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Non-Abelian fractional quantum Hall states and chiral coset conformal field theories

D.C. Cabra[a,b,1], **E. Fradkin**[c], **G.L. Rossini**[a,1],
and
F.A. Schaposnik[a,2]

[a]*Departamento de Física, Universidad Nacional de La Plata,*
C.C. 67 (1900) La Plata, Argentina
[b] *Facultad de Ingeniería, Universidad Nacional de Lomas de Zamora,*
Cno. de Cintura y Juan XXIII, (1832), Lomas de Zamora, Argentina,

[c] *Department of Physics, University of Illinois at Urbana-Champaign*
1110 W. Green St. , Urbana, IL 61801, USA

Abstract

We propose an effective Lagrangian for the low energy theory of the Pfaffian states of the fractional quantum Hall effect in the bulk in terms of non-Abelian Chern-Simons (CS) actions. Our approach exploits the connection between the topological Chern-Simons theory and chiral conformal field theories. This construction can be used to describe a large class of non-Abelian FQH states.

[1]CONICET, Argentina
[2]CICPBA, Argentina

1 Introduction

The Pffafian fractional quantum Hall (FQH) state has received much attention recently, having been proposed as the leading candidate to describe the ground state properties of the experimentally observed $\nu = 5/2$ FQH plateau of a single layer two-dimensional electron gas (2DEG). It also bears a close connection with the $\nu = 1/2$ state in bilayer systems. The Pfaffian states [1, 2] belong to a family of FQH states whose excitations exhibit non-Abelian statistics. All of these states involve some sort of pairing (or more generally multi-particle bound states) among the electrons. [3]

The generalized $q-$Pfaffian wave function has the form

$$\Psi_{\rm Pf} \; = \; {\rm Pf}\left(\frac{1}{z_i - z_j}\right) \; \prod_{i>j}(z_i - z_j)^q \;\; e^{-\frac{1}{4\ell_0^2}\sum |z_i|^2}, \tag{1}$$

where ℓ_0 is the cyclotron length, $\ell_0^2 = \hbar c/(eB)$. This wave function, regarded as the ground state wavefunction of a system of fully polarized fermions in a partially filled Landau level with a local Hamiltonian, describes an incompressible ground state of the system. Like in all quantum Hall states, all the excitations in the bulk of the system have a finite energy gap, and the only gapless excitations reside at the physical boundary. Since these boundary or edge excitations are gapless and, and as the external magnetic field breaks time reversal symmetry, the low energy Hilbert space of the system is described by a chiral CFT. A salient feature of non-Abelian quantum Hall states such as the Pfaffian state is that their spectrum contains quasiparticles which exhibit non-Abelian braiding statistics.

The incompressibility of the bulk state, the breaking of time reversal invariance, and the existence of locally conserved currents together imply that the low energy physics of the bulk states should be describable in terms of a Chern-Simons gauge theory with a suitably chosen gauge group [4]. In this way, the correspondence between the Hilbert space of the bulk FQH state and its edge excitations reduces to the correspodence between a Chern-Simons gauge theory and chiral CFT [5, 6]. These considerations led Wen to postulate that for any *Abelian* FQH state, the *edge chiral* CFT is determined by the conformal block structure exhibited by the holomorphic factors of the wave function (this being the origin of the concept of *bulk* CFT) [7, 8]. Thus, for all the Abelian FQH states there is a one-to-one correspondence between the operators that create the excitations in the bulk and the spectrum of primary fields of the associated chiral CFT of the edge states. Much of the current understanding of the physics of the properties of the edge states is based on this correspondence[4].

However, for the non-Abelian FQH states the picture is still incomplete. Wave functions for non-Abelian FQH states have been proposed[1, 3] and many of the properties of the low energy excitation spectra associated with these states have been studied [9, 10]. In a number of cases the edge states of these non-Abelian FQH states are fairly well understood[9, 11]. For a number of specific non-Abelian FQH states, notably the fermionic and the bosonic Pfaffian states and their generalizations, effective Landau-like theories for the bulk states have been proposed and their connections with the pairing correlation of the wave functions and with the physics of the CFT of their edge states have been worked out[12, 13]. Recently, Wen has also proposed a projective construction which is a promising attempt for understanding many of the properties of these states.[14] However, a full classification of the non-Abelian FQH states is lacking. In particular, the *natural* form of the effective theory in the bulk is not known, even though some of its main ingredients are known.

