w^{(2)2} + \frac{1}{12},$$

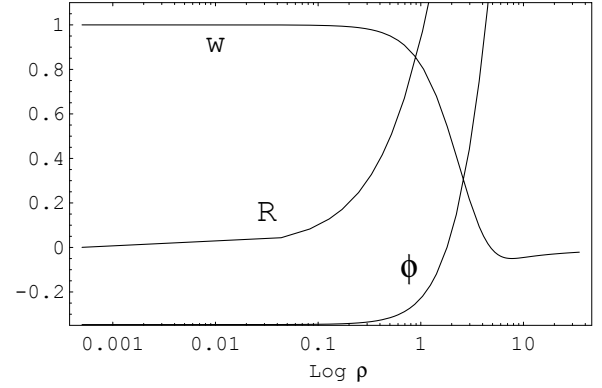


FIG. 1. Globally regular, purely magnetic (non-BPS) solution for $w^{(2)}=0.2$.

$$R^{(3)} = w^{(2)2} + \frac{1}{36}. \quad (31)$$

We are then left with just one shooting parameter $w^{(2)}$.

Concerning spatial infinity, we seek for solutions having the following asymptotic behavior

$$w \sim w_1^{(\infty)} \frac{1}{\rho^{1/2}} + w_2^{(\infty)} \frac{1}{\rho^{3/2}} + \dots + C\rho \exp(-\rho) + \dots,$$

$$\begin{aligned} \phi &\sim \phi_1^{(\infty)} + \frac{\rho}{2} - \frac{1}{4} \log \rho + \phi_2^{(\infty)} \frac{1}{\rho^2} + \dots + D\sqrt{x} \exp(-\rho) \\ &+ \dots, \end{aligned}$$

$$R \sim \sqrt{2\rho} - R_1^{(\infty)} \frac{1}{\rho^{3/2}} + \dots + F\rho \exp(-\rho) + \dots. \quad (32)$$

With this behavior the magnetic charge, as given by Eq. (21), is $Q_M = 1$.

Using these conditions, solutions can be found numerically, varying the shooting parameter $w^{(2)}$ in a given range of values and analyzing the resulting asymptotic behavior. We have found a whole range R_m of values for $w^{(2)}$ for which globally regular solutions exist, which coincides with that

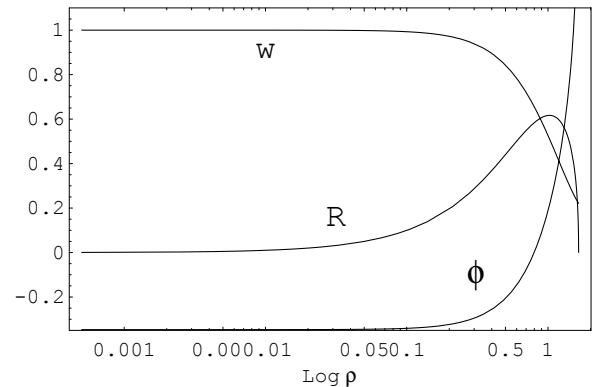


FIG. 2. Bag of gold purely magnetic solution for $w^{(2)}=0.7$.

found in [4], $R_m = (w_a^{(2)} = 0, w_b^{(2)} = 1/2)$. For the particular value $w^{(2)} = 1/6$, the solution coincides with the ChV exact one. For $w_b^{(2)} > 1/2$, as first shown in [4], one finds that R develops a second zero (apart from that at $\rho = 0$) at some finite value ρ_* where the geometry is singular. These kind of solutions are similar to those called bags of gold [20] and are also related to the black hole solutions found in [4]. We shall describe in more detail this behavior below.

