

QUANTUM SUBGROUPS OF SIMPLE TWISTED QUANTUM GROUPS AT ROOTS OF ONE

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ABSTRACT. Let G be a connected, simply connected simple complex algebraic group and let ϵ be a primitive ℓ th root of unity with ℓ odd and coprime with 3 if G is of type G_2 . We determine all Hopf algebra quotients of the twisted multiparameter quantum function algebra $\mathcal{O}_\epsilon^\varphi(G)$ introduced by Costantini and Varagnolo. This extends the results of Andruskiewitsch and the first author, where the untwisted case is treated.

1. INTRODUCTION

Let G be a connected, simply connected complex algebraic group. In this paper we determine all Hopf algebra quotients of the twisted multiparameter quantum function algebra $\mathcal{O}_\epsilon^\varphi(G)$ introduced by Costantini and Varagnolo in [CV1], where ϵ is a primitive ℓ th root of unity with ℓ odd, and if G is of type G_2 , further ϵ is coprime with 3. The dual notion of this was introduced by Reshetikhin [R] to produce multiparameter quantum enveloping algebras of $\mathfrak{g} = \text{Lie}(G)$; see also [Su]. It is constructed as a twist deformation of the topological Hopf algebra $U_\hbar(\mathfrak{g})$ over $\mathbb{C}[[\hbar]]$, where the twist only involves elements of a fixed Cartan subalgebra \mathfrak{h} of \mathfrak{g} . In the dual function algebra, this deformation corresponds to a skew endomorphism φ on the weight lattice of \mathfrak{g} . When $\varphi = 0$, one recovers the standard quantum function algebra on G and the results on this paper reproduce the classification obtained in [AG].

It turns out that $\mathcal{O}_\epsilon^\varphi(G)$ is a 2-cocycle deformation of $\mathcal{O}_\epsilon(G)$; see Lemma 2.14. For this reason, we call $\mathcal{O}_\epsilon^\varphi(G)$ the *twisted quantum function algebra* over G ; heuristically it should correspond to the function algebra over the *twisted* quantum group G_ϵ^φ . This is not an isolated example. The relation between multiparameter quantum function algebras and 2-cocycle deformations has been explained for particular instances of quantum groups; see for example [Ma], [Tk2], [AST], [HLT]. In general, multiparameter quantum groups were intensively studied. They appeared first in the work of Manin [Ma] and were subsequently treated by different authors, among them [AE, BW, CM, DPW, H, HLT, HPR, LS, OY, R, Tk].

An important problem in the theory of quantum groups is the determination of the general properties that a quantum group should have, since up to date there is no axiomatic definition of an *algebraic* quantum group. In this sense, the description of all possible Hopf algebra quotients of the quantum function algebra, seen as algebras of functions over *quantum subgroups*, of the known examples would give some insight on the structure of the quantum group. This can be viewed as the

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quantum version of the classical problem of studying subgroups of a simple algebraic group. This is an actual area of research since the description by P. Podleś [P] of the compact quantum subgroups of Woronowicz's quantum groups $SU_q(2)$ and $SO_q(3)$ for $q \in [-1, 1] \setminus 0$. Besides the result of Podleś, the main contributions are

- ▷ the description of the finite quantum subgroups of $GL_q(n)$ and $SL_q(n)$ for q an odd root of unity by Müller [Mu];
- ▷ the classification in [AG] of the quantum subgroups of G_q , for G a connected, simply connected, complex simple algebraic group G , with q a primitive ℓ th root of unity with ℓ odd and coprime with 3 if G is of type G_2 .
- ▷ the description in [G] of the quantum subgroups of the two-parameter deformation $GL_{\alpha, \beta}(n)$ for $\alpha^{-1}\beta$ a primitive root of unity of odd order;
- ▷ the determination of the compact quantum subgroups of $SO_{-1}(3)$ by Banica and Bichon [BB];
- ▷ the study of the quantum subgroups of $SU_q(2)$ for $q = -1$ in [BN] and for $q \neq -1$ in [FST];
- ▷ the description by Bichon and Dubois-Violette [BD] of the compact quantum subgroups of the half-liberated orthogonal quantum groups O_n^* from [BS];
- ▷ the classification of the quantum subgroups of $SU_{-1}(3)$ by Bichon and Yuncken [BY].

As the reader might have noticed, the problem splits into the algebraic case and the compact case. The latter is reduced mainly to the case when $q = -1$. In this paper, we study the algebraic case, hence we will assume that q is a primitive root of unity of odd order. The main result reads as follows.

Theorem 1. *There is a bijection between*

- (a) *Hopf algebra quotients $q : \mathcal{O}_\epsilon^\varphi(G) \rightarrow A$,*
- (b) *Twisted subgroup data up to equivalence.*

For the definition of the twisted subgroup data see Definition 4.8. We prove Theorem 1 in Section 4 through Theorems 4.9, 3.4 and 4.15. We use the strategy developed in [AG] for the untwisted case, where the Hopf algebra quotients are constructed using commutative diagrams whose rows are central extensions of Hopf algebras. Since $\mathcal{O}_\epsilon^\varphi(G)$ is a 2-cocycle deformation of $\mathcal{O}_\epsilon(G)$, several steps of the construction can be carried out without much effort. On the other hand, special attention has to be paid for certain constructions. To describe them we use the study of $\mathcal{O}_\epsilon^\varphi(G)$ carried out by Costantini and Varagnolo in [CV1] and [CV2], which is in turn a generalization of [DL].

As a consequence of Theorem 1 one would expect the construction of new examples of finite-dimensional Hopf algebras with different properties which might not necessarily be 2-cocycle deformation of Hopf algebra quotients of $\mathcal{O}_\epsilon(G)$. These examples are given by central exact sequences of Hopf algebras. We hope that, using similar methods to those of Ştefan [Ş], who characterized Hopf algebras with certain properties as quotients of the quantum function algebra $\mathcal{O}_q(SL(2))$ and used this to understand the structure of low-dimensional Hopf algebras (see [N] as well), these examples might help to better understand the classification problem for finite-dimensional Hopf algebras over the complex numbers.