Drawing from the classification theory of the Abelian FQH states in terms of generalized multicomponent Abelian Chern-Simons theories [8], it is possible to infer what its natural generalization to the non-Abelian states should look like. In this paper we propose an effective low energy theory for the bulk Pfaffian FQH states. As expected this effective theory contains as one of its main components, non-Abelian Chern-Simons theories with a (quantized) coupling constant (known as the *level*) $k > 1$. We will see, however, that the correct theory has several sectors that need to be glued together carefully. Hence, the effective theory does *not* reduce simply to a suitably chosen non-Abelian Chern-Simons theory. By direct study of the Pfaffian wave function, a number of authors have proposed specific chiral edge theories for several non-Abelian states [9, 10, 11]. For the case at hand, the Pfaffian state, the holomorphic Pfaffian factor in (1) corresponds to the conformal blocks of an Ising CFT [1], while the Laughlin factor corresponds to a Rational Conformal Field Theory (RCFT) of a massless $U(1)$ chiral boson with compactification radius $R = 1/\sqrt{q}$. The non-Abelian structure of the Pfaffian state is encoded in the Ising sector[1, 11]. In particular, the corresponding edge chiral CFT is a direct product of an Ising sector with symmetry $\mathbf{Z}_2$, and a $U(1)$ RCFT with compactification radius $R = 1/\sqrt{q}$,

$$Z_2 \otimes U(1)_q \ , \tag{2}$$

with total central charge $c = 1/2 + 1$.

It has been known for quite some time that as a CFT, the Ising sector can also be described by an $SU(2)_2/U(1)$ coset CFT [15]. Thus, the CFT of the edge states can also be described by a CFT of a set of currents obeying the Kac-Moody symmetry (current algebra)

$$\frac{SU(2)_2}{U(1)} \otimes U(1)_q \tag{3}$$

The spectrum of a coset CFT of this type can be constructed in terms of a (level k) $SU(2)_k$ gauged Wess-Zumino-Witten theory or, equivalently, in terms of a chiral sector of a theory of constrained fermionic currents (see *e.g.* [16] and references therein). Coset constructions of the edge states of FQH states have also been proposed [12, 17]. These constructions are natural at the level of the edge theory, and the cosets appear either as a result of the need to project-out some sector of the theory, or by the need to change the quantum numbers of the edge states (*i. e.* the compactification radius) to get the right spectrum. In both instances the end result is that some subgroup of the symmetry group is gauged. However, in some cases gauging a subgroup at the edge leads to theories with broken symmetries in the bulk, possibly due to a Higgs-like mechanism [12, 13] related with the existence of pairing correlations in the bulk state.

In this paper we propose a simple effective theory for the bulk state which has a natural connection with the known physics of the edge and at the same time it represents a minimal departure from the conventional Chern-Simons construction in the bulk. Guided by the structure of the edge states we see that the problem of finding the effective theory of the bulk is closely connected with the problem of finding a Chern-Simons theory, at some level $k > 1$, whose boundary is a coset CFT. Moreover, as the CFT is a chiral CFT with constraints, the problem also involves finding the origin of the constraints in the bulk physics and the connection with the physics of pairing. Although recent work by several groups [12, 13, 14, 17] has shown that, at least for a few specific cases, it is possible to construct such theories, how general these constructions are is not known. In this work we will use the following alternative and simpler strategy. Given that it is possible to determine the structure of the CFT directly from the structure of the wave function, and this CFT is known to be the chiral coset CFT $(SU(2)_2/U(1)) \otimes U(1)_q$, we will look for a natural bulk Chern-Simons theory whose boundary CFT is this chiral coset. In addition, in order to construct the spectrum of bulk excitations, we have to supply a set of specific rules that will tell us how to put together the representations of the different sectors of the theory to make a physical excitation. In principle these rules should be a consequence of the underlying physics of the bulk that causes the system to have a Pfaffian state as its ground state.