Typical solutions are displayed in Figs. 1 and 2. The shape of solutions for ϕ, w, R as functions of ρ in the whole range R_m is similar to that of the exact ChV solution. Concerning the gauge field, one finds that certain solutions, those corresponding to $w^{(2)} > 1/6$ develop nodes (as it also happens for the Einstein-Yang-Mills monopole solutions in asymptotically AdS space, found in [6]). The magnetic charge, in view of the asymptotic behavior of $w(\rho)$, has value 1 for $w^{(2)} < 1/2$ but becomes noninteger for $w^{(2)} > 1/2$ so that the configuration corresponds in this case to a kind of nontopological soliton (we shall discuss more in detail this issue in the next section). Finally, note that, except for the solution corresponding to $w^{(2)} = 1/6$, all other solutions to the second order equations of motion do not satisfy first order Bogomol'nyi equations.

Let us now study more in detail the behavior of the bag of gold solutions, which in the present purely magnetic configurations correspond to $w^{(2)} > 1/2$. We start by defining a radius R_c as the function multiplying the solid angle $d\Omega^2$ of the spatial geometry [see Eq. (27)],

$$R_c^2(\rho) = \exp(2\phi)R^2(\rho). \quad (33)$$

For bag of gold solutions R_c should vanish at some $\rho = \rho_*$, $R_c(\rho_*) = 0$. To see this, one has to determine the behavior of $R(\rho)$ and $\exp(2\phi)$ when $\rho \sim \rho_*$. This can be done analytically by proposing an asymptotic behavior in powers of $x = \rho_* - \rho$, finding out that

$$w \sim \text{const}, \quad \exp(\phi) \sim Cx^{-1/4}, \quad R \sim Dx^{1/2}, \quad (34)$$

so that one has indeed

$$R_c^{\text{mon}^2}(\rho) \sim D^2 C^2 (\rho_* - \rho)^{1/2}, \quad (35)$$

which corresponds to the behavior of a bag of gold solution.

IV. DYON SOLUTIONS

Let us now extend our analysis to the electrically charged case. Being $a \neq 0$ and $u \neq 0$, symmetry (23) is lost and hence no relation of kind (24) can be established. We are then left with six equations of motion for six unknown functions.

One could try to find BPS solutions by analyzing the supersymmetry variations of fermionic fields when a and u fields are included. The conditions resulting from the vanishing of these variations are highly more involved than those arising in the purely magnetic case. However, by assuming a regular behavior at the origin, we found that the only non-trivial solution to the first order system corresponds to the exact, purely magnetic solution discovered in [1,2]. We are then forced to study the second order equations of motion to

determine whether non-BPS dyon solutions exist.

In analogy with the purely magnetic case, we shall write the system in terms of a new variable ρ and unknown functions in the form

$$r \rightarrow \rho,$$

$$(w, u, \phi, a, N, \sigma) \rightarrow (w, u, \phi, a, R, V), \quad (36)$$

with

$$\exp(2V) = N\sigma^2,$$

$$\exp(2V)R^2(\rho) = r^2. \quad (37)$$

After change (36), metric (5) becomes

$$ds^2 = \exp(2V)[-dt^2 + d\rho^2 + R^2(\rho)d\Omega^2]. \quad (38)$$

Note finally that in the purely magnetic case, due to the global symmetry (23) one can identify V with ϕ according to the relation $V = \phi + \log 2/2$ and then Eq. (38) becomes Eq. (27).

The system of equations can then be brought to the form

$$w'' + 2\phi'w' - \frac{1}{R^2}w(w^2 - 1) + u^2w + 2\exp(-2\phi)uwa' = 0,$$

$$\begin{aligned} \phi'' + 2\phi' \left(V' + \frac{R'}{R} \right) + \exp(2\phi - 2V) \left(u'^2 - 2\frac{w'^2 - u^2w^2}{R^2} \right. \\ \left. - \frac{(1 - w^2)^2}{R^4} \right) - \frac{1}{4}\exp(-2\phi + 2V) \\ + 2\exp(-4\phi)a'^2 = 0, \end{aligned}$$

$$\begin{aligned} V''' + 6\frac{R'}{R}V'' + 5V'^2 - 4\exp(2\phi - 2V) \left(\frac{w'^2}{R^2} + \frac{w^2u^2}{R^2} \right) - \frac{1}{R^2} \\ - \phi'^2 - \exp(-4\phi)a'^2 + \frac{R'^2}{R^2} - \frac{1}{2}\exp(-2\phi + 2V) = 0, \end{aligned}$$

