

# Boson stars with generic self-interactions

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We study boson star configurations with generic, but not non-topological, self-interaction terms, i.e. we do not restrict ourselves just to consider the standard  $\lambda|\psi|^4$  interaction but more general  $U(1)$ -symmetry-preserving profiles. We find that when compared with the usual potential, similar results for masses and number of particles appear. However, changes are of order of few percent of the star masses. We explore the stability properties of the configurations, that we analyze using catastrophe theory. We also study possible observational outputs: gravitational redshifts, rotation curves of accreted particles, and lensing phenomena, and compare with the usual case.

## I. INTRODUCTION

In the past two decades, the study of the universe in its earlier stages has given an important role to scalar fields: conventional models for inflation rely on its possible potential energy and topological defects may form when a scalar field breaks some fundamental symmetry. Both of these processes may be responsible for the formation of the large scale structure we now see. In addition, scalar fields connected with time or space-time variation of fundamental constants have produced the most powerful alternative theories of gravitation, known as scalar-tensor theories, and entered in all unification scheme trials made so far. It must be said, however, that no fundamental cosmological scalar field has been directly observed yet.

If one accepts the need of scalar fields in the cosmological scenario, one interesting question naturally arise: may these scalars be the seed of astrophysical structures or of observable phenomena that could signal their existence? Objects made up of scalar massive particles were introduced as early as 1968 by Kaup [1] and Ruffini and Bonazzola [2]. These configurations are now known as boson stars. These stars, contrary to the more common neutron or fermion stars, are not supported by Pauli's exclusion but by Heisenberg's uncertainty principle, which effectively keeps scalars from being localized to within their Compton wavelength and prevents their collapse to a black hole.

The first investigations with a nonlinear  $\psi^6$  potential were carried out by Mielke and Scherzer [3]. They calculated this potential from a nonlinear Heisenberg-Pauli-Weyl spinor equation and found the first scalar field solutions with nodes. More recently, Colpi et al. [4] proved that the existence of a self-interaction in the boson Lagrangian could yield higher values for the masses of the configurations. This means that for some values of the free parameters existing in the theory, stellar structures of appreciable masses and extremely high density may arise. This could have incredible effects on our understanding of the non-baryonic content of the universe and have different observational effects: affecting usual objects as galaxies [5] or even light trajectories in powerful,

degree-ranged, microlensing phenomena [6]. These kind of features, as well as the important point of star rotation (see [7] among others), was thoroughly studied in the past few years, including gauge charged stars, boson-fermion models (see [8] for reviews) and models of stars in alternative theories of gravity [9].

In all these works, the self-interaction was always choiced to be like  $\lambda|\psi|^4$ , thus giving a matter sector provided by,

$$\mathcal{L}_m = -\frac{1}{2}g^{\mu\nu} \partial_\mu \psi^* \partial_\nu \psi - \frac{1}{2}m^2|\psi|^2 - \frac{1}{4}\lambda|\psi|^4, \quad (1)$$

where  $\lambda$  is a constant. However, a priori, there is no reason to maintain just the form adopted for the self-interaction and not a more generic potential term  $V(\psi)$ . Although in order to retain the  $U(1)$  symmetry of the whole Lagrangian, this potential must be a function of  $|\psi|^2$ , in principle, it may have any other particular form. In this work we would like to explore whether substantial variations in the form of the potential may yield appreciable changes in the final configurations, the binding energies, or the masses of boson stars. We also analyze the changes that a different potential introduces in gravitational redshifts and microlensing phenomena. This could be important to discover if different scenarios for dark matter candidates, explanation of galactic rotation curves [5], or new observational gravitational effects may appear [10]. Additionally, new potentials may strongly affect gravitational memory and evolution scenarios [11] within theories with time variation of Newton's constant (for the most recent account of this see Ref. [12]).

Knowledge about that modifications in the boson potential may yield to profound changes in the structure of the stellar objects comes from the works of Friedberg, Lee, and Pang [13–15] on non-topological soliton stars. However, we point out that a systematic study on their possible observational signatures is still absent. We shall briefly mention their features and compare with our potential choices below. However, *our main aim here is to study if potentials which are not non-topological ones may still introduce significant changes in the configurations.*

This work is organized as follows. In the next section we introduce the models for boson stars and search, with

different self-interaction terms, for their configurations and stability properties. Section III compares the results of these generic boson star models, concerning observational outputs, with those obtained for the usual  $\lambda|\psi|^4$  term. We give our conclusions and provide a brief discussion in the final section.

## II. THE MODELS

We shall take into account the following boson Lagrangian

$$\mathcal{L}_m = -\frac{1}{2}g^{\mu\nu}\partial_\mu\psi^*\partial_\nu\psi - \frac{1}{2}U(|\psi|^2), \quad (2)$$

with the potential

$$U(|\psi|^2) = m^2|\psi|^2 + \frac{1}{2}V(|\psi|^2). \quad (3)$$

This action possesses an invariance under the global  $U(1)$  transformation  $\psi \rightarrow e^{i\theta}\psi$ , which gives rise to a conserved current

$$J^\mu = ig^{\mu\nu}(\psi^*\partial_\nu\psi - \psi\partial_\nu\psi^*), \quad (4)$$

and a corresponding conserved charge

$$N = \int dx^3 \sqrt{-g} J^0. \quad (5)$$

This conserved quantity can be identified with the number of bosons present in the stellar structure. We shall consider the usual spherically symmetric metric