The natural candidate for such an effective theory should be a topological quantum field theory on a manifold with boundary, supplied with a set of appropriate boundary conditions. The boundary conditions play two roles: (a) they *define* the actual *dynamics* at the edge (and hence its energetics), and (b) they specify how the excitations are actually built up. This approach is a natural generalization of the well known $U(1)$ Chern-Simons description of the Abelian FQH states (see *e. g.* [7]). In this case it is well established that the full low energy dynamics is described by a bulk (multicomponent) Abelian CS theory, with edge modes described by a (set of) chiral $U(1)$ RCFT of free bosons. Although this description works for generic Abelian FQH states, it is not sufficient to reproduce the non-Abelian structure of *e. g.* the Ising sector of the Pfaffian state. The main purpose of the present paper is to generalize those previous schemes to the Pfaffian state and other non-Abelian FQH systems.

Here we use the general approach to non-Abelian CS topological field theories on manifolds with boundaries, as developed by Moore and Seiberg [18], suitably extended to include the boundary contitions relevant to the physics of the Pfaffian state. Our approach should work for any non-Abelian FQH state with a boundary CFT determined by a chiral current algebra (as in the chiral coset CFT's). We note, however, that there exist non-Abelian FQH states which can be obtained by projecting a generic multicomponent Abelian theory [19, 20] and their CFT's are not generated by a current algebra. The relation between these projected Abelian theories and the Pfaffian (and its generalizations) is an interesting open question. This paper is organized as follows. In section 2 we review the Chern-Simons-CFT within the path-integral framework of Moore and Seiberg although with a new type of boundary conditions suitable to the physics of quantum Hall systems. In particular we derive in detail the form of the coset construction associated with these boundary conditions. In section 3 we use the results of section 2 to determine the form of the effective theory of the bulk FQH Pfaffian states. Here we show how the observables of the Pfaffian state are represented in this theory and show that this theory is consistent with the known properties of the Pfaffian states. In Section 4 we generalize of this construction to the newly propsed parafermionic states. Finally, in section 5 we summarize our results and give a discussion of its validity.

2 The Chern-Simons Conformal Field Theory connection in the path integral framework

We will summarize here the way to connect a CS theory on a 3-dimensional manifold Ω with boundary (say $D \times R$) with a CFT on the boundary $\partial\Omega$ ($S_1 \times R$). The bulk CS theory is a topological field theory (in the sense that the CS action is independent of the metric of the manifold Ω). Therefore, on a manifold with a boundary, the definition of the theory must include boundary conditions, and different conditions will in principle lead to different theories. We will adopt as a general criterion for boundary conditions that surface terms should not appear in the equations of motion [18]. However, unlike reference [18], we will manage to control the propagation velocity of the chiral modes at the boundary in the spirit of [7], so as to end up with non-trivial boundary dynamics.

The treatment of the bulk CS partition function that we present below has a first step in which a component of the gauge field is recasted as a Lagrange multiplier and integrated out. Then, the delta function condition that arises is solved explicitly via an adequate parameterization of the degrees of freedom, and all 3-dimensional integrals are shown to depend only on the boundary fields, *i. e.* they are total divergences or topological WZW terms. We modify the procedure presented in [18] for the case of chiral coset CFT's by choosing suitable boundary conditions in the sense of getting the right boundary dynamics, and establish the connection between the effective theory of the bulk and the corresponding chiral CFT at the boundary. As a byproduct, we will find a set of rules that characterizes the physical observables.

2.1 Pure CS theories

Consider the CS action on Ω

$$S_{CS} = \frac{1}{4\pi}\int_\Omega d^3x\epsilon^{\mu\nu\lambda}\mathrm{Tr}_{\hat{G}}(A_\mu\partial_\nu A_\lambda + \frac{2}{3}A_\mu A_\nu A_\lambda), \tag{4}$$

where the gauge field is taken in the Lie algebra of a group G. We use the conventions

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \tag{5}$$

$$A_\mu \;=\; A^a_\mu \tau_a\,,\; A^a_\mu \in \Re \tag{6}$$

with $\tau_a^\dagger = -\tau_a$, $[\tau_a, \tau_b] \;=\; f_{abc}\,\tau_c$ and $\mathrm{Tr}_{\hat{G}}(\tau_a\tau_b) = -\frac{1}{2}\delta_{ab}$ for the non-Abelian $SU(N)$ case. In components the CS action reads

$$S_{CS} = -\frac{1}{8\pi}\int_\Omega d^3x\epsilon^{\mu\nu\lambda}(A^a_\mu\partial_\nu A^a_\lambda + \frac{1}{3}f_{abc}A^a_\mu A^b_\nu A^c_\lambda), \tag{7}$$