$$\begin{aligned} R'' - 6V'R' - \frac{R'^2}{R} + \frac{1}{2}\exp(-2\phi + 2V)R + 2R(\phi'^2 - 3V'^2) \\ + 6\exp(2\phi - 2V) \left(\frac{w'^2 + u^2w^2}{R} \right) + \frac{1}{R} \\ + 2\exp(-4\phi)Ra'^2 = 0, \\ -u'' - 2u' \left(\phi' + \frac{R'}{R} \right) + \frac{2uw^2}{R^2} + \frac{2}{R^2}\exp(-2\phi)a' \\ \times (w^2 - 1) = 0, \end{aligned}$$

$$\begin{aligned}
& a'' + 2a' \left(V' + \frac{R'}{R} - 4\phi' \right) - 8a\phi' \left(\frac{R'}{R} + V' - 2\phi' \right) \\
& + \frac{2}{R^2} \exp(4\phi - 2V) [(w^2 - 1)u' + 2ww'u] - 4a\phi'' = 0.
\end{aligned} \tag{39}$$

It is useful to notice that, apart from the generalization of invariance (30) already present in the purely magnetic case,

$$\begin{aligned}
\rho &\rightarrow \lambda\rho, \\
\phi &\rightarrow \phi + \log \lambda, \quad u \rightarrow \lambda^{-1}u, \quad w \rightarrow w, \\
a &\rightarrow \lambda^2 a \quad V \rightarrow V, \quad R \rightarrow \lambda R,
\end{aligned} \tag{40}$$

system (39) exhibits now a second invariance

$$\begin{aligned}
\rho &\rightarrow \mu\rho, \\
\phi &\rightarrow \phi, \quad u \rightarrow \mu^{-1}u, \quad w \rightarrow w, \\
a &\rightarrow a, \quad V \rightarrow V - \log \mu, \quad R \rightarrow \mu R,
\end{aligned} \tag{41}$$

and this reduces the number of shooting parameters to fix when seeking for a numerical solution.

For the case of dyon solutions, we have to supplement Eq. (29) with conditions at the origin for u , a , and V ,

$$\begin{aligned}
u &= u^{(0)}\rho + u^{(3)}\rho^3 + O(\rho^5), \\
a &= a^{(0)} + a^{(2)}\rho^2 + O(\rho^4), \\
V &= V^{(2)}\rho^2 + O(\rho^4).
\end{aligned} \tag{42}$$

Note that, in view of symmetry (41) one can fix $V(0) = 0$. Moreover, Eqs. (39) now allow us to write $a^{(2)}$, $V^{(2)}$, $u^{(3)}$, $f^{(2)}$, and $R^{(3)}$ in terms of $w^{(2)}$, $u^{(0)}$, and $a^{(0)}$:

$$\begin{aligned}
a^{(2)} &= \frac{1}{6}(2a^{(0)} - 6a^{(0)}u^{(0)2} - 3u^{(0)}w^{(2)} + 24a^{(0)}w^{(2)2}), \\
V^{(2)} &= \frac{1}{12}(1 + 3u^{(0)2} + 12w^{(2)2}), \\
f^{(2)} &= \frac{1}{12}(1 - 3u^{(0)2} + 12w^{(2)2}), \\
R^{(3)} &= \frac{1}{36}(1 + 9u^{(0)2} + 36w^{(2)2}), \\
u^{(3)} &= \frac{1}{90}(-u^{(0)} + 27u^{(0)3} + 48a^{(0)}w^{(2)} + 36u^{(0)}w^{(2)} \\
&\quad - 144a^{(0)}u^{(0)2}w^{(2)} - 36u^{(0)}w^{(2)2} + 576a^{(0)}w^{(2)3}).
\end{aligned} \tag{43}$$