This paper is organized as follows. In Section 2 we recall the definition and general properties of the twisted quantum enveloping algebra $U_q^\varphi(\mathfrak{g})$, its divided power algebra, the twisted quantum function algebra $\mathcal{O}_q^\varphi(G)$ and their specializations at roots of unity, and we show that $\mathcal{O}_\epsilon^\varphi(G)$ is a 2-cocycle deformation of $\mathcal{O}_\epsilon(G)$. In Section 3 we describe the twisted Frobenius–Lusztig kernels $\mathbf{u}_\epsilon^\varphi(\mathfrak{g})$ and all the Hopf algebra quotients of $\mathbf{u}_\epsilon^\varphi(\mathfrak{g})^*$. We also prove that $\mathbf{u}_\epsilon^\varphi(\mathfrak{g})$ is a twist deformation of $\mathbf{u}_\epsilon(\mathfrak{g})$. Finally, in Section 4 we prove the main theorem.

Conventions and preliminaries. Our references for the theory of Hopf algebras are [Mo], [Ra]. We use standard notation for Hopf algebras; the comultiplication, counit and antipode are denoted by Δ , ε , and \mathcal{S} , respectively. Let \mathbb{k} be a field. The set of group-like elements of a coalgebra C is denoted by $G(C)$. We also denote by $C^+ = \text{Ker } \varepsilon$ the augmentation ideal of C . Let H be a Hopf algebra. H^{op} denotes the Hopf algebra with the same coalgebra structure but opposite multiplication and H^{cop} denotes the Hopf algebra with the same algebra structure but opposite comultiplication. Let $g, h \in G(H)$, the set of (g, h) -primitive elements is given by $P_{g,h}(H) = \{x \in H : \Delta(x) = x \otimes g + h \otimes x\}$. We call $P_{1,1}(H) = P(H)$ the set of primitive elements.

Recall that a convolution invertible linear map σ in $\text{Hom}_{\mathbb{k}}(H \otimes H, \mathbb{k})$ is a *normalized multiplicative 2-cocycle* if

$$(1) \quad \sigma(b_{(1)}, c_{(1)})\sigma(a, b_{(2)}c_{(2)}) = \sigma(a_{(1)}, b_{(1)})\sigma(a_{(2)}b_{(2)}, c)$$

and $\sigma(a, 1) = \varepsilon(a) = \sigma(1, a)$ for all $a, b, c \in H$; see [Mo, Sec. 7.1]. In particular, the inverse of σ is given by $\sigma^{-1}(a, b) = \sigma(\mathcal{S}(a), b)$ for all $a, b \in H$. Using a 2-cocycle σ it is possible to define a new algebra structure on H by deforming the multiplication, which we denote by H_σ . Moreover, H_σ is indeed a Hopf algebra with $H = H_\sigma$ as coalgebras, deformed multiplication $m_\sigma = \sigma * m * \sigma^{-1} : H \otimes H \rightarrow H$ given by

$$m_\sigma(a, b) = a \cdot_\sigma b = \sigma(a_{(1)}, b_{(1)})a_{(2)}b_{(2)}\sigma^{-1}(a_{(3)}, b_{(3)}) \quad \text{for all } a, b \in H,$$

and if we write $Q^\sigma = \sigma(\text{Id} \otimes \mathcal{S})$ and $Q^{\sigma^{-1}} = \sigma^{-1}(\mathcal{S} \otimes \text{Id})$, then the antipode $\mathcal{S}_\sigma = Q^\sigma * \mathcal{S} * Q^{\sigma^{-1}} : H \rightarrow H$ is given by (see [Do] for details)

$$\mathcal{S}_\sigma(a) = \sigma(a_{(1)}, \mathcal{S}(a_{(2)}))\mathcal{S}(a_{(3)})\sigma^{-1}(\mathcal{S}(a_{(4)}), a_{(5)}) \quad \text{for all } a \in H.$$

Remark 1.1. Let H be a Hopf algebra, let I be a Hopf ideal, let $A = H/I$ and let $\pi : H \rightarrow A$ be the canonical map. Clearly, any 2-cocycle on A can be lifted through π to a 2-cocycle on H . Let $\sigma : H \otimes H \rightarrow \mathbb{k}$ be a normalized multiplicative 2-cocycle on H such that $\sigma|_{I \otimes H + H \otimes I} = 0$. Then the map $\hat{\sigma} : A \otimes A \rightarrow \mathbb{k}$ given by $\hat{\sigma}(\pi(h), \pi(k)) = \sigma(h, k)$ for all $h, k \in H$ defines a normalized multiplicative 2-cocycle on A and the induced map $\pi_\sigma : H_\sigma \rightarrow A_{\hat{\sigma}}$ is a Hopf algebra map. In particular, if B is a central Hopf subalgebra of H such that $\sigma|_{I \otimes H + H \otimes I} = 0$ with $I = B^+H$, then the formula above defines a 2-cocycle on $A = H/B^+H$.

Let H be a Hopf algebra and let $J \in H \otimes H$ be an invertible element. We say that J is a *normalized twist* if

$$(\Delta \otimes \text{id})(J)(J \otimes 1) = (\text{id} \otimes \Delta)(J)(1 \otimes J) \quad \text{and} \quad (\varepsilon \otimes \text{id})(J) = 1 = (\text{id} \otimes \varepsilon)(J).$$

Given a twist J for H , one can define a new Hopf algebra H^J with the same algebra structure and counit as H , but different comultiplication and antipode

$$\Delta^J(h) = J^{-1}\Delta(h)J, \quad \mathcal{S}^J(h) = Q_J^{-1}\mathcal{S}(h)Q_J,$$

for all $h \in H$, where we denote $J = J^{(1)} \otimes J^{(2)}$ and $Q_J = \mathcal{S}(J^{(1)})J^{(2)}$. We say that H^J is a twist deformation of H .