The variation of the action is

$$\delta S_{CS} = -\frac{1}{8\pi}\int_\Omega d^3x\,\epsilon^{\mu\nu\lambda}\delta A^a_\mu F^a_{\nu\lambda} + \frac{1}{8\pi}\int_{\partial\Omega} dS_\mu\,\epsilon^{\mu\nu\lambda}A^a_\nu\delta A^a_\lambda, \tag{8}$$

where dS_μ is a surface element normal to $\partial\Omega$. The boundary conditions should be such that the surface term vanishes for any allowed variation of the gauge field. In our geometry $D \times R$ different possibilities are to set $A_t = 0$, or $A_x = 0$, or any linear combination such as $A_t + vA_x = 0$ (being $\partial\Omega = S_1 \times R$, we use x for the compact direction S_1 and t for the real line R).

Let us first consider $A_t = 0$ on $\partial\Omega$ [21]. In this case one can write

$$S_{CS} = -\frac{1}{4\pi}\int_\Omega d^3x\epsilon^{ij}A^a_i\partial_t A^a_j + \frac{1}{2\pi}\int_\Omega d^3x\epsilon^{ij}A^a_t(\partial_i A^a_j + f_{abc}A^b_i A^c_j), \tag{9}$$

where now i, j indicate coordinates x, y on D. Surface terms have been discarded *using the boundary condition.*

The partition function

$$\mathcal{Z} = \int \mathcal{D}A_\mu \exp(iS_{CS}) \tag{10}$$

can now be easily transformed in the following way: A_t is integrated out, imposing the condition $F_{ij} = 0$. Then the only contributing configurations are explicitely parameterized as a 2D pure gauge,

$$A_i = -\partial_i g g^{-1}, \qquad (i = x, y) \tag{11}$$

It is worth noting that the Jacobians arising from the change of variables $A_1 \to g$ cancel out exactly with the one coming from the delta functional written in terms of the new fields. The remaining effective action reads

$$S_{eff} = -\frac{1}{4\pi}\int_\Omega d^3x \epsilon^{ij} \mathrm{Tr}_{\hat{G}} \left(\partial_i g g^{-1} \partial_0 (\partial_j g g^{-1})\right). \tag{12}$$

After some algebra, the partition function reads

$$\mathcal{Z} = \int \mathcal{D}g \exp(i\tilde{S}^{Ch}_{WZW}[g]) \tag{13}$$

where $\tilde{S}^{Ch}_{WZW}$ is a kind of chiral Wess-Zumino-Witten (WZW) action,

$$\begin{aligned} \tilde{S}^{Ch}_{WZW}[g] &= \frac{1}{4\pi}\int_{\partial\Omega} d^2x \; \mathrm{Tr}_{\hat{G}} \left(g^{-1}\partial_x g g^{-1} \partial_t g\right) - \\ & \frac{1}{12\pi}\int_\Omega d^3x \; \epsilon^{\mu\nu\alpha} \mathrm{Tr}_{\hat{G}} \left(g^{-1}\partial_\mu g g^{-1}\partial_\nu g g^{-1}\partial_\alpha g\right). \end{aligned} \tag{14}$$

However, this result presented in [21] is not satisfactory for our present interest in the sense that the action (14) has no propagating degrees of freedom. In fact, the quadratic term gives just the simplectic structure of the 2D theory, no kinetic term is present in the action. This structure is a consequence of the fact the the Chern-Simons theory is topological (in the sense of being invariant under transformations of the metric). The gauge choice $A_t = 0$ is consistent with general covariance and thus the effective theory has no dynamics. However, in a general gauge and/or for a general choice of boundary conditions general covariance will be broken. Clearly, different boundary conditions lead to different dynamics of the effective theory. Which choice is physically correct cannot be determined from Chern-Simons theory alone.

2.1.1 Modification of the boundary conditions

Following the approach introduced by Wen in the abelian theories[7], we will now change the theory by modifying the boundary conditions to $A_t + vA_x = 0$.

