(1+1)-dimensional Galilean supersymmetry in ultracold quantum gases

Gustavo S. Lozano,[1,*] Oliver Piguet,[2] Fidel A. Schaposnik,[3,†] and Lucas Sourrouille[1]
[1]*Departamento de Física, FCEyN, Universidad de Buenos Aires, Pab.1, Ciudad Universitaria, 1428, Ciudad de Buenos Aires, Argentina*
[2]*Universidade Federal do Espírito Santo, UFES, Vitória, ES, Brazil*
[3]*Departamento de Física, Facultad de Ciencias Exactas, Universidad Nacional de La Plata, C.C. 67, 1900 La Plata, Argentina*
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We discuss a (1+1)-dimensional Galilean invariant model recently introduced in connection with ultracold quantum gases. After showing its relation to a nonrelativistic (2+1) Chern-Simons matter system, we identify the generators of the supersymmetry and its relation with the existence of self-dual equations.

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I. INTRODUCTION

The study of supersymmetry (SUSY) started more than 30 years ago in the context of *relativistic* field theories. They play today a major role in the understanding of several properties of high energy physics, ranging from phenomenological aspects of the standard model of electroweak interactions to mathematical consistency of superstring theories [1].

From a mathematical point of view, supersymmetry is related to a grading of the group of space-time symmetries of the theory and as such, the concept of supersymmetry can be extended to other groups beyond Poincaré.

The case where the space-time group is the Galilean group is particularly relevant as many condensed matter systems display explicitly this symmetry. The (3+1)-dimensional case was first considered in Ref. [2]. In Ref. [3], the supersymmetric extension of the Galilean group in (2+1)-dimensions was discussed and a field theoretical model possessing this symmetry was explicitly built. The model is interesting as it incorporates gauge fields. The dynamics of these gauge fields is dictated by a Chern-Simons term, which due to its topological character has a larger symmetry than the usual Maxwell action governing electromganetism. It is also possible to write for the bosonic sector of the model self-dual or Bogolnyi-Prassad-Sommerfeld (BPS) equations [4], indicating that the relation between BPS equations and supersymmetry, characteristic of relativistic theories [5], also extends to the nonrelativistic domain. Galilean supersymmetry in (2+1)-dimensions was further studied in [6]. Notice that in the models considered in Refs. [3,6,7], the invariance group is indeed enlarged by the presence of dilatation and conformal symmetries leading to the study of Schroedinger superalgebras.

Another area where Galilean symmetry can be relevant is that concerned with the study of perfect fluids. Also in this case fermionic degrees of freedom can be incorporated in the theory and supersymetric models in (2+1)- and (1+1)-dimensions can be built [8–10].

Very recently, a Galilean model in (1+1)-dimension was considered in Refs. [11,12] in the context of ultracold boson-fermion mixture in one-dimensional optical lattices. This is indeed a very interesting proposal since according to these authors it is possible to tune experimentally the parameters of the model in such a way that the effective Hamiltonian describing the dynamics of the vortex is indeed a (1+1)-dimensional Galilean supersymmetric theory.

In this paper we will study in more detail the structure of the supersymmetry behind the model presented in [11,12]. In particular, we will find the correct supersymmetry transformations that leave the action invariant. A basic feature of supersymmetric theories is that the *full* Hamiltonian can be written as an anticommutator of supersymmetry generators. We will show that the charges generating these transformations in this nonrelativistic model also satisty this basic property. This should be contrasted with the results in [12] where the anticommutator of the charges equals just the free Hamiltonian.

In order to study the supersymmetric structure of the theory, we will make use of a relation of the (1+1) model presented in [11,12] with the (2+1) model studied in [3]. This connection, which was already known for the bosonic sector of the theory [13], is not necessary for establishing the main result of this paper (the correct supersymmetry algebra), but in our opinion, the existence of such a connection makes the model even more interesting. On the other hand, it will lead us in a natural way to the discussion of the relation between supersymmetry and self-dual equations.