The notion of a 2-cocycle and a twist are dual of each other. If H is finite-dimensional, then J is a twist for H if and only if J^* is a 2-cocycle on H^* .

Definition 1.2. A Hopf pairing between two Hopf algebras U and H over a ring R is a bilinear form $b : H \times U \rightarrow R$ such that, for all $u, v \in U$ and $f, h \in H$,

$$\begin{aligned} \text{(i)} \quad & b(h, uv) = b(h_{(1)}, u)b(h_{(2)}, v); & \text{(iii)} \quad & b(1, u) = \varepsilon(u); \\ \text{(ii)} \quad & b(fh, u) = b(f, u_{(1)})b(h, u_{(2)}); & \text{(iv)} \quad & b(h, 1) = \varepsilon(h). \end{aligned}$$

It follows that $b(h, \mathcal{S}(u)) = b(\mathcal{S}(h), u)$ for all $u \in U$, $h \in H$. Given a Hopf pairing, one has Hopf algebra maps $U \rightarrow H^\circ$ and $H \rightarrow U^\circ$, where H° and U° are the Sweedler duals. The pairing is called *perfect* if these maps are injections.

Let G be a connected, simply connected simple complex algebraic group and let $\mathfrak{g} = \text{Lie}(G)$ be the Lie algebra of G . We fix $\mathfrak{h} \subseteq \mathfrak{g}$ as a Cartan subalgebra and Φ as the root system associated to \mathfrak{h} with simple roots $\Pi = \{\alpha_1, \dots, \alpha_n\}$, where $n = \text{rk}(\mathfrak{g}) := \dim(\mathfrak{h})$. Let $(-, -)$ be the symmetric bilinear form over \mathfrak{h}^* induced by the Killing form. Then, the Cartan matrix $A = (a_{ij})_{1 \leq i, j \leq n} \in \mathbb{Z}^{n \times n}$ associated with \mathfrak{g} is given by $a_{ij} = \frac{2(\alpha_j, \alpha_i)}{(\alpha_i, \alpha_i)}$. If we write $d_i = \frac{(\alpha_i, \alpha_i)}{2}$ and $D = \text{diag}(d_1, \dots, d_n)$, then $(\alpha_i, \alpha_j) = d_i a_{ij}$ and DA is symmetric. The fundamental weights $\omega_1, \dots, \omega_n$ are given by the property $(\omega_i, \alpha_j) = d_i \delta_{ij}$ for all $1 \leq i \leq n$. Then, $\alpha_i = \sum_{j=1}^n a_{ji} \omega_j$ for all $1 \leq i \leq n$. We denote by $P = \sum_{i=1}^n \mathbb{Z} \omega_i$ the weight lattice, P_+ the positive weights, $Q = \sum_{i=1}^n \mathbb{Z} \alpha_i$ the root lattice, Q_+ the positive roots and \mathcal{W} the Weyl group associated to Φ . The bilinear form $(-, -)$ defines a \mathbb{Z} -pairing over $P \times Q$.

Let q be an indeterminate, let $R = \mathbb{Q}[q, q^{-1}]$ and let $\mathbb{Q}(q)$ be its field of fractions. Let ϵ be an ℓ th root of unity with $\text{ord } \epsilon = \ell$ odd and $3 \nmid \ell$ if G is of type G_2 . If $\chi_\ell(q)$ denotes the ℓ th cyclotomic polynomial, then $R/[\chi_\ell(q)R] = \mathbb{Q}(\epsilon)$. We denote $q_i = q^{d_i}$ for all $1 \leq i \leq n$.

For $n > 0$ define

$$\begin{aligned} (n)_q &= \frac{q^n - 1}{q - 1} = q^{n-1} + \dots + q + 1, & (n)_q! &= (n)_q(n-1)_q \cdots (2)_q(1)_q, & (0)_q &= 1, \\ [n]_q &= \frac{q^n - q^{-n}}{q - q^{-1}}, & [n]_q! &= [n]_q[n-1]_q \cdots [1]_q & \text{and} & [0]_q = 1, \\ \binom{n}{k}_q &= \frac{(n)_q}{(k)_q(n-k)_q}, & \left[\begin{matrix} n \\ k \end{matrix} \right]_q &= \frac{[n]_q!}{[k]_q![n-k]_q!}. \end{aligned}$$

2. TWISTED QUANTUM GROUPS

In this section we recall the definition of the twisted (multiparameter simply connected) quantum enveloping algebra $\check{U}_q^\varphi(\mathfrak{g})$, its divided power algebra, and the twisted quantum function algebra $\mathcal{O}_q^\varphi(G)$. The former is isomorphic to the multiparameter quantum enveloping algebra defined in [R] and [CKP] (see [CV1]), and the latter is introduced by Costantini and Varagnolo in [CV2]. We mainly follow [CV2] for the description.

These algebras depend on a \mathbb{Q} -linear map on the weight lattice that induces a deformation on the coproduct of $\check{U}_q(\mathfrak{g})$ and on the product of $\mathcal{O}_q(G)$. This deformation is given by a multiplicative 2-cocycle on $\mathcal{O}_q(G)$ and *resembles* a twist deformation on $\check{U}_q(\mathfrak{g})$. For this reason, we call them *twisted* quantum function

algebras and *twisted* quantum enveloping algebras, respectively. We also describe the corresponding objects at roots of unity and some basic properties such as PBW basis, a Hopf algebra pairing, and the quantum Frobenius map. In particular, twisted quantum function algebras at roots of unity fit into an exact sequence of Hopf algebras.