The previous computation can be copy cut if we define new coordinates on Ω as

$$\begin{aligned}\tilde{x} &= x + vt \\ \tilde{y} &= y \\ \tilde{t} &= t\end{aligned} \tag{15}$$

so that the covariant vectors change as

$$\begin{aligned}\tilde{A}_x &= A_x \\ \tilde{A}_y &= A_y \\ \tilde{A}_t &= A_t - vA_x\end{aligned} \tag{16}$$

The action (4) is invariant under this change of coordinates, so that all the previous computations proceed in the same way, writing $\tilde{A}$ instead of A. At the end of the procedure we get as effective action the chiral WZW action as studied in [22]:

$$\begin{aligned}S^{Ch}_{WZW} &= \frac{1}{4\pi}\int_{\partial\Omega} d^2x \ \mathrm{Tr}_{\hat{G}}\left(g^{-1}\partial_x g g^{-1}(\partial_t - v\partial_x)g\right) - \\ &\quad \frac{1}{12\pi}\int_{\Omega} d^3x \ \epsilon^{\mu\nu\alpha}\mathrm{Tr}_{\hat{G}}\left(g^{-1}\partial_\mu g g^{-1}\partial_\nu g g^{-1}\partial_\alpha g\right),\end{aligned} \tag{17}$$

which describes propagating edge excitations as desired.

For later convenience, let us write out the particular Abelian ($\hat{G} = U(\hat{1})$) form of eq.(17),

$$S^{Ch}_{U(\hat{1})} = -\frac{1}{4\pi}\int_{\partial\Omega} d^2x \ \partial_x\phi(\partial_t - v\partial_x)\phi, \tag{18}$$

which is of course nothing but the usual chiral boson action [23, 24].

2.2 The CS-CFT connection for Coset theories

As we pointed out in the introduction, the Ising CFT of interest in this paper can be formulated as a $SU(2)_2/U(1)$ coset CFT. For convenience, we will extend the previous treatment to the case in which the boundary theory is a coset $\hat{G}_{k_G}/\hat{H}_{k_H}$, $k_G = lk_H$, with l being the index of the embedding of $\hat{H}$ in $\hat{G}$. We will follow the approach in [18] as well as a suitable modification in the boundary conditions so as to study the physical realization of the Pfaffian state [12].

We begin with the action of two gauge fields A and B in the bulk in $2+1$ dimenions, where A is in the Lie algebra of $\hat{G}_{k_G}$ and B is in the Lie algebra of $\hat{H}_{k_H}$. The action S is just the difference of the CS action of $\hat{G}_{k_G}$ and that of $\hat{H}_{k_H}$,

$$S = k_G S_{CS}[A, \hat{G}] - k_H S_{CS}[B, \hat{H}], \tag{19}$$

supplied with the boundary conditions

$$\mathcal{P}_{\hat{H}^\perp} A_t = \quad 0, \tag{20}$$

$$k_G \mathcal{P}_{\hat{H}} A_i - k_H B_i = \quad 0, \quad (i = x, t) \quad \text{on } \partial\Omega \tag{21}$$

These boundary conditions do not break general covariance and consequently do not give dynamis to the effective theory. We consider again $\Omega = D \times R$ and hence $\partial\Omega = S_1 \times R$. $\mathcal{P}_{\hat{H}}$ and $\mathcal{P}_{\hat{H}^\perp}$ denote the projectors on the Lie algebra $\hat{H}$ and its orthogonal complement respectively. These boundary conditions satisfy the general criterion of having no boundary terms in the equations of motion, and allow to write each CS term in the form of eq.(9) as boundary terms arising from the integration by parts vanish.

After integrating out A_t and B_t the partition function takes the form

$$\begin{aligned} \mathcal{Z} &= \int DA_i DB_i \delta\left[\epsilon^{ij} F_{ij}[A]\right] \delta\left[\epsilon^{ij} F_{ij}[B]\right] \\ & \exp -\frac{i}{4\pi} \int_\Omega d^3x \epsilon^{ij} \left(k_G \mathrm{Tr}_{\hat{G}}(A_i \partial_t A_j) - k_H \mathrm{Tr}_{\hat{H}}(B_i \partial_t B_j)\right), \end{aligned} \tag{22}$$

where i, j are the indices corresponding to the coordinates on D.