Using these conditions one can numerically integrate Eqs. (39), with $w^{(2)}$, $u^{(0)}$, and $a^{(0)}$ as shooting parameters. As we shall describe in detail below, the space of dyon solutions is radically different to the pure monopole case discussed above. Indeed, all dyon solutions are bag of gold type: function $R(\rho)$ always has two zeros: one at $\rho = 0$ and the second one at some finite value $\rho_*(w^{(2)}, u^{(0)}, a^{(0)})$. One can see

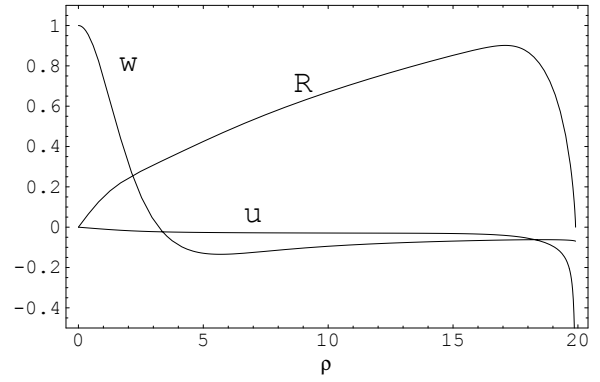


FIG. 3. Dyon solution for gauge field and R for $w^{(2)} = 0.3$, $u^{(0)} = -0.01$, $a^{(0)} = 0.00414$.

numerically that this happens for all positive $w^{(2)}$ values (negative ones lead to an unbounded magnetic field at $\rho = \rho_*$).

To see this in more detail, let us start by considering the region $0 < w^{(2)} < 1/2$ (where regular pure monopole solutions were found). For small values of $u^{(0)}$ and $a^{(0)}$ ($u^{(0)} \sim a^{(0)} \approx 0.1$) all (w, R, V, ϕ) solutions are, at small ρ , very similar to the monopole ones but, as one increases ρ , all except w start deviating from such a behavior. Indeed, the behavior of w for the dyon solution is much like that of the monopole solution in the whole range $0 \leq \rho \leq \rho_*$. In contrast, although R starts growing as in the monopole solution, it reaches in this case a maximum value and then decreases monotonically, until it vanishes at some fixed value ρ_* . As we already discussed in the case of the monopole solution (for $w^{(2)} > 1/2$), this behavior corresponds to a bag of gold. We have found that the electric and axion fields, $u(\rho)$ and $a(\rho)$, have no nodes for $w^{(2)} < 1/6$ while in the region $w^{(2)} > 1/6$ they have a growing number of nodes. Typical behaviors are shown in Figs. 3 and 4.

As one approaches $w^{(2)} = 1/2$ (the critical monopole solution value), w , u , and a develop more and more nodes (this oscillatory behavior was already present for w in the monopole case). For $w^{(2)} > 1/2$, where also the pure monopole solution was bag of gold type, one finds that adjusting the shooting parameters $a^{(0)}$ and $u^{(0)}$ one can make ρ_*^{dyon}

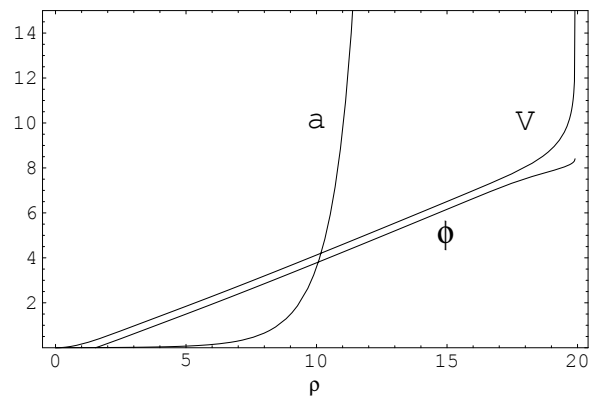


FIG. 4. Axion, dilaton, and V solutions for $w^{(2)} = 0.3$, $u^{(0)} = -0.01$, $a^{(0)} = 0.00414$.