Throughout we omit the supraindex φ when $\varphi = 0$ on the quantum function algebras and on the corresponding maps if no possible confusion may arise.

2.1. The twisting map φ . Consider the \mathbb{Q} -linear space $\mathbb{Q}P = \sum_{i=1}^n \mathbb{Q}\omega_i$ spanned by the weights and define a \mathbb{Q} -linear map satisfying

$$(2) \quad \begin{cases} (\varphi x, y) = -(x, \varphi y) & \forall x, y \in \mathbb{Q}P, \\ \varphi \alpha_i = \delta_i = 2\tau_i, & \tau_i \in P, i = 1, \dots, n, \\ \frac{1}{2}(\varphi \lambda, \mu) \in \mathbb{Z} & \forall \lambda, \mu \in P, \end{cases}$$

where $(-, -)$ is considered as a linear extension of the \mathbb{Z} -pairing over $P \times Q$ to a symmetric bilinear form $\mathbb{Q}P \times \mathbb{Q}P \rightarrow \mathbb{Q}$. In particular, φ is antisymmetric with respect to this form.

According to the first two conditions we have that $(2\tau_i, \alpha_j) = -(2\tau_j, \alpha_i)$. Writing $\tau_i = \sum_{j=1}^n x_{ji}\omega_j = \sum_{j=1}^n y_{ji}\alpha_j$ with $x_{ji}, y_{ji} \in \mathbb{Z}$ for all $1 \leq i, j \leq n$, it follows that

$$d_j x_{ji} = \left(\sum_{k=1}^n x_{ki}\omega_k, \alpha_j \right) = - \left(\sum_{l=1}^n x_{lj}\omega_l, \alpha_i \right) = - \sum_{l=1}^n x_{lj}(\omega_l, \alpha_i) = -d_i x_{ij}.$$

If we denote $X = (x_{ij})_{1 \leq i, j \leq n}$, $Y = (y_{ij})_{1 \leq i, j \leq n} \in \mathbb{Z}^{n \times n}$, then $AY = X$ and $DX = (d_i x_{ij})_{1 \leq i, j \leq n}$ is antisymmetric. In particular, $x_{ii} = 0$ for all $1 \leq i \leq n$ and φ depends on at most $\frac{n(n-1)}{2}$ integer parameters. Moreover, by [CV1, Lemma 2.1] the matrix $A+2X$ is invertible and the maps $1 \pm \varphi : \mathbb{Q}P \rightarrow \mathbb{Q}P$ are \mathbb{Q} isomorphisms that satisfy that

$$((1 + \varphi)^{\pm 1} \lambda, \mu) = (\lambda, (1 - \varphi)^{\pm 1} \mu) \quad \text{for all } \lambda, \mu \in P.$$

Write $r = (1 + \varphi)^{-1}$, $\bar{r} = (1 - \varphi)^{-1}$. Note that if $\lambda \in r(P)$ and $\mu \in P$, then $(\lambda, \mu) \in \frac{1}{\det(A+2X)}\mathbb{Z}$. Let u be an element contained in the algebraic closure of $\mathbb{Q}(q)$ such that $q = u^{\det(A+2X)}$. If $z \in \frac{1}{\det(A+2X)}\mathbb{Z}$, then we write q^z for $u^{z \det(A+2X)}$.

2.2. Twisted quantum enveloping algebras. Let $Q \subseteq M \subseteq P$ be a lattice. For convenience, we recall the definition of the one-parameter quantum enveloping algebras $U_q(\mathfrak{g}, M)$; see [BG, I.6.5].

Definition 2.1. $U_q(\mathfrak{g}, M)$ is the $\mathbb{Q}(q)$ -algebra generated by the elements $\{E_i, F_i\}_{i=1}^n$ and $\{K_\lambda : \lambda \in M\}$ satisfying the relations

$$\begin{aligned} K_0 &= 1, & K_\lambda K_\mu &= K_{\lambda+\mu} = K_\mu K_\lambda & \text{for all } \lambda, \mu \in M, \\ K_\lambda E_j K_{-\lambda} &= q^{(\lambda, \alpha_j)} E_j, \\ K_\lambda F_j K_{-\lambda} &= q^{-(\lambda, \alpha_j)} F_j, \\ E_i F_j - F_j E_i &= \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \\ \sum_{l=0}^{1-a_{ij}} (-1)^l \begin{bmatrix} 1-a_{ij} \\ l \end{bmatrix}_{q_i} E_i^{1-a_{ij}-l} E_j E_i^l &= 0 \quad (i \neq j), \\ \sum_{l=0}^{1-a_{ij}} (-1)^l \begin{bmatrix} 1-a_{ij} \\ l \end{bmatrix}_{q_i} F_i^{1-a_{ij}-l} F_j F_i^l &= 0 \quad (i \neq j). \end{aligned}$$

It is well known that $U_q(\mathfrak{g}, M)$ is a Hopf algebra with its comultiplication defined by setting E_i to be $(1, K_{\alpha_i})$ -primitive and F_i to be $(K_{-\alpha_i}, 1)$ -primitive for all $1 \leq i \leq n$. Using the map φ , one may define a different coproduct, counit, and antipode on $U_q(\mathfrak{g}, M)$ as follows (see [CV2, §1.3]):

$$\begin{cases} \Delta_\varphi(E_i) = E_i \otimes K_{\tau_i} + K_{\alpha_i - \tau_i} \otimes E_i, \\ \Delta_\varphi(F_i) = F_i \otimes K_{-\alpha_i - \tau_i} + K_{\tau_i} \otimes F_i, \\ \Delta_\varphi(K_\lambda) = K_\lambda \otimes K_\lambda, \end{cases} \quad \begin{cases} \varepsilon_\varphi(E_i) = 0, \\ \varepsilon_\varphi(F_i) = 0, \\ \varepsilon_\varphi(K_\alpha) = 1, \end{cases} \quad \begin{cases} \mathcal{S}_\varphi(E_i) = -K_{-\alpha_i} E_i, \\ \mathcal{S}_\varphi(F_i) = -F_i K_{\alpha_i}, \\ \mathcal{S}_\varphi(K_\lambda) = K_{-\lambda}. \end{cases}$$