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2d\Omega^2, \quad (6)$$

and shall also demand a spherically symmetric form for the field which describe the boson, *i.e.* we adopt the ansatz:

$$\psi(r, t) = \chi(r) \exp[-i\varpi t]. \quad (7)$$

This ensures that a static solution is still possible. To establish that this time dependence is the lowest energy solution at a fixed number of particles, it is necessary to make a first order variation  $\delta(E - \varpi N)$ , where in general  $\varpi$  is a Lagrangian multiplier associated with the conservation of  $N$  and  $E$  is the energy (mass) of the system. See for instance the appendix of Ref. [14] for a detailed derivation. This form for  $\psi$  is independent of the potential  $U$ . For a more detailed derivation of the basic model we refer the reader to Refs. [1,2,4,8]. Using the Einstein gravitational action we obtain the field equations,

$$\begin{aligned} \frac{2M'}{x^2} &= \sigma^2 \left( \frac{\Omega^2}{B} + 1 \right) + \frac{\sigma'^2}{A} + \\ &\frac{1}{2} \left( \frac{4\pi}{m^2 M_{\text{Pl}}^2} \right) V(\sigma M_{\text{Pl}}/\sqrt{4\pi}), \end{aligned} \quad (8)$$

$$\begin{aligned} \frac{B'}{ABx} - \frac{1}{x^2} \left( 1 - \frac{1}{A} \right) &= \sigma^2 \left( \frac{\Omega^2}{B} - 1 \right) + \\ &\frac{\sigma'^2}{A} - \frac{1}{2} \left( \frac{4\pi}{m^2 M_{\text{Pl}}^2} \right) V(\sigma M_{\text{Pl}}/\sqrt{4\pi}), \end{aligned} \quad (9)$$

and the Klein-Gordon equation for  $\psi$ ,

$$\begin{aligned} \sigma'' + \sigma' \left( \frac{2}{x} - \frac{A'}{2A} + \frac{B'}{2B} \right) + A\sigma \left( \frac{\Omega^2}{B} - 1 \right) \\ - \frac{1}{4} \left( \frac{4\pi}{m^2 M_{\text{Pl}}^2} \right) \frac{dV(\sigma M_{\text{Pl}}/\sqrt{4\pi})}{d\sigma} = 0. \end{aligned} \quad (10)$$

In these equations, we have used dimensionless units, which are common to those introduced in the paper by Colpi et al.,

$$x = mr, \quad (11)$$

for the radial coordinate ( $'$  stands for the derivatives with respect to  $x$ ) and,

$$\Omega = \frac{\varpi}{m}, \quad \sigma = \sqrt{4\pi} \frac{\chi(r)}{M_{\text{Pl}}}, \quad (12)$$

where  $M_{\text{Pl}} \equiv G^{-1/2}$  is the Planck mass.

In order to consider the total amount of mass contained within a radius  $x$  we have changed the function  $A$  in the metric to its Schwarzschild form,

$$A(x) = \left( 1 - \frac{2M(x)}{x} \right)^{-1}. \quad (13)$$

Note that the terms corresponding to the potential are correctly normalized. From the Lagrangian (2) one may see that  $V$  has dimensions of  $[\text{Energy}]^4$ , and all of them are divided by two squares of masses. However, in order to avoid the explicit appearance of the boson mass  $m$  and the Planck mass  $M_{\text{Pl}}$ , we need to define the form of  $V$ . We shall look at several choices.

The first expansion to nonlinear potentials can be found in the 1981 paper by Mielke & Scherzer [3]. They constructed a potential for the Klein-Gordon equation from the Heisenberg-Pauli-Weyl nonlinear spinor equation. It has the general form  $U = m^2|\psi|^2 - \alpha_1|\psi|^4 + \alpha_2|\psi|^6$ , where  $\alpha_1$  and  $\alpha_2$  are two positive constants. They presented solutions with nodes for the first time.

The standard (CSW [4]) choice is  $V(\psi) = \lambda|\psi|^4$ , with  $\lambda$  a constant. Here, the usual adimensionalization appears,

$$\begin{aligned} \frac{4\pi}{m^2 M_{\text{Pl}}^2} V(\sigma M_{\text{Pl}}/\sqrt{4\pi}) &= \\ \frac{4\pi}{m^2 M_{\text{Pl}}^2} \sigma^4 \lambda \frac{M_{\text{Pl}}^4}{(4\pi)^2} &= \lambda \frac{M_{\text{Pl}}^2}{4\pi m^2} \sigma^4 = \Lambda \sigma^4, \end{aligned} \quad (14)$$

where  $\Lambda = \lambda M_{\text{Pl}}^2/4\pi m^2$ . As we stated in the introduction, with this choice, the order of magnitude of boson star masses is deeply enhanced. It grows from

$M \sim M_{\text{P1}}^2/m$  when  $\Lambda = 0$  to  $M \sim M_{\text{P1}}^3/m^2$  when  $\Lambda \neq 0$ . Recall that the mass of a neutron star is roughly given by the Chandrasekhar mass  $M_{\text{Ch}} \sim M_{\text{P1}}^3/m_n^2$  which is close to a solar mass ( $m_n$  is the neutron mass).

Other three options we would like to explore are (note that these are options for  $U(|\psi|^2)$ , given in Eq. (3), not only for  $V(|\psi|^2)$  as will be clear below):

- Cosh-Gordon potential:

$$U_{\text{cosh}} = \alpha m^2 [\cosh(\beta\sqrt{|\psi|^2}) - 1]$$

- Sine-Gordon potential:

$$\begin{aligned} U_{\text{sin}} &= \alpha m^2 [\sin(\pi/2[\beta\sqrt{|\psi|^2} - 1]) + 1] \\ &= \alpha m^2 [1 - \cos(\pi/2\beta\sqrt{|\psi|^2})] \end{aligned}$$

- $U(1)$ -Liouville potential:

$$U_{\text{exp}} = \alpha m^2 [\exp(\beta^2|\psi|^2) - 1]$$

The usual Liouville potential  $\exp(\beta\psi)$  has to be changed so that a  $U(1)$  symmetry is ensured.