II. THE MODEL

Let us then start by considering the (2+1)-dimensional Chern-Simons model coupled to nonrelativistic bosonic (ϕ) and fermionic (down-spinor ψ) matter governed by the action

$$S = S_{cs} + \int d^3x\biggl(i\phi^\dagger D_t\phi + i\psi^\dagger D_t\psi - \frac{1}{2m}(D_i\phi)^\dagger D_i\phi - \frac{1}{2m}(D_i\psi)^\dagger D_i\psi - \frac{e}{2m}\psi^\dagger F_{12}\psi + \lambda_1(\phi^\dagger\phi\phi^\dagger\phi) + \lambda_2(\phi^\dagger\phi\psi^\dagger\psi)\biggr), \qquad (1)$$

where the Chern-Simons action is given by

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*Associated with CONICET.
†Associated with CICBA.

$$S_{cs} = \frac{\kappa}{4}\int d^3x \epsilon^{\mu\nu\lambda}A_\mu F_{\nu\lambda} = \kappa\int d^3x(A_0F_{12} + A_2\partial_t A_1), \quad (2)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad D_\mu = \partial_\mu + ieA_\mu. \quad (3)$$

The metric tensor is $g^{\mu\nu}=(1,-1,-1)$ and $\epsilon^{\mu\nu\lambda}$ is the totally antisymmetric tensor such that $\epsilon^{012}=1$. We are choosing units such that $\hbar=1$. We are including a Pauli term for the fermion corresponding to a down-spinor.

The bosonic sector of the ($\psi=0$) model was first considered in Refs. [14,4]. It is an explicit example of a gauge theory possessing Galilean invariance and it provides a field theoretical formulation of the Aharonov Bohm problem. Notice that there is no massless particle associated to the vector field. For the particular relation of coupling constants

$$\lambda_1 = \frac{e^2}{2m\kappa} \quad (4)$$

the model has self-dual equations and many of the properties of its soliton solutions can be established in detail [4]. The fermionic generalization of this model has been analyzed in [3] where it was shown that for a particular choice of coupling constants,

$$\lambda_1 = \frac{e^2}{2m\kappa}, \quad \lambda_2 = 3\lambda_1, \quad (5)$$

it exhibits an extended Galilean symmetry. Thus the usual connection between the existence of BPS equations and extended supersymmetry [5] also holds in this nonrelativistic example.

In Ref. [13] a reduction of the purely bosonic model to (1+1)-dimensions was considered by assuming that the fields do not depend on one of the spatial coordinates, say x_2. One of the main purposes in that work was to explore if the peculiar static properties of the parent (2+1)-dimensional model were inherited by the (1+1) reduction. The answer for this is negative and, as it was discussed in detail in [13], the gauge fields can be eliminated from the theory without changing the physical structure of the model.

Renaming A_2 as B, and extending this procedure to the full model leads to an action that can be written as

$$S = S_{rcs} + \int dxdt\Bigg\{ i\phi^\dagger D_t\phi + i\psi^\dagger D_t\psi - \frac{1}{2m}(D_x\phi)^\dagger D_x\phi - \frac{1}{2m}(D_x\psi)^\dagger D_x\psi - \frac{e}{2m}\partial_x B\rho_f - \frac{e^2}{2m}B^2\rho + \lambda_1\rho_b^2 + \lambda_2\rho_b\rho_f\Bigg\}, \quad (6)$$

where S_{rcs} is the reduced Chern-Simons action, or "BF" term

$$S_{rcs} = \kappa\int dxdt(B\partial_t A_1 + A_0\partial_1 B) = \kappa\int dxdt BF_{01}, \quad (7)$$

and we have introduced the matter densities,

$$\rho_b = \phi^\dagger\phi, \quad \rho_f = \psi^\dagger\psi, \quad \rho = \rho_b + \rho_f. \quad (8)$$