Note that the coproduct is well defined by (2). With this new structure, $U_q(\mathfrak{g}, M)$ is again a Hopf algebra which is denoted by $U_q^\varphi(\mathfrak{g}, M)$. Clearly, $U_q^\varphi(\mathfrak{g}, M) = U_q(\mathfrak{g}, M)$ if $\varphi = 0$. We write $U_q^\varphi(\mathfrak{g}, P) = \check{U}_q^\varphi(\mathfrak{g})$ and $U_q^\varphi(\mathfrak{g}, Q) = U_q^\varphi(\mathfrak{g})$.

Remark 2.2. From the defining relations we have that $K_{\tau_i} E_i = E_i K_{\tau_i}$ and $K_{\tau_i} F_i = F_i K_{\tau_i}$ for all $1 \leq i \leq n$. Indeed,

$$\begin{aligned} K_{\tau_i} E_i &= \prod_{j=1}^n K_{\omega_j^{x_{ji}}} E_i = \left(\prod_{j=1}^n q_j^{x_{ji}(\omega_j, \alpha_i)} \right) E_i \left(\prod_{j=1}^n K_{\omega_j^{x_{ji}}} \right) \\ &= \left(\prod_{j=1}^n q_j^{d_j x_{ji} \delta_{ij}} \right) E_i K_{\tau_i} = E_i K_{\tau_i}. \end{aligned}$$

The second assertion follows analogously.

Definition 2.3 ([CV2, §1.4]). Let $\check{U}_q^\varphi(\mathfrak{b}_+)$ and $U_q^\varphi(\mathfrak{b}_+)$ be the Hopf subalgebras of $\check{U}_q^\varphi(\mathfrak{g})$ generated by the elements K_λ, E_i with $\lambda \in P$ and $\lambda \in Q$, respectively. Similarly, let $\check{U}_q^\varphi(\mathfrak{b}_-)$ and $U_q^\varphi(\mathfrak{b}_-)$ be the Hopf subalgebras generated by the elements K_λ, F_i , with $\lambda \in P$, and $\lambda \in Q$, respectively. The algebra $U_q^\varphi(\mathfrak{g})$ is the Hopf subalgebra generated by K_λ, E_i, F_i with $\lambda \in Q$ and $1 \leq i \leq n$.

2.3. Pairings, Borel subalgebras and integer forms. By [CV1, § 2] (see also [CV2, §1.4], [DL, §3]) there exist perfect Hopf pairings $\pi_\varphi : \check{U}_q^\varphi(\mathfrak{b}_-)^{\text{cop}} \times \check{U}_q^\varphi(\mathfrak{b}_+) \rightarrow$

$\mathbb{Q}(u)$ and $\bar{\pi}_\varphi : \check{U}_q^\varphi(\mathfrak{b}_+)^{\text{cop}} \times \check{U}_q^\varphi(\mathfrak{b}_-) \rightarrow \mathbb{Q}(u)$. These are given by

$$\begin{cases} \pi_\varphi(K_\lambda, K_\mu) = q^{(r(\lambda), \mu)}, \\ \pi_\varphi(K_\lambda, E_i) = \pi_\varphi(F_i, K_\lambda) = 0, \\ \pi_\varphi(F_i, E_j) = \frac{\delta_{ij}}{q_i - q_i^{-1}} q^{(r(\tau_i), \tau_i)}, \end{cases} \quad \begin{cases} \bar{\pi}_\varphi(K_\lambda, K_\mu) = q^{-(\bar{r}(\lambda), \mu)}, \\ \bar{\pi}_\varphi(E_i, K_\lambda) = \bar{\pi}_\varphi(K_\lambda, F_i) = 0, \\ \bar{\pi}_\varphi(E_i, F_j) = \frac{\delta_{ij}}{q_i^{-1} - q_i} q^{-(\bar{r}(\tau_i), \tau_i)}, \end{cases}$$

for $\lambda, \mu \in P$ and $1 \leq i, j \leq n$. The pairing $\bar{\pi}_\varphi$ can be obtained from $\pi_{-\varphi}$ by the conjugation of the Hopf algebra \mathbb{Q} -anti-isomorphism $\zeta_\varphi : \check{U}_q^\varphi(\mathfrak{g}) \rightarrow \check{U}_q^{-\varphi}(\mathfrak{g})$ given by $E_i \mapsto F_i$, $F_i \mapsto E_i$, $K_\lambda \mapsto K_{-\lambda}$, and $q \mapsto q^{-1}$. Clearly, ζ_φ maps $\check{U}_q^\varphi(\mathfrak{b}_+)$ into $\check{U}_q^{-\varphi}(\mathfrak{b}_-)$ and $\bar{\pi}_\varphi = \zeta_\varphi \circ \pi_{-\varphi} \circ (\zeta_\varphi \otimes \zeta_\varphi)$. If $\varphi = 0$, denote by π_0 the corresponding bilinear form. Using these pairings we will define four R -Hopf algebras that will be needed later.