Let us first consider a series expansion of these potentials. In order to do so we shall consider a value of  $\beta$  such that, when going from  $\psi$  to the dimensionless  $\sigma$ , the arguments of the functions are not affected. The parameter  $\beta$  is arbitrary, it enlarges the parameter space of the solutions, as was the case with  $\Lambda$  in CSW's solutions. The appearance of the factor  $\beta$  is just because dimensional grounds, while  $\alpha$  can be used to get a simple mass term in the boson Lagrangian, as we shall see below. Taking this into account, the series expansions are,

$$U_{\text{cosh}} = \alpha m^2 [\cosh(\beta\sigma) - 1] = \alpha m^2 \times \left[ \frac{\beta^2\sigma^2}{2} + \frac{\beta^4\sigma^4}{24} + \frac{\beta^6\sigma^6}{720} + \frac{\beta^8\sigma^8}{40320} + \dots \right], \quad (15)$$

$$U_{\text{sin}} = \alpha m^2 [\sin(\pi/2(\beta\sigma - 1)) + 1] = \alpha m^2 \times \left[ \frac{\beta^2\pi^2\sigma^2}{8} - \frac{\pi^4\beta^4\sigma^4}{384} + \frac{\pi^6\beta^6\sigma^6}{46080} - \frac{\pi^8\beta^8\sigma^8}{10321920} + \dots \right], \quad (16)$$

$$U_{\text{exp}} = \alpha m^2 [\exp(\beta^2\sigma^2) - 1] = \alpha m^2 \times \left[ \beta^2\sigma^2 + \frac{1}{2}\beta^4\sigma^4 + \frac{1}{6}\beta^6\sigma^6 + \frac{1}{24}\beta^8\sigma^8 + \dots \right]. \quad (17)$$

Note that, from each expansion, we are recognizing a usual mass term (proportional just to  $m^2$ ). This term is very important: without it, it is impossible to find solutions with exponential decrease of the scalar field, something relevant for the definition of the star radius.

Then, in order to be consistent with equations (3) and (8-10), the field equations and the definition of  $U$ , and to avoid a useless double counting of the mass term, we must take particular choices for  $\alpha$ ; the parameter  $\beta$  is still free for choice.

- Cosh-Gordon potential:

$$\alpha = 2(M_{\text{P1}}^2/4\pi)/\beta^2$$

- Sine-Gordon potential:

$$\alpha = (8/\pi^2)(M_{\text{P1}}^2/4\pi)/\beta^2$$

- $U(1)$ -Liouville potential:

$$\alpha = (M_{\text{P1}}^2/4\pi)/\beta^2$$

In this way, the potential  $V$  is everything but the first terms in each of the previous series. Hence, it is a series of attractive-repulsive self-interactions in the case of the Sine-Gordon potential and a series of repulsive potentials in the Cosh-Gordon case. The case of the  $U(1)$ -Liouville potential is reminiscent of the Cosh-Gordon one, in the sense of being a series of repulsive power law self-interactions, just the coefficients differ. The form of these potentials and others mentioned above are shown in Fig. 1.

For the numerical procedure, it is best to make a redefinition of the scalar field mass

$$\tilde{m}^2 = \alpha m^2, \quad (18)$$

and with this also redefine the coordinate  $x$  and the frequency  $\Omega$ . Notice that  $\beta$  still appears within the differential equations while  $\alpha$  does not.

## A. Soliton stars

A potential with symmetry breaking was investigated by Lee et al. [13–15]. They called the solutions non-topological soliton stars, and found that the mass has units of  $M_{\text{P1}}^4/(m\sigma_0^2)$  which is huge in comparison with a boson or neutron star (for the case of comparable boson and fermion masses). The potential investigated was  $U = m^2|\psi|^2(1 - |\psi|^2/\sigma_0^2)^2$  where  $\sigma_0$  is a constant; this belongs to the more general forms of potentials derived in [3].

Compared with the usual boson star case, non-topological soliton stars have to fulfill two characteristics:

1. The Lagrangian must be invariant under a global  $U(1)$  transformation.
2. In the absence of gravity, the theory must have non-topological solutions; i.e. solutions with a finite mass, confined to a finite region of space, and non-dispersive.

In general, boson stars accomplish the requirement 1. but not 2. Invariance under  $U(1)$  only requires that the potential be a function of  $\psi^*\psi$ , and in order to fulfill condition 2.,  $U$  must contain attractive terms. This is why the coefficient of  $(\psi^*\psi)^2$  of Lee's potential has a negative sign. Finally, when  $|\psi| \rightarrow \infty$ ,  $U$  must be positive, which leads, minimally, to a sixth order function of  $\psi$  for the self-interaction. It is then clear that CSW's,  $U_{\text{cosh}}$ , and  $U_{\text{exp}}$  choices are not non-topological potentials. Neither of them have attractive terms. This is why, a priori, we may say that the order of magnitudes for the boson star masses remains the same as in CSW's case. The Sine-Gordon potential  $U_{\text{sin}}$  has, on the contrary, a similar series expansion, up to the sixth order, to that corresponding to Lee's potential. But here, what happens is that  $U_{\text{sin}}$ , in the absence of gravity and for a real scalar field, has not a non-topological soliton solution, instead, it has a topological one. It has a degenerate vacuum: an infinite set of  $\sigma$  values for which  $U_{\text{sin}} = 0$ . For a detailed account of this, we refer the reader to Lee's book, especially Chapter 7 and exercise 7.1 [16]. Then,  $U_{\text{sin}}$  neither is a non-topological soliton potential.

## B. Numerical solutions

Fig. 2 shows the usual plot of boson star configurations for the case of the Cosh-Gordon potential. The maximal mass is slightly higher than in the standard case due to the additional higher-order repulsive terms in the potential. For  $\beta = 1$ , we find  $M_{\text{max}} = 0.638 M_{\text{Pl}}^2/m$  and  $N_{\text{max}} = 0.658 M_{\text{Pl}}^2/m^2$  what for  $M$  and  $N$  is higher by 0.5%. Fig. 3 shows the stability analysis, which can be done using catastrophe theory [17–19]. One necessary condition for the configurations to be stable is a negative binding energy. However, this is not sufficient. Fig. 3 shows the appearance of two cusps signaling that there is a change in the star stability. The first branch is the only stable one, while the second and third are both unstable.