Notice that the Gauss law constraint

$$\partial_x B = \frac{e}{\kappa}\rho \quad (9)$$

can be solved as

$$B(x) = \frac{e}{2\kappa}\int dz\epsilon(x-z)\rho(z), \quad (10)$$

where $\epsilon(x)=\theta(x)-\theta(-x)$ is the odd step function

When A_1, A_0, and B are set to zero the system has a "trivial" supersymmetry where bosons and fermions are interchanged according to

$$\delta_1\phi = \sqrt{2m}\eta_1^\dagger\psi, \quad \delta_1\psi = -\sqrt{2m}\eta_1\phi, \quad (11)$$

if $\lambda_2=2\lambda_1$. Here, η_1 is an infinitesimal Grassmann parameter.

This supersymmetry survives the incorporation of the additional interactions if

$$\delta_1 A_1 = \delta_1 B = 0, \quad \delta_1 A_0 = \frac{e}{\sqrt{2m}\kappa}(\eta_1\phi\psi^\dagger - \eta_1^\dagger\psi\phi^\dagger) \quad (12)$$

and

$$\frac{e^2}{2m\kappa} + 2\lambda_1 - \lambda_2 = 0. \quad (13)$$

The model is also invariant under a second, less obvious, supersymmetry transformation given by

$$\delta_2\phi = \frac{i}{\sqrt{2m}}\eta_2^\dagger(D_x\psi - eB\psi), \quad \delta_2\psi = -\frac{i}{\sqrt{2m}}\eta_2(D_x\phi + eB\phi),$$

$$\delta_2 A_1 = -\frac{e}{\sqrt{2m}\kappa}(\eta_2\phi\psi^\dagger - \eta_2^\dagger\psi\phi^\dagger),$$

$$\delta_2 B = \frac{ie}{\sqrt{2m}\kappa}(\eta_2\phi\psi^\dagger + \eta_2^\dagger\psi\phi^\dagger),$$

$$\delta_2 A_0 = \frac{ie}{(2m)^{3/2}\kappa}[\eta_2\phi(D_x\psi - eB\psi)^\dagger + \eta_2^\dagger(D_x\psi - eB\psi)\phi^\dagger] \quad (14)$$

provided that the coupling constants satisfy

$$\lambda_1 = \frac{e^2}{2mk}. \quad (15)$$

These transformations can be obtained via the dimensional reduction of the supersymmetry of the parent (2+1)-dimensional model [3] which in turn can be obtained via the nonrelativistic limit (contraction) of the associated relativistic (2+1) model [15].

We will discuss next the relation of the model described by the action Eq. (6) with the model discussed in Ref. [12] where no fields A_0, A_1, or B appear. The action of the model of Ref. [12] without the chemical potential terms is

$$S = \int d^2x \left\{ i\phi^\dagger \partial_t \phi + i\psi^\dagger \partial_t \psi - \frac{1}{2m}(\partial_x \phi)^\dagger \partial_x \phi - \frac{1}{2m}(\partial_x \psi)^\dagger \partial_x \psi + \lambda_1 \rho^2 \right\}. \quad (16)$$

It was shown in [13,16], for the bosonic sector of the (1+1)-dimensional model given by Eq. (6), that the field A_1 can be eliminated via a gauge transformation. This property is also valid after the incorporation of fermions. Indeed, after transforming the matter fields as

$$\phi(x) \to e^{-i\alpha(x)}\phi(x), \quad \psi(x) \to e^{-i\alpha(x)}\psi(x) \quad (17)$$

with

$$\alpha(x) = \frac{e}{2}\int dz \epsilon(x-z) A_1(z) \quad (18)$$

and using the Gauss law, i.e, using the explicit form of B given by Eq. (10), the action can be written simply as

$$S = \int dxdt \left\{ i\phi^\dagger \partial_t \phi + i\psi^\dagger \partial_t \psi - \frac{1}{2m}(\partial_x \phi)^\dagger \partial_x \phi - \frac{1}{2m}(\partial_x \psi)^\dagger \partial_x \psi - \frac{e^2}{2m\kappa}\rho\rho_f + \lambda_1 \rho_b^2 + \lambda_2 \rho_b \rho_f - \frac{e^2}{2m}B^2 \rho \right\}. \quad (19)$$