Fix a reduced expression of the longest element $w_0 = s_{i_1} \cdots s_{i_N}$ in the Weyl group \mathcal{W} and consider the total ordering on Φ_+ given by

$$\beta_1 = \alpha_{i_1}, \quad \beta_2 = \alpha_{i_1} \alpha_{i_2}, \quad \dots \quad \beta_N = \alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_N}.$$

The braid group $B_{\mathcal{W}}$ associated to \mathcal{W} acts on $\check{U}_q^\varphi(\mathfrak{g})$ via the Lusztig automorphisms T_{i_j} for $1 \leq j \leq N$, and one may define the root vectors

$$E_{\beta_k} = T_{i_1} T_{i_2} \cdots T_{i_{k-1}}(E_k) \quad \text{and} \quad F_{\beta_k} = T_{i_1} T_{i_2} \cdots T_{i_{k-1}}(F_k).$$

For $s \in \mathbb{N}$, $1 \leq i \leq n$, $1 \leq k \leq N$, and $G_i = E_i$ or F_i define

$$G_i^{(s)} = \frac{G_i^s}{[s]_{q_i}!} \quad \text{and} \quad G_{\beta_k}^{(s)} = T_{i_1} T_{i_2} \cdots T_{i_{k-1}}(G_k^{(s)}).$$

For $\alpha \in \Phi_+$, let

$$\begin{aligned} q_\alpha &= q^{\frac{(\alpha, \alpha)}{2}}, & \tau_\alpha &= \frac{1}{2} \varphi(\alpha), & e_\alpha^\varphi &= (q_\alpha^{-1} - q_\alpha) E_\alpha K_{-\tau_\alpha}, \\ e_i^\varphi &= e_{\alpha_i}^\varphi, & f_i^\varphi &= f_{\alpha_i}^\varphi, & f_\alpha^\varphi &= (q_\alpha - q_\alpha^{-1}) F_\alpha K_{-\tau_\alpha}. \end{aligned}$$

Definition 2.4. Denote by $R_q^\varphi[B_-]'$ and $R_q^\varphi[B_-]''$ the R -subalgebras of $\check{U}_q^\varphi(\mathfrak{b}_+)^{\text{op}}$ and $\check{U}_q^\varphi(\mathfrak{b}_+)^{\text{cop}}$, respectively, generated by the elements e_α^φ and $K_{(1-\varphi)\omega_i}$ for $1 \leq i \leq n$ and $\alpha \in \Phi_+$. Similarly, let $R_q^\varphi[B_+]'$ and $R_q^\varphi[B_+]''$ be the R -subalgebras of $\check{U}_q^\varphi(\mathfrak{b}_-)^{\text{op}}$ and $\check{U}_q^\varphi(\mathfrak{b}_-)^{\text{cop}}$, generated by the elements f_α^φ and $K_{(1+\varphi)\omega_i}$ for $1 \leq i \leq n$ and $\alpha \in \Phi_+$.

By restriction, we get the following pairings:

$$\begin{aligned} \pi'_\varphi : \check{U}_q^\varphi(\mathfrak{b}_-) \otimes_R R_q^\varphi[B_-]' &\rightarrow \mathbb{Q}(q), & \pi''_\varphi : R_q^\varphi[B_+]'' \otimes_R \check{U}_q^\varphi(\mathfrak{b}_+) &\rightarrow \mathbb{Q}(q), \\ \bar{\pi}'_\varphi : \check{U}_q^\varphi(\mathfrak{b}_+) \otimes_R R_q^\varphi[B_+] &\rightarrow \mathbb{Q}(q), & \bar{\pi}''_\varphi : R_q^\varphi[B_-]'' \otimes_R \check{U}_q^\varphi(\mathfrak{b}_-) &\rightarrow \mathbb{Q}(q). \end{aligned}$$

They are given by

$$\begin{aligned} (3) \quad \pi'_\varphi(K_\lambda, K_{(1-\varphi)\mu}) &= q^{(\lambda, \mu)}, & \pi'_\varphi(F_j, e_i^\varphi) &= -\delta_{ij}, \\ \pi''_\varphi(K_{(1+\varphi)\mu}, K_\lambda) &= q^{(\mu, \lambda)}, & \pi''_\varphi(f_i^\varphi, E_j) &= \delta_{ij}, \\ \bar{\pi}'_\varphi(K_\lambda, K_{(1+\varphi)\mu}) &= q^{-(\lambda, \mu)}, & \bar{\pi}'_\varphi(E_j, f_i^\varphi) &= -\delta_{ij}, \\ \bar{\pi}''_\varphi(K_{(1-\varphi)\mu}, K_\lambda) &= q^{-(\mu, \lambda)}, & \bar{\pi}''_\varphi(e_i^\varphi, F_j) &= \delta_{ij}. \end{aligned}$$

By [L], one may take as bases of $U_q^\varphi(\mathfrak{b}_+)$ and $U_q^\varphi(\mathfrak{b}_-)$, respectively, the elements

$$\xi_{m,t} = \prod_{j=N}^1 E_{\beta_j}^{(m_j)} \prod_{i=1}^n \binom{K_{\alpha_i}; 0}{t_i} K_{\alpha_i}^{-\left[\frac{t_i}{2}\right]}, \quad \eta_{m,t} = \prod_{j=N}^1 F_{\beta_j}^{(m_j)} \prod_{i=1}^n \binom{K_{\alpha_i}; 0}{t_i} K_{\alpha_i}^{-\left[\frac{t_i}{2}\right]},$$

where $[\cdot]$ represents the integer part function and $\binom{K_{\alpha_i}; 0}{t_i} = \prod_{s=1}^{t_i} \frac{K_{\alpha_i}^{-s+1}-1}{q^s-1}$.

Divided power algebras. We now describe an integer form of $U_q^\varphi(\mathfrak{g})$, which is used to define the algebra $U_\epsilon^\varphi(\mathfrak{g})$ at the root of unity ϵ .