Fig. 4 represents similar profiles, but for the Sine-Gordon potential. In this case, the maximal values of mass and particle number are below the pure mass potential case. The influence of the higher order attractive terms is noticeably. For  $M$  and  $N$ , the maximal values are lower by about 2%:  $M_{\text{max}} = 0.620 M_{\text{Pl}}^2/m$  and  $N_{\text{max}} = 0.639 M_{\text{Pl}}^2/m^2$ . Fig. 5 shows the bifurcation plot for this case. The diagram in Fig. 5 shows cusps again, where is a change in stability. A remark on the calculation of mass is in order. We check that the calculation of the mass is correct by applying two different mass definitions. The first is the Schwarzschild mass, which is defined by the energy density  $\rho$  and which also appears in the asymptotic spherically symmetric spacetime,  $B(r) \rightarrow 1 - 2M/r$ , where  $M$  is the mass of the boson star. The formula for the Schwarzschild mass is

$$M_S = 4\pi \int_0^\infty \rho r^2 dr \quad (19)$$

$$= 4\pi \int_0^\infty \left( \varpi^2 \frac{\chi(r)^2}{B} + \left( \frac{d\chi(r)}{dr} \right)^2 \frac{1}{A} + U \right) r^2 dr. \quad (20)$$

A second mass formula can be derived for a general quasi-static isolated mass insula where a time-like Killing vector  $\xi^\alpha = 2n^\alpha$  exists [7]. Tolman's expression [20] is:

$$M_T = \int (2T_0^0 - T_\mu^\mu) \sqrt{|g|} d^3x. \quad (21)$$

For an asymptotically flat spherically symmetric spacetime, both masses agree with each other.

Finally, Figs. 6 and 7 shows the behavior of the  $U(1)$ -Liouville potential. It is both, qualitatively and quantitatively similar to the usual CSW's case. We have stable and unstable branches. The maximal mass and particle number are higher by about 5% which are the largest deviations from the standard case. The repulsive potential terms yield larger contributions in comparison with the Cosh-Gordon potential.

## III. OBSERVATIONAL OUTPUTS

At this stage, we have succeeded in proving that a change in the form of the self-interaction among the bosons yields appreciable -but small- changes in the form of the star configurations. We shall discuss below the feasibility of detecting observational consequences of these results concerning gravitational phenomena. In particular, we are interested in seeing if appreciable differences appear in the computation of gravitational redshifts, rotation curves, and gravitational lensing features. For the Sine-Gordon potential, and because of the smaller masses it produces, gravitational phenomena are diminished. So we shall be mainly interested in Cosh-Gordon and  $U(1)$ -Liouville cases.

### A. Gravitational redshifts

In this section, we follow Ref. [10], and make use of the assumption that the scalar particles have no interaction -other than gravitational- with baryonic matter. Thus, this *normal* matter can penetrate the boson star up to the center and if it emits radiation there, well within the gravitational potential, we expect the spectral features to be redshifted.

The gravitational redshift  $z$  of our static line element is given by

$$1 + z = \sqrt{\frac{B(\infty)}{B(int)}}, \quad (22)$$

where  $int$  stands for the position of the emitter particle with respect to the star center. As the receiver is practically at infinity,  $B(\infty) \sim 1$ . The maximum possible redshift is obtained when the particle emits exactly at the center of the boson star, where the metric deviates maximal from outside vacuum space-time.

We are only interested in stable configurations, the maximum redshift is then provided by the maximum value of  $\sigma(0)$ , which gives the biggest mass. As it was shown in Ref. [10], the simple mass term produced a maximum redshift of 0.45 while for CSW's choice, with  $\Lambda$  tending to infinity, one gets 0.69. Here we find  $z_{max} = 0.46$  for the Cosh-Gordon potential and  $z_{max} = 0.49$  for the  $U(1)$ -Liouville potential. For comparison, we quote the result  $z_{max} = 0.47$  for neutron stars.

### B. Rotation curves

Another gravitational effect considered for CSW's choice [10], and which we would like to compare with the more generic potentials here studied are the rotation curves of test particles moving around boson stars. For our metric, circular geodesics have a rotation velocity (as measured by an observer at infinity) given by,

$$v_\varphi^2 = \frac{x B'}{2}. \quad (23)$$

In Fig. 8 we compare the rotation curves for the case  $\Lambda = 0$ ,  $\Lambda = 100$  of CSW's choice, and our potentials: Cosh-Gordon and  $U(1)$ -Liouville. Already from the usual case, it was shown that the possible velocities that particles can reach are a notable amount of the speed of light, and matter can have an impressive kinetic energy. If such kinetic energy were transferred to radiation, we could expect very luminous boson stars, orders of magnitude more luminous than the Sun. This provides speculative alternatives to accretion disks around black holes, and would make boson stars almost indistinguishable from its final effects. This is currently being analyzed, especially concerning the possibility of having boson stars at the center of some galaxies, as was proposed with neutrino balls. For the new potentials we are analyzing here, we obtain that  $U_{\cosh}$  and  $U_{\exp}$  produce similar angular velocities to the  $\Lambda = 0$  case. Their maximum velocity happens for values  $x \sim 5$  and have typical magnitudes of 100 000 km  $s^{-1}$ .

### C. Gravitational lensing

Boson star microlensing effects were first investigated by Dąbrowski and Schunck [6], for CSW's  $\Lambda = 0$  case, also known as the mini-boson star. Their procedure, which we closely follow, consist in studying the photon trajectories along the curved (boson star generated) space-time. For particular details of the derivation of

quoted formulae see Refs. [6,21]. In Ref. [21], a related study on microlensing features was made, taking a scalar-field-generated naked singularity as lens. It has the property of producing both, positive and negative binding angles; in this later case, in a way similar to the recently studied wormhole microlensing scenario [22].

The light traveling from a distant source is deflected, because of the presence of the boson star, with a deflection angle given by (see Fig. 9 for a schematic drawing of the geometry),

$$\hat{\alpha} = \Delta\varphi - \pi, \quad (24)$$

where,

$$\Delta\varphi = 2 \int_{r_0}^{\infty} \frac{dr}{r} \frac{\sqrt{AB}}{\sqrt{r^2/b^2 - B}}, \quad (25)$$

with impact parameter  $b = r_0 \sqrt{1/B(r_0)}$  ( $r_0$  is the closest distance between the light ray and the center of the boson star: the first point where the square root in the denominator is non-negative). The lens equation can be expressed as the difference between the true angular position,  $\beta$ , and the image positions,  $\vartheta$ , as [23,24]

$$\sin(\vartheta - \beta) = \frac{D_{ps}}{D_{os}} \sin \hat{\alpha}, \quad (26)$$