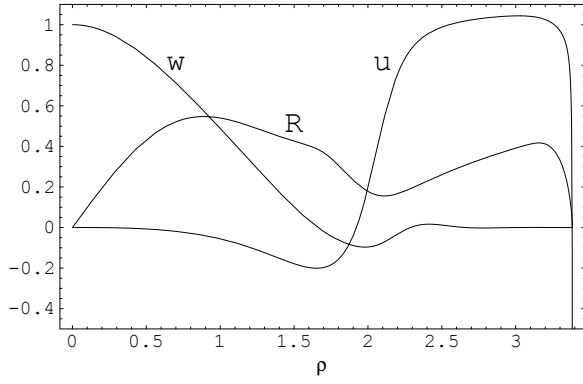


FIG. 5. Dyon solution for $w^{(2)}=0.7$, $u^{(0)}=-0.01$, $a^{(0)}=0.233$.

$> \rho_{*}^{\text{monopole}}$. This behavior is shown in Fig. 5 for $w^{(2)}=0.7$. As mentioned, we see by comparing Figs. 2 and 5 that ρ_{*} has grown from $\rho_{*}=1.647$ for the purely magnetic solution to $\rho_{*}=3.384$ for the dyon solution.

Let us extend the analysis of purely magnetic bag of gold solutions presented at the end of Sec. III to the case of dyons. The analytic determination of asymptotic behaviors becomes in this case more involved. One can however find bounds which can be used to study crucial properties of the solutions and can also be confronted with the results obtained numerically. We start by assuming the following asymptotic behavior:

$$\begin{aligned} w &\sim w_0, \quad u \sim Bx^\beta, \\ \exp(-V) &\sim Kx^\delta, \quad R \sim Ax^\alpha. \end{aligned} \quad (44)$$

Using these expressions in the equations for V and R , Eq. (39), and requiring the dominant terms to be of the order x^{-2} (the only consistent possibility), one gets the following conditions relating exponents α and δ :

$$\begin{aligned} (\delta - 6\alpha\delta + 5\delta^2 + \alpha^2)x^{-2} - 4F^2(\rho) - G^2(\rho) - H^2(\rho) &= 0, \\ (-\alpha + 6\alpha\delta - 6\delta^2)x^{-2} + 6F^2(\rho) + 2G^2(\rho) + H^2(\rho) &= 0, \end{aligned} \quad (45)$$

where

$$\begin{aligned} F^2 &= \frac{1}{R^2} \exp(2\phi - 2V)(w'^2 + w^2u^2), \\ G^2 &= \phi'^2 + \exp(-4\phi)a'^2, \\ H^2 &= \frac{1}{2} \exp(-2\phi + 2V) \end{aligned} \quad (46)$$

can at most be of the order of x^{-2} . Consistency of Eqs. (45), together with reality of F , G , and H , imposes certain conditions on exponents α and δ . One of the possibilities that results from Eqs. (45) is

$$\alpha \geq \delta, \quad \alpha + \delta \leq 1, \quad (47)$$

$$\alpha \leq 2\delta, \quad \alpha - \delta \leq \frac{1}{2}, \quad (48)$$

from which one finds

$$0 < \alpha \leq \frac{2}{3}, \quad 0 < \delta \leq \frac{1}{2}. \quad (49)$$

There are other three possibilities which can be discarded by comparison with the numerical solutions. Indeed two of them lead to $\alpha \leq \delta$ while the third one implies $\delta \leq 1/4$. Neither of these conditions is verified by the numerical solutions. We are then left with Eqs. (47)–(49) as the sole possibility.

These bounds are very useful in analyzing some crucial properties of the solutions. Indeed, from Eq. (38) and the analogous of formula (33) for dyons,

$$R_c^{\text{dyon}2}(\rho) = \exp(2V)R^2(\rho). \quad (50)$$

one can write

$$R_c^{\text{dyon}2} \sim \frac{A^2}{K^2} x^{2(\alpha - \delta)} \quad (51)$$

so that from Eq. (47) one has

$$R_c^{\text{dyon}}(\rho_{*}) = 0. \quad (52)$$

Then, the three parameter family of solutions corresponds to a geometry characterized by compact spatial sections (of radius ρ_{*}). In order to have a better understanding of the properties of such a geometry, let us evaluate the affine length \mathcal{L} associated to a massless particle, moving along a null radial geodesic, from $\rho=0$ to $\rho=\rho_{*}$. To this end, we start from the null geodesic condition (we call v the affine parameter)

$$-\left(\frac{dt}{dv}\right)^2 g_{tt} + \left(\frac{d\rho}{dv}\right)^2 g_{\rho\rho} = 0. \quad (53)$$