After elimination of the gauge field A_1 and use of the Gauss law, the SUSY transformations can be written as

$$\delta_2 \phi = \frac{i}{\sqrt{2m}}\eta_2^\dagger(\partial_x \psi - eB\psi) + i\delta_2 \alpha \phi,$$

$$\delta_2 \psi = -\frac{i}{\sqrt{2m}}\eta_2(\partial_x \phi + eB\phi) + i\delta_2 \alpha \psi. \quad (20)$$

The quantities B and α should be considered a functional of the matter fields, see Eqs. (10) and (18).

Notice that the last term in the action is a constant of motion. Indeed

$$\frac{e^2}{2m}\int dx_1 B^2(x_1)\rho(x_1) = \frac{e^4}{8m\kappa^2}\int dx_1 dx_2 dx_3 \epsilon(x_1-x_2)\epsilon(x_1-x_3)\rho(x_1)\rho(x_2)\rho(x_3) = \frac{e^4}{24m\kappa^2}\int dx_1 dx_2 dx_3 \rho(x_1)\rho(x_2)\rho(x_3) = \frac{e^4 N^3}{24m\kappa^2}, \quad (21)$$

where $N = \int dx \rho(x)$, and use has been made of the identity

$$\epsilon(x_1-x_2)\epsilon(x_1-x_3) + \epsilon(x_2-x_3)\epsilon(x_2-x_1) + \epsilon(x_3-x_1)\epsilon(x_3-x_2) = 1. \quad (22)$$

Thus dropping this term, the action at the supersymmetric point [given by Eqs. (13) and (15)], can be written as

$$S = \int d^2x \left\{ i\phi^\dagger \partial_t \phi + i\psi^\dagger \partial_t \psi - \frac{1}{2m}(\partial_x \phi)^\dagger \partial_x \phi - \frac{1}{2m}(\partial_x \psi)^\dagger \partial_x \psi + \lambda_1 \rho^2 \right\}. \quad (23)$$

The model written in this way is the one considered recently in [12] in the context of the dynamics of vortices in boson-fermion mixtures. In fact, this model was originally considered by Lai and Yang [17] (see also [18]).

III. THE SUPERSYMMETRY ALGEBRA

In discussing the supersymmetry algebra behind this model, the following generators were given in Ref. [12] (up to normalization)

$$Q_1 = -i\sqrt{2m}\int dx \psi^\dagger \phi, \quad R = -\frac{1}{\sqrt{2m}}\int dx \psi^\dagger \partial_x \phi. \quad (24)$$

The Q_1 generator is related to the first of the supersymmetries (12), and can be obtained via the Noether theorem. Poisson brackets can be defined as

$$\{F,G\}_{PB} = i\int dr \left(\frac{\delta F}{\delta \phi^\dagger(r)}\frac{\delta G}{\delta \phi(r)} - \frac{\delta F}{\delta \phi(r)}\frac{\delta G}{\delta \phi^\dagger(r)} - \frac{\delta^r F}{\delta \psi^\dagger(r)}\frac{\delta^l G}{\delta \psi(r)} - \frac{\delta^r F}{\delta \psi(r)}\frac{\delta^l G}{\delta \psi^\dagger(r)} \right), \quad (25)$$

where the subscripts r and l refer to right and left derivatives. In particular,

$$\{\phi(x_1,t), \phi^*(x_2,t)\} = -i\delta(x_1-x_2),$$

$$\{\psi(x_1,t), \psi^*(x_2,t)\} = -i\delta(x_1-x_2). \quad (26)$$

It is easy to show that

$$\{Q_1, Q_1^\dagger\} = -2im\int dx \rho \equiv -2iM. \quad (27)$$

Nevertheless, the second generator R defined in [12] does not give transformations (14). This R charge is related to a symmetry of the free part of the theory (no interactions) and its anticommutator gives only the free Hamiltonian,

$$\{R, R^\dagger\} = -iH_{free} = \frac{i}{2m}\int dx (\phi^\dagger \partial_x^2 \phi + \psi^\dagger \partial_x^2 \psi). \quad (28)$$