where  $D_{ps}$  ( $D_{os}$ ) stands for the angular distance between the point P close to the lens and the source (the observer and the source). Also, from the geometry of the lens we have  $\sin \vartheta = b/D_{ol}$ . Hence, choosing  $\vartheta$  and the distance, we have  $b$ , and  $\Delta\varphi$  may be computed afterwards. In the numerical program, we use again the dimensionless quantities of Eqs. (11-12) and instead of the impact parameter  $b$ , we follow [6] and take  $\vartheta$ . The term  $(r/b)^2$  in (25) is then  $x^2/(\varpi^2 D_{ol}^2 \sin^2 \vartheta)$  or just  $x^2/(\varpi^2 D_{ol}^2 \vartheta^2)$  for small  $\vartheta$ , respectively. Our numerical program uses always the correct  $\sin \vartheta$  without any abbreviations so that also angles in the degree regime can be calculated. The examples in Figs. 10 and 11 apply  $\vartheta$  in arc-seconds having the additional unit factor 1/206265 for one arc-second in radians. The change to other units can then be explained by an additional *distance factor*  $n$ . For instance, if  $\vartheta$  has to be measured in milli-arc-seconds,  $n$  equals  $10^{-3}$ . In order to get rid of the distance factor within the numerical program, it is chosen  $D_{ol} = 206265/(\varpi n)$ .

Furthermore, the reduced angular deflection angle is defined to be

$$\alpha = \vartheta - \beta = \arcsin \left( \frac{D_{ps}}{D_{os}} \sin \hat{\alpha} \right). \quad (27)$$

A second lens equation can be derived for large deflection angles, where  $D_{Is}$  cannot be considered as being similar to  $D_{ps}$ . Of course, by its construction the source is always within a plane with constant distance to the observer, and studying the diagrammatic view depicted in

Fig. 9, it can be obtained that (for a detailed derivation see Appendix of Ref. [6]),

$$\sin \alpha = \frac{D_{ls}}{D_{os}} \cos \vartheta \cos \left[ \arcsin \left( \frac{D_{os}}{D_{ls}} \sin(\vartheta - \alpha) \right) \right] \times \\ \times [\tan \vartheta + \tan(\hat{\alpha} - \vartheta)] , \quad (28)$$

where  $D_{ls}$  stands for the angular distance between the lens and the source. Lens equation (26) requires only the proportion between  $D_{ps}$  and  $D_{os}$  so that, in general, the position of source S describes a more complicated surface. For the physical situation considered in our paper, the differences for  $\alpha$  amounts a few parts per thousand of a degree at most so that our Figs. 10 and 11 describe both cases.

Assuming that the boson star lens is half-way between the observer and the source, such that  $D_{ls}/D_{os} = 1/2$  and  $D_{ps}/D_{os} = 1/2$ , we performed numerical computations of the reduced deflection angle for our new potentials, which we show in Figs. 10 and 11. The difference among these cases and the simple mass term (corresponding to the mini-boson star) is clearly observable. We have taken for the plot the maximum central density (which produces the maximum deflection angle). In the case of the mini-boson star, the biggest possible value of  $\alpha$  is 23.03 degrees with an image at about  $\vartheta = n \times 2.88$  arc-secs with the distance factor  $n = n(D_{ol}, \Omega)$  which is a function of the distance from the observer to the lens and the scalar field frequency, whose inverse can be associated with the star radius. In our examples we assume that  $n = 1$ , which fixes  $\vartheta$  to be measured in arc-sec.

The characteristics of the boson stars produce the deflection angles which depend on the observer-to-lens-distance  $D_{ol}$  and the mass. As mentioned above, if  $\vartheta$  is chosen to be of the order of arc-secs (distance factor  $n = 1$ ), then the distance  $D_{ol}$  is measured in units of  $206265/\varpi$ . Under the assumption that the mass of the boson star is  $10^{10}M_{\odot}$  one has  $\varpi \sim 10^{-15}\text{cm}^{-1}$ . Then, the distance  $D_{ol}$  is about 100pc. If the distance factor is  $n = 10^{-3}$ , and so  $\vartheta$  measured in milli-arc-secs, the boson-star-lens is at about 100kpc.

We assumed that the boson star is transparent and calculated the deflection angles. All qualitative features of a non-singular spherically symmetric transparent lens can be revealed using Figs. 10 and 11: the lens curve for the maximal boson star. Three images exist, two of them being inside the Einstein radius and one outside. An Einstein ring with infinite tangential magnification (tangential critical curve) is found, and also a radial critical curve for which two internal images merge. The appearance of the radial critical curve distinguish boson stars from other extended and non-transparent lenses. For a black hole or a neutron star, the radial critical curve does not exist because it is inside the event horizon or the star. Two bright images near the center of the boson star and the third image at some very large distance from the center are found. For non-relativistic cases smaller angles will be found. An interesting point can be made if one

considers an extended source. In such a case one finds the two radially and tangentially elongated images very close to each other. Then, looking along the line defined by these two images the third one can be detected at a very large distance.

For the Cosh-Gordon potential, we obtain as maximal reduced deflection angle 23.229 degrees, and for  $U_{\text{exp}}$ , an even larger deviation, the biggest possible being 24.391 degrees with an image about at the same place. Differences among these cases and the usual one is between 0.2 and 0.4 degrees.

#### IV. DISCUSSION

In a recent communication, Ho et al. [25] have studied the maximum masses of boson stars formed with different self-interaction terms (all of them, however, of power law form and of positive sign). By just comparing the order of magnitude of the terms involved in the self-interaction with the mass term, and asking for them to be of the same order, they were able to see that the contribution of higher order terms goes as power of  $(m/M_{\text{Pl}})^2$ . The fact that the contribution to the maximum masses of these different boson stars decreases (if  $m < M_{\text{Pl}}$ ), does not automatically yield to unobservable effects, as we have discussed in the previous sections.

The star masses maintain the order of magnitude, for equal single boson masses, when compared to those cases studied by CSW [4], which is in agreement with the results of Ho et al. [25]. This also stems from the fact that all Lagrangians analyzed are not non-topological ones. Changes are of order of several percent of the star mass.

For the first time, we have investigated the systems of differential equations of Einstein-Cosh-Gordon, Einstein-Sine-Gordon, and Einstein- $U(1)$ -Liouville. The different potentials studied so far showed similar gravitational redshifts and rotational curves, with high angular velocities, when compared among them, and between them and the CSW's case. However, large deviations in the maximum angular deflection have appeared, with differences amounting appreciable parts of a degree. Some observational effects distinguish boson stars from other non-transparent compact objects.