The zero curvature constraints are easily solved on the simply connected 2D manifold D in terms of group valued fields $g \in G$, $h \in H$

$$A_i = -\partial_i g g^{-1}, \qquad B_i = -\partial_i h h^{-1}. \qquad (i = x, y) \tag{23}$$

The boundary conditions on the x components have to be explicitly implemented. This is done by means of a Lagrange multiplier $\lambda \in \hat{H}$

$$\int D\lambda \exp\left(i \int d^2x \mathrm{Tr}_{\hat{H}}\left(\lambda(k_G \partial_x g g^{-1} - k_H \partial_x h h^{-1})\right)\right). \tag{24}$$

Putting all these things together we get

$$\begin{aligned} \mathcal{Z} = \int DgDhD\lambda \exp\Big(& i k_G S^{Ch}_{WZW}[g] - i k_H S^{Ch}_{WZW}[h] + \\ & \frac{ik}{2\pi} \int d^2x \mathrm{Tr}_{\hat{H}}\left(\lambda(k_G \partial_x g g^{-1} - k_H \partial_x h h^{-1})\right)\Big). \end{aligned} \tag{25}$$

Now, being $k_G = lk_H$, we change variables to

$$U = gh^{-1}, \quad \lambda = C_t, \quad -\partial_x h h^{-1} = \frac{1}{\sqrt{l}} C_x, \tag{26}$$

where C_x, C_t are gauge fields in $\hat{H}$. Using the Polyakov-Wiegmann identity we arrive to an effective action in the form of a Chiral gauged WZW action

$$\begin{aligned} k_G \tilde{S}^{Ch,gauged}_{WZW}[U, C_i] &= k_G \tilde{S}^{Ch}_{WZW}[U] \\ &+ \frac{k_G}{2\pi} \int_{\partial\Omega} dt\ dx\ Tr_{\hat{H}}(-C_x U^{-1}\partial_t U + C_t \partial_x U U^{-1} - C_t U C_x U^{-1} + C_t C_x) \end{aligned} \tag{27}$$

for the fields $U \in G$, $C_i \in \hat{H}$ (the functional integral over h is factored out as the volume of the group H). This effective action corresponds to the coset theory $\hat{G}_{k_G}/\hat{H}_{k_G}$.

Here again, we use the tilde notation $\tilde{S}^{Ch,gauged}_{WZW}$ in order to emphasize that this is a kind of chiral theory with no propagating degrees of freedom. Once again, we will modify minimally the boundary conditions in order to break general covariance, thus giving dynamics to the effective theory.

2.2.1 Modification of the boundary conditions for the coset theory

We can easily modify the spectrum of the theory and introduce chiral propagating excitations by a suitable modification of the boundary conditions for the CS fields. In fact, it is enough to use the coordinates in eqs.(15, 16) and impose the boundary conditions (21) for the tilde components of the gauge fields. The modified boundary conditions read

$$\mathcal{P}_{\hat{H}^\perp}(A_t + vA_x) = \quad 0, \tag{28}$$

$$k_G \mathcal{P}_{\hat{H}} A_i - k_H B_i = \quad 0, \quad (i = x, t) \text{ on } \partial\Omega \tag{29}$$

The resulting effective action reads

$$\begin{aligned} k_G S^{Ch,gauged}_{WZW}[U, C_i] &= k_G S^{Ch}_{WZW}[U] \\ &+ \frac{k_G}{2\pi} \int_{\partial\Omega} dt\ dx\ Tr_{\hat{H}}(-C_{\tilde{x}} U^{-1}\partial_{\tilde{t}} U + C_{\tilde{t}} \partial_{\tilde{x}} U U^{-1} - C_{\tilde{t}} U C_{\tilde{x}} U^{-1} + C_{\tilde{t}} C_{\tilde{x}}) \end{aligned} \tag{30}$$

The kinetic term here insures that the action in Eq. (30) describes a propagating chiral CFT for the $\hat{G}_{k_G}/\hat{H}_{k_G}$ coset theory.

We conclude this section by describing the content of physical observables in the coset CS theory: the gauge group of the coset CS theory is not simply $G \times H$ due to the imposed boundary conditions. In fact, the common center of G and H has to be mod out. Then, the observables in the coset CS theory are built up as products of Wilson loop operators in representations Λ in G and λ in H satisfying the following rules [18]:

1. both representations should transform in the same way under the common center so that the product of Wilson loops is invariant under its action.