“Energy” conservation implies

$$\frac{dt}{dv} = \frac{E}{g_{tt}}, \quad (54)$$

so that, from Eq. (53) one can write

$$\frac{d\rho}{dv} = \frac{E}{\sqrt{g_{tt}g_{\rho\rho}}} = \frac{E}{\exp(2V)} \quad (55)$$

and then the affine length takes the form

$$\mathcal{L} = \int_0^{\rho_{*}} dv = \frac{1}{E} \int_0^{\rho_{*}} \exp(2V) d\rho. \quad (56)$$

Since $\exp(2V)$ is regular in the interval $0 \leq \rho < \rho_{*}$ the only possibility of having an infinite affine length (so as to have a geodesically complete space) should come from the singularity at $\rho = \rho_{*}$. Then, using the asymptotic expression (44) for V , one gets

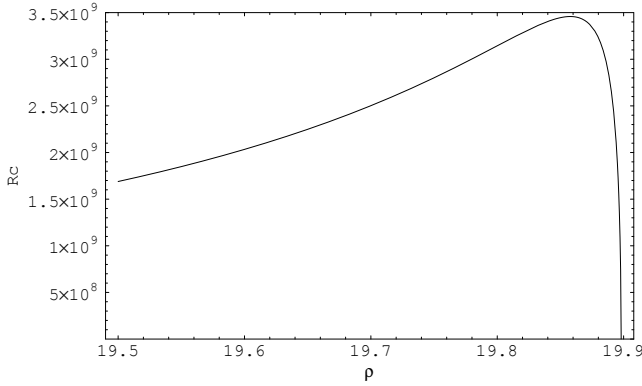


FIG. 6. The radius $R_c(\rho)$ for the dyon solution with $w^{(2)} = 0.3$, $u^{(0)} = -0.01$, $a^{(0)} = 0.00414$.

$$\mathcal{L} = \text{const} - \lim_{\rho \rightarrow \rho_*} \frac{C^2}{1 - 2\delta} (\rho_* - \rho)^{1-2\delta}. \quad (57)$$

Now, we have found analytically that $0 < \delta \leq 1/2$ so that the affine length is finite and then $\rho = \rho_*$ corresponds to a real singularity of space-time (incomplete geodesic). Let us end this discussion by analyzing the scalar curvature \mathcal{R} at $\rho = \rho_*$. From the explicit form of metric (38), one has

$$\mathcal{R} = -2 \exp(-2V) R^{-2} [-1 + R'^2 + 6RR'V' + 3R^2(V' + V'') + 2RR'']. \quad (58)$$

Now, using the asymptotic behaviors (44) one can see that, to leading order,

$$\mathcal{R} \sim [\alpha^2 - 6\alpha\delta + 2\alpha(\alpha - 1) + 3\delta] x^{2\delta-2}. \quad (59)$$

Using the allowed ranges (49) for α and δ as well as the numerical results one can easily see that the coefficient of $x^{2\delta-2}$ does not vanish, and hence the curvature diverges at $\rho = \rho_*$, so that ρ_* corresponds to a scalar curvature singularity. Concerning numerical results one can extract the following values for β , δ and α (for a typical case corresponding to $w^{(2)} = 0.3$, $u^{(0)} = -0.01$, and $a^{(0)} = 0.00414$):

$$\beta \sim 0.39, \quad \delta \sim 0.34, \quad \alpha \sim 0.65, \quad (60)$$

which, as we see, agree with the bounds that one can obtain analytically. We show in Fig. 6 the function $R_c(\rho)$ for this solution.