However, fair is to say that if an observable determination proves the existence of a boson star, the effective form of the Lagrangian may be hidden within the percentage of possible errors. In that case, facing with the problem of degeneracy –i.e. different physical theories giving the same observational effects– Occam's razor would probably lead us to consider just the CSW's choice. Only a detailed knowledge of the boson involved, and the average form of the interactions, all of them encompassed in the self-interaction term, may shed light on the explicit model for the Lagrangian.

The definition of the actual boson which participates in the construction of the boson star may also influence the

transparent consideration for gravitational lensing phenomena. For instance, if the star is made up of Higgs particles, we may expect some kind of interaction apart from the gravitational one that may yield the star to be a non-transparent object.

Furthermore, we become aware of the possibility that some potentials may give rise of tunneling of parts of the scalar field. Of course, this effect can only occur in the quantum regime, hence, if boson stars are in the order of magnitude of atoms or even atomic nuclei. Our first preliminary results for the Newtonian case show that especially the form of potentials of Lee et al. and of Sine-Gordon can lead to instability due to tunneling. The effect could mean two things: (i) The boson (soliton) star is destroyed: it disperses or it forms a black hole. (ii) The boson (soliton) star experiences an internal rearrangement. We expect to report on these issues on a forthcoming article.

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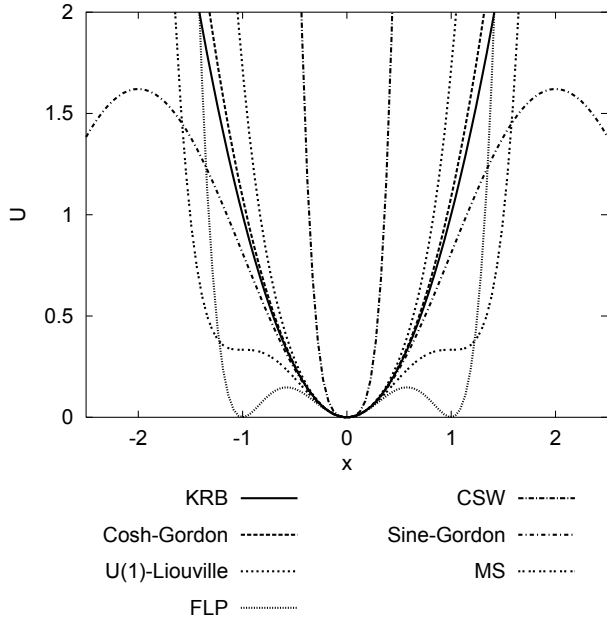


FIG. 1. Comparison of different potentials. KRB (Kaup-Ruffini-Bonazzola):  $x^2$ , Cosh-Gordon:  $2(\cosh(x) - 1)$ , U(1)-Liouville:  $\exp(x^2) - 1$ , FLP (Friedberg-Lee-Pang):  $x^2(1 - x^2/2)^2$ , CSW (Colpi-Shapiro-Wasserman):  $x^2 + 100x^4/2$ , Sine-Gordon:  $(8/\pi^2)\{\sin(\pi/2[x - 1]) + 1\}$ , MS (Mielke-Scherzer):  $x^2 - x^4 + x^6/3$ .

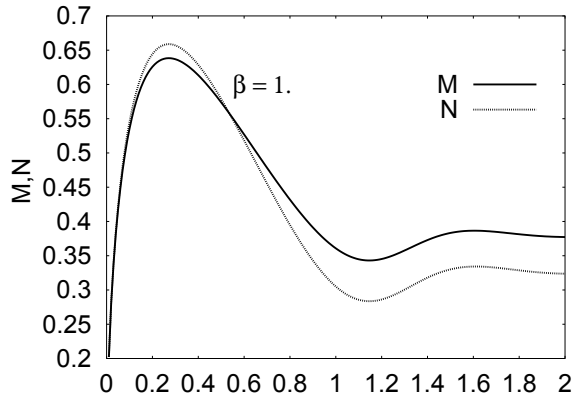


FIG. 2. Configurations for boson stars self-interacting via a Cosh-Gordon potential for  $\beta = 1$ .

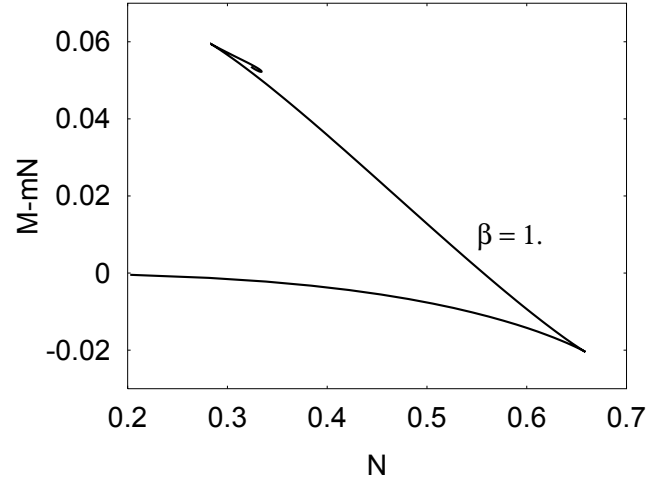


FIG. 3. Stability analysis for Cosh-Gordon configurations.

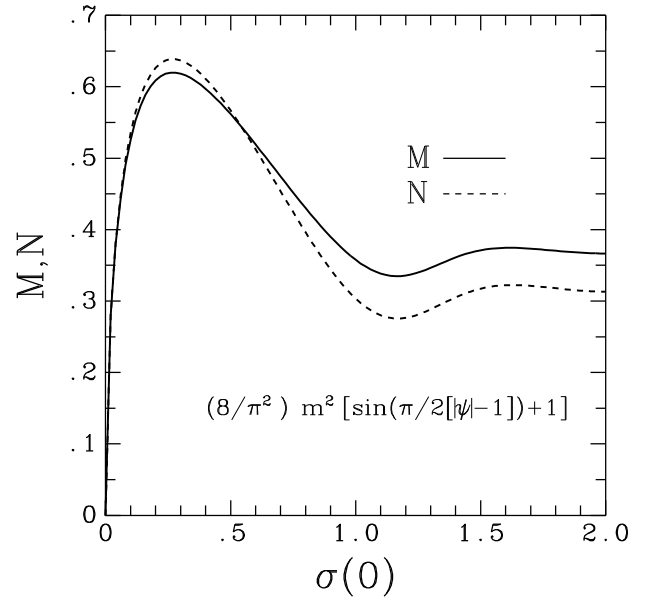


FIG. 4. Configurations for boson stars self-interacting via a Sine-Gordon potential.