2. representations related by the spectral flow associated to the center should be identified.

Hence, gluing the theories at the boundary, as is done through the boundary conditions (21), insures that only the physical representations are allowed to appear in the bulk, and that the allowed representations of the bulk states are in one-to-one correspondence with the allowed states at the edge. This property insures that the observables of the coset CS theory correspond to the correct integrable representations of the coset CFT [18].

3 Bulk theory for the $\nu = 1/q$ Pffafian FQHE state

The mathematical setting described above can be now readily applied to construct an effective action for the $\nu = 1/q$ Pffafian FQHE state (q even). The *action* of the effective theory is given by the sum of two terms,

$$S^{bulk}_{Pfaff} = S_{Z_2} + S_{U(1)_q}, \tag{31}$$

corresponding to the Ising and $U(1)$ sectors respectively. However, the Hilbert space of the theory is smaller since the only allowed states are those that at the boundary satisfy the gluing conditions. In particular only the representations that are invariant under the simultaneous actions of the center Z_2 of both groups survive (see below).

The action for the Ising sector $(SU(2)_2/U(1))$ is given by eq. (30), with $k_G = k_H = 2$, $G = SU(2)$ and $H = U(1)$

$$S_{Z_2} = 2S_{CS}[A, S\hat{U}(2)] - 2S_{CS}[B, U\hat{(1)}], \tag{32}$$

together with the boundary conditions (29).

The effective action for the $U(1)$ sector simply reads

$$S_{U(1)_q} = qS_{CS}[C, U\hat{(1)}], \tag{33}$$

with the boundary conditions $C_t + vC_x = 0$, which insure the chiral boson properties for the charge degree of freedom on the edge and do not couple the C-field to the Ising sector.

The main point in this paper is that the bulk partition function

$$\mathcal{Z}_{bulk} = \int \mathcal{D}A_\mu \mathcal{D}B_\mu \mathcal{D}C_\mu \exp(-S^{bulk}_{Pfaff}) \tag{34}$$

with the imposed boundary conditions can be identified, as shown throughout this work, with that of the edge theory

$$\mathcal{Z}_{edge} = \int \mathcal{D}U\mathcal{D}C_i\mathcal{D}\phi \exp(-2S^{Ch,gauged}_{WZW}[U, C_i] - qS^{Ch}_{\hat{U}(1)}[\phi]). \tag{35}$$

Moreover, we are in conditions to give a detailed description of the observables of the proposed theory:

1. For the case of the $SU(2)_2/U(1)$ coset chiral edge CFT, the gauge group of the coset CS theory is

   $$\frac{SU(2) \times U(1)}{Z_2}. \tag{36}$$

   The CS observables correspond to Wilson loop operators in the integrable representations of the loop group [5]. In the case of $SU(2)_2$ they are labeled by the spin j and restricted to $\Lambda_0, \Lambda_{1/2}, \Lambda_1$, [25]. For the $U(1)$ CS, the corresponding representations λ_r are labeled by the $U(1)$ charge which is restricted to the set $r = 0, 1, 2$ [18]. Being Z_2 the common center for these groups, transformations by the center just mean parity and the rule (1) matches together (Λ_0, λ_0), (Λ_1, λ_0) and

$(\Lambda_{1/2}, \lambda_1)$, the other pairs being redundant by rule (2) (*i.e.* the associated spectral flow relates $(\Lambda_j, \lambda_r) \to (\Lambda_{1-j}, \lambda_{r+2})$). This leads to the correct representations for the edge chiral CFT, which respectively correspond to the identity operator, the Majorana fermion (ψ_M) and the spin operator (σ).

2. In the case of a direct-product CS theory, (such as the present $Z_2 \times U(1)_q$ in the Pfaffian), the boundary conditions do not relate observables, and we still have to combine together the integrable representations of the Z_2 sector with those of the level $2q$ $U(1)$ sector to build up the integrable representations of the whole theory. All different possible products of the integrable representations of the sectors should correspond to the allowed ones in the direct-product CS theory. However, as discussed in [26], not all the operators in the direct product theory are physical. In particular, for $q = 2$, the electron operator is given by $\psi_M \times \exp(i\sqrt{q}\phi)$ and the quasiparticle operator by $\sigma \times \exp(i\phi/(2\sqrt{q}))$ [1, 26].