Definition 2.5. Let $\Gamma^\varphi(\mathfrak{b}_+)$ and $\Gamma^\varphi(\mathfrak{b}_-)$ be the R -submodules of $U_q^\varphi(\mathfrak{g})$ given by

$$\begin{aligned} \Gamma^\varphi(\mathfrak{b}_+) &= \{u \in U_q^\varphi(\mathfrak{b}_+) \mid \pi''_\varphi(R_q^\varphi[B_+]'^{\text{cop}} \otimes u) \subset R\}, \\ \Gamma^\varphi(\mathfrak{b}_-) &= \{u \in U_q^\varphi(\mathfrak{b}_-) \mid \pi'_\varphi(u \otimes R_q^\varphi[B_-]'^{\text{op}}) \subset R\}. \end{aligned}$$

It is known that the sets $\{\xi_{m,t}\}$ and $\{\eta_{m,t}\}$ are R -bases of $\Gamma^\varphi(\mathfrak{b}_+)$ and $\Gamma^\varphi(\mathfrak{b}_-)$, respectively. This implies that both are algebras isomorphic to $\Gamma(\mathfrak{b}_+)$ and $\Gamma(\mathfrak{b}_-)$. Moreover, they are also subcoalgebras with the coproduct given by

$$(4) \quad \begin{cases} \Delta_\varphi E_i^{(p)} = \sum_{r+s=p} q_i^{-rs} E_i^{(r)} K_{s(\alpha_i - \tau_i)} \otimes E_i^{(s)} K_{r\tau_i}, \\ \Delta_\varphi F_i^{(p)} = \sum_{r+s=p} q_i^{-rs} F_i^{(r)} K_{s\tau_i} \otimes F_i^{(s)} K_{-r(\alpha_i + \tau_i)}, \\ \Delta_\varphi \binom{K_i; 0}{t} = \sum_{r+s=t} K_i^s q_i^{-rs} \binom{K_i; 0}{r} \otimes \binom{K_i; 0}{s}. \end{cases}$$

Again by restriction, we get the Hopf pairings

$$\begin{aligned} \pi'_\varphi : \Gamma^\varphi(\mathfrak{b}_-) \otimes_R R_q^\varphi[B_-]' &\rightarrow R, & \pi''_\varphi : R_q^\varphi[B_+]'' \otimes_R \Gamma^\varphi(\mathfrak{b}_+) &\rightarrow R, \\ \bar{\pi}'_\varphi : \Gamma^\varphi(\mathfrak{b}_+) \otimes_R R_q^\varphi[B_+] &\rightarrow R, & \bar{\pi}''_\varphi : R_q^\varphi[B_-]'' \otimes_R \Gamma^\varphi(\mathfrak{b}_-) &\rightarrow R. \end{aligned}$$

By [CV2, Lemma 1.12], the algebras $R_q^\varphi[B_\pm]'$, $R_q^\varphi[B_\pm]''$ admit a Hopf algebra structure such that the pairings above become perfect Hopf algebra pairings. Moreover, we have that $R_q^\varphi[B_\pm]' \simeq R_q^\varphi[B_\pm]''$ as Hopf algebras.

Definition 2.6 ([DL, §3.4]). The algebra $\Gamma^\varphi(\mathfrak{g})$ is the R -subalgebra of $U_q^\varphi(\mathfrak{g})$ generated by $\Gamma^\varphi(\mathfrak{b}_+)$ and $\Gamma^\varphi(\mathfrak{b}_-)$. In particular, it is generated by the elements

$$\begin{aligned} K_{\alpha_i}^{-1} & \quad (1 \leq i \leq n), & E_i^{(t)} & \quad (t \geq 1, 1 \leq i \leq n), \\ \binom{K_{\alpha_i}; 0}{t} & \quad (t \geq 1, 1 \leq i \leq n), & F_i^{(t)} & \quad (t \geq 1, 1 \leq i \leq n). \end{aligned}$$

2.4. Twisted quantum function algebras. In this subsection, we introduce the dual algebras $\mathcal{O}_q^\varphi(G)$ of $U_q^\varphi(\mathfrak{g})$ and $R_q^\varphi[G]$ of $\Gamma^\varphi(\mathfrak{g})$. They are obtained as the submodules generated by the matrix coefficients of representations of type one.

Let \mathcal{C}_φ be the full faithful subcategory in $U_q^\varphi(\mathfrak{g})$ -mod consisting of finite-dimensional modules on which the elements K_{α_i} act diagonally by powers of q . Then \mathcal{C}_φ is a tensor category which is strict. Denote by $\mathcal{O}_q^\varphi(G)$ the $\mathbb{Q}(q)$ -submodule of $\text{Hom}_{\mathbb{Q}(q)}(U_q^\varphi(\mathfrak{g}), \mathbb{Q}(q))$ spanned by all the matrix coefficients of objects in \mathcal{C}_φ . Then $\mathcal{O}_q^\varphi(G)$ is a $\mathbb{Q}(q)$ -Hopf algebra with the usual structure. Given $V \in \mathcal{C}_\varphi$,

$v \in V$, and $f \in V^*$, then the matrix coefficient $c_{f,v} : U_q^\varphi(\mathfrak{g}) \rightarrow \mathbb{Q}(q)$ is defined by $c_{f,v}(x) = f(x \cdot v)$ for all $x \in U_q^\varphi(\mathfrak{g})$. Then we have

$$\Delta(c_{f,v})(x \otimes y) = c_{f,v}(xy) \quad \text{and} \quad m_\varphi(c_{f,v} \otimes c_{g,w}) = c_{f \otimes g, v \otimes w},$$

for $V, W \in \mathcal{C}_\varphi$, $v \in V$, $f \in V^*$, $w \in W$, $g \in W^*$, and $x, y \in U_q^\varphi(\mathfrak{g})$.