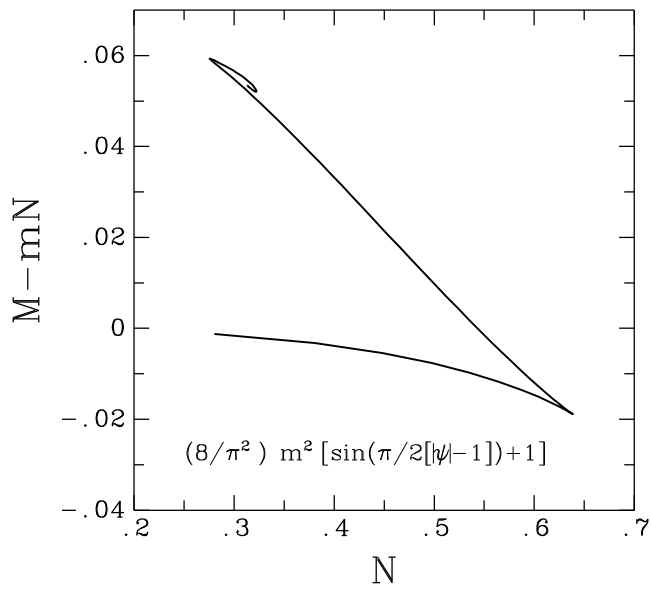


FIG. 5. Bifurcation diagram for the Sine-Gordon potential configurations.

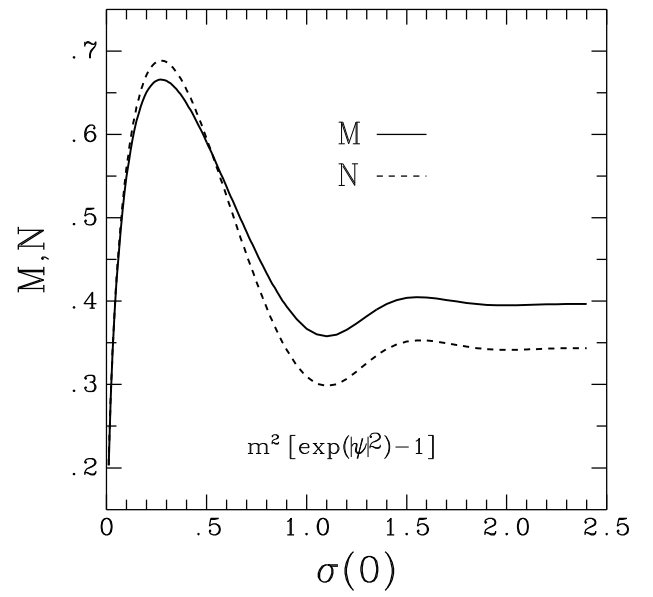


FIG. 6. Configurations for boson stars self-interacting via a  $U(1)$ -Liouville potential.

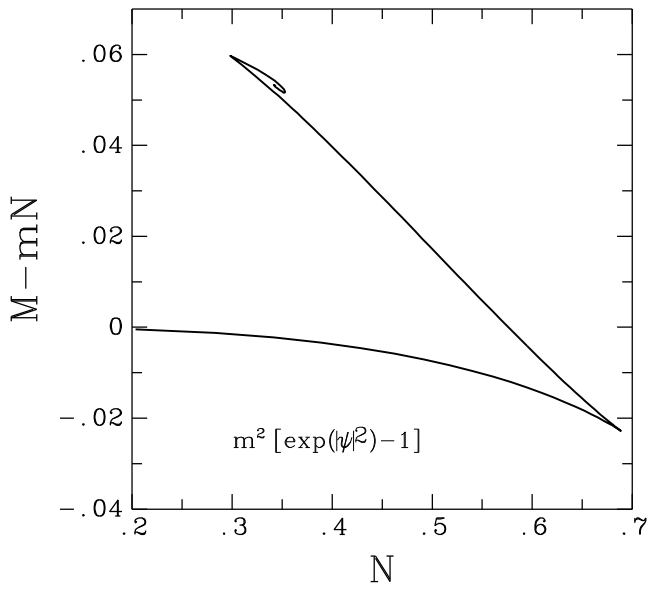


FIG. 7. Stability analysis for the  $U(1)$ -Liouville potential configurations. The binding energy changes sign and the presence of cusps signals changes in stability. The first branch is the only stable one.

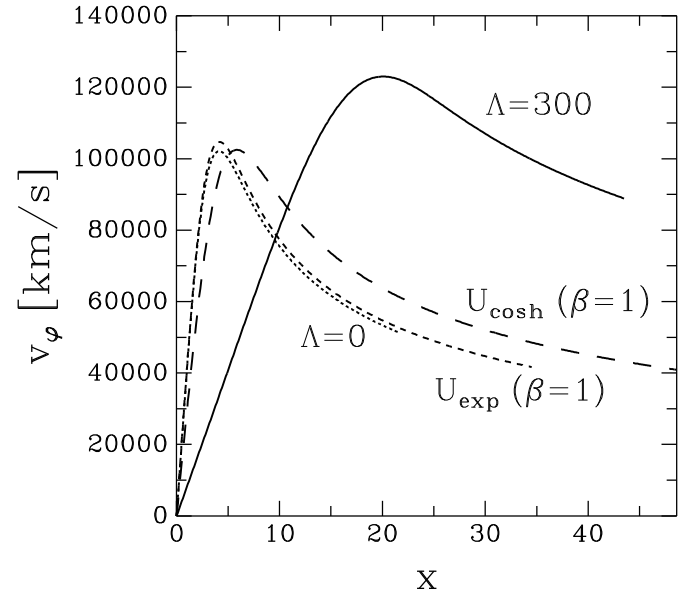


FIG. 8. Rotation curves for non-interacting boson stars and boson stars with generic potentials. Shown in the Figure are curves corresponding to  $\Lambda = 0, 300$  of the CSW's choice, taken from the work of Schunck and Liddle, and our new results for the potentials  $U_{\text{cosh}}$  and  $U_{\text{exp}}$ . The maximal velocities are: 122 990km/s at  $x = 20.1$  for  $\Lambda = 300$ , 102 073km/s at  $x = 4.1$  for  $\Lambda = 0$ , 104 685km/s at  $x = 4.2$  for  $U_{\text{exp}}$ , and 102 459km/s at  $x = 5.9$  for  $U_{\text{cosh}}$ .