The topological properties of the proposed effective action characterize completely and faithfully the universality class corresponding to the Pfaffian state. In fact, the theory predicts the same topological properties as that proposed in [12]. In particular it gives the correct the ground state degeneracy on the torus, the degeneracy of the quasihole states and the correct non-Abelian braiding statistics. This last result is a direct consequence of the fact that the non-trivial non-Abelian structure is encoded in the Ising factor, which inherits its structure from the $SU(2)_2$ theory from which it is derived.

4 Parafermionic states

We can also apply our method to construct the effective action for the parafermionic states, recently proposed in [27], which are described by the wave function

$$\Psi^M_{Para}(z_1, z_2, .., z_N) = \langle \psi_1(z_1)\psi_1(z_2)...\psi_1(z_N)\rangle \prod_{i<j}(z_i - z_j)^{M+2/k}, \qquad (37)$$

where ψ_1 stands for the basic parafermion field in the Z_k CFT, M odd (even) corresponds to a fermionic (bosonic) state and the filling factor is given by $\nu = 1/(M + 2/k)$.

From this wave function, (assuming as usual that bulk and boundary CFT's coincide), one can read off the boundary chiral CFT, which is given by the direct product

$$Z_k \otimes U(1)_{N/2}, \tag{38}$$

where Z_k stands for the parafermionic CFT [28, 29] and the bosonic RCFT has level $N = k(kM+2)/2$.

As for the Ising boundary CFT, we use the coset construction $Z_k = SU(2)_k/U(1)$ for the parafermionic sector, and then the whole previous analysis applies, the only change being in the level k of the $SU(2)$ WZW sector. As discussed in [1], rules (1) and (2) described above reproduce the right operator content of the Z_k parafermionic CFT. In particular, the basic parafermion operator ψ_1, (which is the generalization of the Majorana fermion for $k > 2$), and the basic spin field σ_1 are built up as (Λ_1, λ_0) and $(\Lambda_{1/2}, \lambda_1)$ respectively.

Then, the electron and quasiparticle operators proposed in [27] are identified in our notation with $\psi_1 \times \exp(i\sqrt{M+2/k}\phi)$ and the quasiparticle operator by $\sigma_1 \times \exp(i\phi/(k\sqrt{M+2/k}))$.

5 Conclusions

In summary, in this paper we proposed a general method to construct effective actions for non-Abelian FQHE states, following mainly the techniques developed in [18] for CS topological field theories on manifold with boundaries. We showed that suitably modified boundary conditions lead to the correct edge description for the chiral (gapless) physical propagating modes. We also study the issue of the allowed representations in such theories which provide a one-to-one correspondence between bulk and edge observables. In particular, we have seen that for a given action describing the physics of the edge, *e. g.* in the case of the q-Pfaffian state which action is given by

$$S^{edge}_{Pfaffian} = S_{Z_2} + S_{U(1)_q}, \tag{39}$$

supplied with a suitable set of boundary conditions that insure that the gauge group of the coset theory is actually $(SU(2) \times U(1))/Z_2$, we can construct the corresponding bulk effective field theory action, which is given by

$$S^{bulk}_{Pfaffian} = 2S_{CS}[A, S\hat{U(2)}] - 2S_{CS}[B, U\hat{(1)}] + qS_{CS}[C, U\hat{(1)}]. \tag{40}$$

We have proven this connection through the equality of the corresponding partition functions

$$\mathcal{Z}_{bulk} \equiv \mathcal{Z}_{edge} \tag{41}$$

and we have furthermore shown a one-to-one correspondence between observables (physical operators) in the edge and in the bulk. The proposed bulk theory satisfies all the requirements needed to describe the physics of the q-Pfaffian state, in the following sense:

1. The boundary chiral CFT corresponds to the direct product $\frac{SU(2)_2}{U(1)} \otimes U(1)_q$
2. It has the right number of propagating degrees of freedom as indicated by the central charge $c = 3/2$.
3. It has the right content of physical observables.

One of the main advantages of the present approach is that it can be extended to other non-Abelian FQHE states, such as those recently proposed in references [14] and [30]. The connection between the proposed effective action with those given by previous authors [12], [14], and the microscopic origin of such an effective action are open points that remain to be studied.

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