The expression for the charge generating the second set of transformations (14), which correspond to a supersymmetry of the full hamiltonian, can be easily obtained by considering the dimensional reduction of the corresponding charge in the (2+1) model. This leads to

$$Q_2 = -\frac{1}{\sqrt{2m}}\int dx\psi^\dagger(\partial_x + eB)\phi$$
$$= -\frac{1}{\sqrt{2m}}\int dx\psi^\dagger(x)\left(\partial_x + \frac{e^2}{2\kappa}\int dz\epsilon(x-z)\rho(z)\right)\phi(x). \quad (29)$$

Using the definition of Poisson brackets, we can now calculate

$$\{Q_2,Q_2^\dagger\} = -\frac{i}{2m}\int dx(\partial_x - eB)\psi^\dagger(\partial_x - eB)\psi + (\partial_x + eB)\phi^\dagger(\partial_x + eB)\phi - \frac{2e^2}{\kappa}\phi^\dagger\phi\psi^\dagger\psi \quad (30)$$

or

$$\{Q_2,Q_2^\dagger\} = -\frac{i}{2m}\int dx\bigg(\partial_x\psi^\dagger\partial_x\psi + e^2B^2\rho_f + \partial_x\phi^\dagger\partial_x\phi + e^2B^2\rho_b - \frac{2e^2}{\kappa}\rho_f\rho_b - e\partial_xB\rho_b + e\partial_xB\rho_f\bigg), \quad (31)$$

which, after using the Gauss law [Eq. (10)] becomes

$$\{Q_2,Q_2^\dagger\} = -iH. \quad (32)$$

Thus the anticommutator of the charges is related to the *full* Hamiltonian.

Finally, the only remaining nonvanishing bracket gives

$$\{Q_1,Q_2^\dagger\} = -\frac{1}{2}\int d^2x\{\phi^\dagger(\partial_x - eB)\phi - [(\partial_x - eB)\phi]^\dagger\phi + \psi^\dagger(\partial_x - eB)\psi - [(\partial_x - eB)\psi]^\dagger\psi\}$$
$$= -\frac{1}{2}\int d^2x\phi^\dagger\partial_x\phi - \partial_x\phi^\dagger\phi + \psi^\dagger\partial_x\psi - \partial_x\psi^\dagger\psi = -iP_1, \quad (33)$$

P_1 being the momentum. We have used that

$$\int dxB(x)\rho = \frac{e}{2\kappa}\int dxdz\epsilon(x-z)\rho(x)\rho(z) = 0. \quad (34)$$

Then, with the use of the generators Q_1 and Q_2, the correct algebra is obtained,

$$\{Q_1,Q_1^\dagger\} = -2im\int dx\rho = -2iM,$$
$$\{Q_2,Q_2^\dagger\} = -iH, \quad \{Q_1,Q_2^\dagger\} = -iP_1. \quad (35)$$

In deriving the algebra, we have used canonical variables after the implementation of Gauss law. In this way, the SUSY charge Q_2 is a nonlocal function of the fields. We could have arrived at the same algebra by keeping A_1 and B as conjugate variables,

$$\{A_1(x_1,t),B(x_2,t)\} = \frac{1}{k}\delta(x_1 - x_2). \quad (36)$$

IV. DISCUSSION

To summarize, the connection of the Yang-Lai model to the (2+1)-dimensional Chern-Simons theory has helped us in identifying the correct supersymmetry transformations and charges of the model. If we consider the Yang-Lai model as a gauge "BF" theory, then the SUSY transformations are local in space time. If we instead eliminate the gauge fields A_0 and A_1 together with the scalar field B, then the Hamiltonian of the theory becomes simpler, but the supersymmetry charges become more complicated nonlocal functions of the physical fields.