The fact that $R(\rho_*) = 0$ corresponds to compact spatial sections and has important consequences on the magnetic and electric charges. Indeed, in order to compute the electric and magnetic charges, one should use formulas (21) and (22) and integrate over ρ from $\rho = 0$ to $\rho = \rho_*$. Then, one finds for the magnetic charge

$$Q_M = 1 - w^2(\rho_*) < 1, \quad (61)$$

which gives a noninteger value, showing that the solution is nontopological in the sense discussed in [5,6] for Einstein-Yang-Mills dyon solutions in asymptotically AdS space.

Concerning the electric charge, formula (22) gives

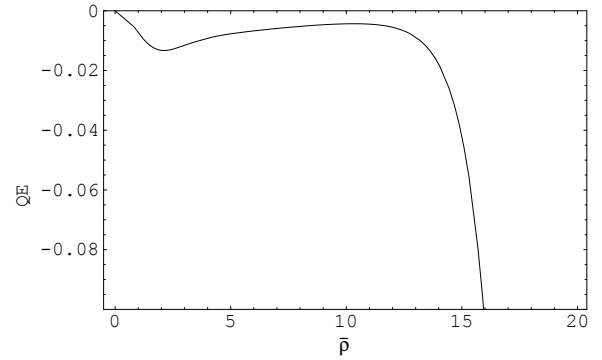


FIG. 7. The electric charge enclosed in a sphere of radius $\bar{\rho}$ for the dyon solution with $w^{(2)} = 0.3u^{(0)} = -0.01$, $a^{(0)} = 0.00414$.

$$Q_E = R^2 \frac{du}{d\rho} \Big|_{\rho=\rho_*}. \quad (62)$$

We represent $Q_E(\bar{\rho})$, the charge enclosed in a sphere of radius $\bar{\rho}$, for $0 \leq \bar{\rho} < \rho_*$ in Fig. 7. We see that as $\bar{\rho}$ approaches ρ_* , the charge grows and finally diverges. From the numerically determined values for β and α , its behavior is given by

$$Q_E(\bar{\rho}) \sim (\rho_* - \bar{\rho})^{-0.09}. \quad (63)$$

It is important to stress that although we were not able to fix the exponents α and δ analytically, inequalities (47) allow us to ensure that all dyon solutions correspond to bags of gold. Moreover, there is no loss of generality in our results because we have put $g = 1$ from the beginning: they are valid for arbitrary g . Then, in a sense we can say that the bag of gold singular behavior of dyon solutions is “universal.”

V. SUMMARY

We have discussed in this work classical monopole and dyon solutions to the equations of motion of $d=4$, $\mathcal{N}=4$ Freedman-Schwarz supergravity. Purely magnetic solutions were already found in [1–4]. Apart from the exact BPS solution, the authors in Ref. [4] constructed, numerically, some non-BPS magnetic monopole solutions noting the existence of singular (bag of gold) solutions in some range of the shooting parameter. We have confirmed this result, showing analytically a set of exponents characterizing the singularity.

We have then extended the analysis to allow for electrically charged dyon solutions. The first important result is that one can see that there is no possibility of finding dyon configurations making the supersymmetry variations vanish. Then, we conclude that BPS solutions cannot be electrically charged. Concerning the second order equations of motion, we have shown analytically that they all correspond to bag of gold configurations (in contrast with the purely magnetic case where both regular and singular solutions were allowed). In fact, we have computed the affine length \mathcal{L} along radial null geodesics showing that \mathcal{L} is finite and hence $\rho = \rho_*$ corresponds to a real singularity, of the space-time. We have studied the asymptotic behavior near the singularity finding that the magnetic charge is not an integer; the solu-

tion then corresponds to a nontopological soliton as those discussed in the case of the Einstein-Yang-Mills theory in asymptotically AdS space [5,6]. As stressed in the Introduction, soliton solutions in the bosonic sector of $d=4$ supergravity models are of interest, after uplifting to $d=10$ dimensions, in the context of the AdS/CFT correspondence. An interesting question in this context is whether the $d=4$ space-time singularity in $\rho=\rho_*$ survives the uplifting procedure and, in the affirmative, how it affects the physical outcome of the $d=10$ theory.

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