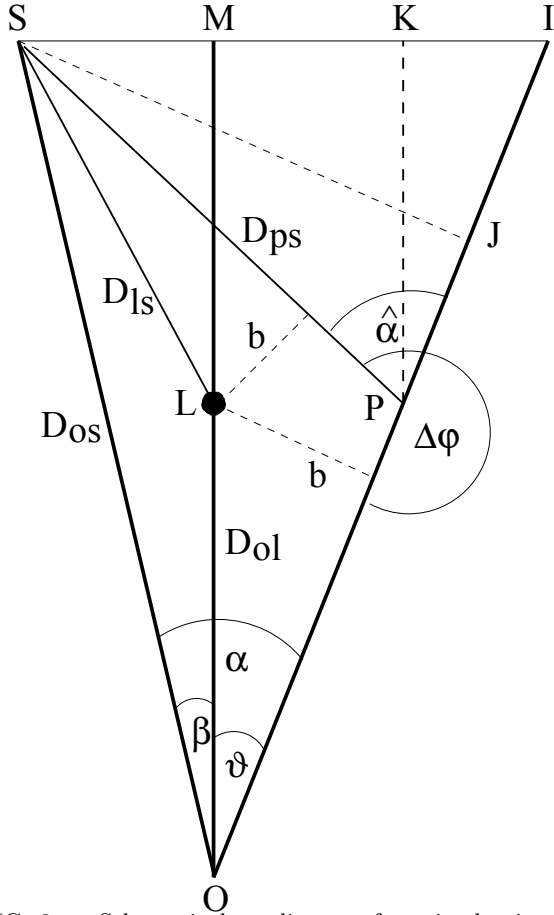


FIG. 9. Schematic lens diagram for microlensing phenomenon.  $D_{ol}$  is the distance from the observer (O) to the lens (L),  $D_{os}$  is the observer-source distance, and  $D_{ls}$  the distance from the lens to the source (S). The angle  $\beta$  denotes the true angular position of the source whereas  $\vartheta$  measures the angle of the image position.  $D_{ps}$  is the distance from the point P in the source plane to the source (for small deflection angles,  $D_{ps} \sim D_{ls}$ , as well as  $D_{os} \sim D_{ol} + D_{ls}$ ). For the meaning and use of all other parameters see text and the Appendix of the paper by Dąbrowski and Schunck.

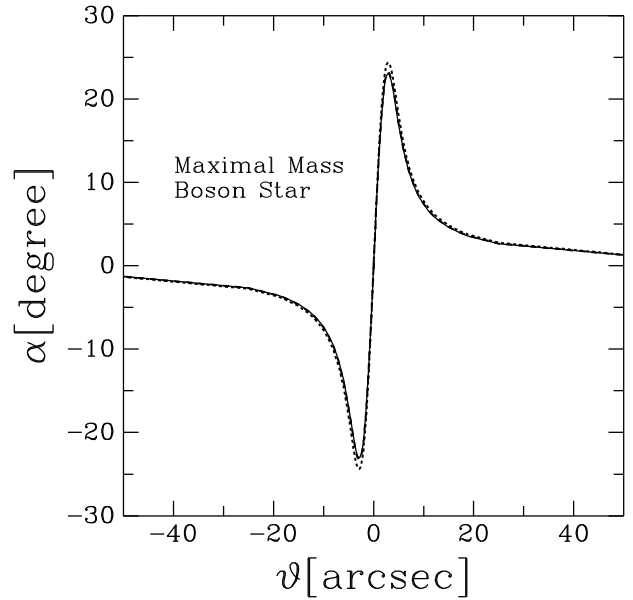


FIG. 10. The reduced deflection angle (difference between the true and the image angular position) as a function of the the image position for the different potentials.  $U_{\text{exp}}$  produces the largest deflection angle in comparison with the simple mass term potential (CSW's choice with  $\Lambda = 0$ ) and the Cosh-Gordon potential. We have chosen here the stable maximal mass boson star.

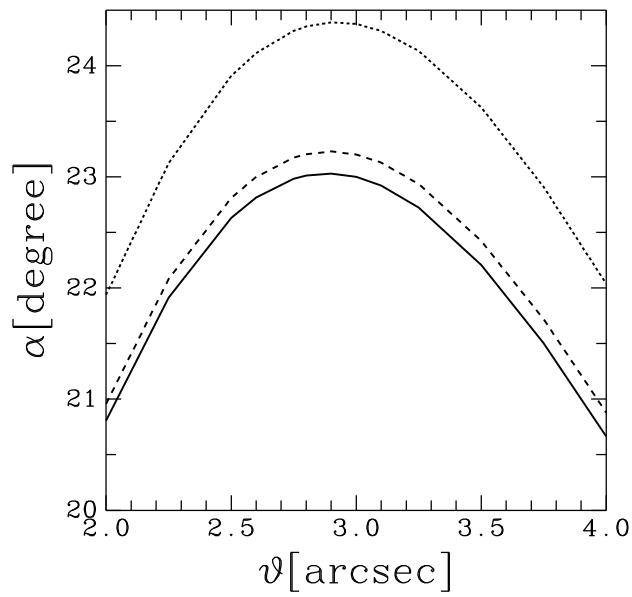


FIG. 11. Comparison for all three potentials at the maximal reduced deflection angle. The upper curve is for  $U_{\text{exp}}$ , the middle curve is for  $U_{\text{cosh}}$  and the lower one for  $\Lambda = 0$  CSW's choice.