The introduction of chemical potentials can be achieved by considering an effective potential

$$\Omega = H - \mu_bN_b - \mu_fN_f. \quad (37)$$

Then, as far as $\mu_b=\mu_f$, the transformations generated by Q_1 and Q_2 are still formally a symmetry $\{Q_i,\Omega\}=0$

As discussed by [16] the B field plays also an important role in the derivation of the self-dual equations. Indeed, writing the action S after the implementation of the Gauss law,

$$S = \int d^2x\bigg\{i\phi^\dagger\partial_t\phi + i\psi^\dagger\partial_t\psi - \frac{1}{2m}|(\partial_1 + e\gamma_bB)\phi|^2 - \frac{1}{2m}|(\partial_1 + e\gamma_fB)\psi|^2 - \frac{e}{2m}\partial_1B\rho_f + \frac{e}{2m}(-\gamma_b\partial_1B\rho_b - \gamma_f\partial_1B\rho_f) + \lambda_1\rho_b^2 + \lambda_2\rho_b\rho_f\bigg\}, \quad (38)$$

leads to a Hamiltonian of the form

$$H = \int d^2x\bigg\{\frac{1}{2m}|(\partial_1 + e\gamma_bB)\phi|^2 + \frac{1}{2m}|(\partial_1 + e\gamma_fB)\psi|^2 + \frac{e^2}{2m\kappa}\rho_T\rho_f + \frac{e^2}{2m\kappa}(\gamma_b\rho_T\rho_b + \gamma_f\rho_T\rho_f) - \lambda_1\rho_b^2 - \lambda_2\rho_b\rho_f\bigg\}, \quad (39)$$

$$H = \int d^2x\bigg\{\frac{1}{2m}|(\partial_1 + e\gamma_bB)\phi|^2 + \frac{1}{2m}|(\partial_1 + e\gamma_fB)\psi|^2 + \left(\frac{e^2\gamma_b}{2m\kappa} - \lambda_1\right)\rho_b^2 + \left(\frac{e^2}{2m\kappa}(1 + \gamma_f + \gamma_b) - \lambda_2\right)\rho_b\rho_f\bigg\}. \quad (40)$$

Thus for

$$\lambda_1 = \gamma_b\frac{e^2}{2m\kappa}, \quad (41)$$

$$\lambda_2 = (1 + \gamma_f + \gamma_b)\frac{e^2}{2m\kappa}, \quad (42)$$

the Hamiltonian is written as the sum of squares and minimum energy configurations are such that they satisfy self-dual or BPS equations,

$$(\partial_1 + e\gamma_b B)\phi = 0, \tag{43}$$

$$(\partial_1 + e\gamma_f B)\psi = 0. \tag{44}$$

The particular case $\gamma_b = \gamma_f = +1$ leads to the supersymmetric case. Solitons solutions to this equations can be found when $\lambda_1 \geqslant 0$, i.e., attractive self-interactions, and they have been considered in detail in [13,16].

We have thus discussed how a simple nonrelativistic supersymmetric model in (1+1)-dimensions is related to a model of matter interacting with gauge fields whose dynamics is governed by a "BF" term. Although the gauge fields can be eliminated in terms of the matter fields, their presence helps in identifying the correct supersymmetry transformations which in this case became a *nonlocal* functional of the matter fields. Working without gauge fields would require one to guess nonlinear terms in the SUSY transformations. These terms were not considered in Ref. [12] and as a result the supercharge anticommutator equals just the free Hamiltonian.

Due to the low dimensionality, the model here discussed might provide the simplest theoretical realization of Galilean supersymmetry, and according to the authors of Refs. [11,12], the model could be accessible experimentally. From the theoretical point of view, it can also provide an interesting playground where to explore further the relation between BPS states, supersymmetry, and integrability.

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