For $\Lambda \in P_+$, let $L(\Lambda)$ be a simple highest weight module of $U_q^\varphi(\mathfrak{g})$. Then, $L(\Lambda) = \bigoplus L(\Lambda)_\lambda$ is a graded module and by the Peter–Weyl Theorem we have that $\mathcal{O}_q(G) = \bigoplus_{\Lambda \in P_+} L(\Lambda) \otimes L(\Lambda)^*$, where $L(\Lambda)^* \simeq L(-\omega_0 \Lambda)$. If $v \in L(\Lambda)_\mu$, $f \in L(\Lambda)_{-\lambda}$, then write $\Delta(c_{f,v}) = \sum_i c_{f,v}^{-\lambda, \nu} \otimes c_{f,v}^{-\nu, \mu} \in \bigoplus_\nu \mathcal{O}_q(G)_{-\lambda, \nu} \otimes \mathcal{O}_q(G)_{-\nu, \mu}$. Since $\mathcal{O}_q^\varphi(G)$ equals $\mathcal{O}_q(G)$ as a coalgebra, we keep this notation for the coproduct on $\mathcal{O}_q^\varphi(G)$.

Lemma 2.7 ([LS]). *For $i = 1, 2$ and $\Lambda_i \in P_+$, $v_i \in L(\Lambda_i)_{\mu_i}$, $f_i \in L(\Lambda_i)_{-\lambda_i}$ we have*

$$m_\varphi(c_{f_1, v_1} \otimes c_{f_2, v_2}) = q^{\frac{1}{2}((\varphi(\mu_1), \mu_2) - (\varphi(\lambda_1), \lambda_2))} m(c_{f_1, v_1} \otimes c_{f_2, v_2}).$$

□

Remark 2.8. Following [HLT, §2], the quantum function algebra $\mathcal{O}_q(G)$ is a P -bigraded Hopf algebra. In particular, if $f \in L(\Lambda)_\lambda^*$, $v \in L(\Lambda)_\mu$, and $f(v) \neq 0$ we have that $f(v) = f(1 \cdot v) = f(K_{\alpha_i}^{-1} K_{\alpha_i} \cdot v) = f(\mathcal{S}(K_{\alpha_i}) K_{\alpha_i} \cdot v) = (K_{\alpha_i} \cdot f)(K_{\alpha_i} \cdot v) = q_i^{(\lambda, \alpha_i) + (\mu, \alpha_i)} f(v)$ for all $1 \leq i \leq n$, which implies that $\lambda = -\mu$ if $f(v) \neq 0$.

If we define the antisymmetric bicharacter $p : P \times P \rightarrow \mathbb{Q}(q)$ by $p(\lambda_1, \lambda_2) = q^{-\frac{1}{2}(\varphi(\lambda_1), \lambda_2)}$, then it induces a group 2-cocycle \tilde{p} on $P \times P$ given by

$$\tilde{p}((\lambda_1, \mu_1), (\lambda_2, \mu_2)) = p(\lambda_1, \lambda_2) p(\mu_1, \mu_2)^{-1} = q^{\frac{1}{2}((\varphi(\mu_1), \mu_2) - (\varphi(\lambda_1), \lambda_2))},$$

and by [HLT, Theorem 2.1], $\mathcal{O}_q^\varphi(G)$ is isomorphic to the deformed P -bigraded Hopf algebra $\mathcal{O}_q(G)_p$, where the product is given

$$m_\varphi(c_{f_1, v_1} \otimes c_{f_2, v_2}) = p(\lambda_1, \lambda_2) p(\mu_1, \mu_2)^{-1} m(c_{f_1, v_1} \otimes c_{f_2, v_2}),$$

for $\Lambda_i \in P_+$, $v_i \in L(\Lambda_i)_{\mu_i}$, $f_i \in L(\Lambda_i)_{-\lambda_i}$.

Corollary 2.9. $\mathcal{O}_q^\varphi(G)$ is a 2-cocycle deformation of $\mathcal{O}_q(G)$. The 2-cocycle $\sigma : \mathcal{O}_q(G) \otimes \mathcal{O}_q(G) \rightarrow \mathbb{Q}(q)$ is given by the formula

$$\sigma(c_{f_1, v_1}, c_{f_2, v_2}) = \varepsilon(c_{f_1, v_1}) \varepsilon(c_{f_2, v_2}) q^{-\frac{1}{2}(\varphi(\lambda_1), \lambda_2)}$$

for $\Lambda_i \in P_+$, $v_i \in L(\Lambda_i)_{\mu_i}$, $f_i \in L(\Lambda_i)_{-\lambda_i}$, and $i = 1, 2$.

Proof. Denote $\chi(\lambda_1, \lambda_2) = q^{-\frac{1}{2}(\varphi(\lambda_1), \lambda_2)}$. Clearly, $\sigma(x, 1) = \sigma(1, x) = \varepsilon(x)$ for all $x \in \mathcal{O}_q(G)$. We first prove the condition of (1). For $1 \leq i \leq 3$, let $c_{f_i, v_i} \in \mathcal{O}_q(G)$ with $f_i \in L(\Lambda_i)_{\lambda_i}$ and $v_i \in L(\Lambda_i)_{\mu_i}$. On one hand, we have

$$\begin{aligned} & \sigma((c_{f_2, v_2})_{(1)}, (c_{f_3, v_3})_{(1)}) \sigma((c_{f_1, v_1})_{(2)}, (c_{f_3, v_3})_{(2)}) \\ &= \sum_{\nu_1, \nu_2} \sigma(c_{f_2, v_2}^{-\lambda_2, \nu_1}, c_{f_3, v_3}^{-\lambda_3, \nu_2}) \sigma(c_{f_1, v_1}^{-\nu_1, \mu_2}, c_{f_3, v_3}^{-\nu_2, \mu_3}) \\ &= \sum_{\nu_1, \nu_2} \varepsilon(c_{f_2, v_2}^{-\lambda_2, \nu_1}) \varepsilon(c_{f_3, v_3}^{-\lambda_3, \nu_2}) \chi(\lambda_2, \lambda_3) \varepsilon(c_{f_1, v_1}) \varepsilon(c_{f_2, v_2}^{-\nu_1, \mu_2}) \varepsilon(c_{f_3, v_3}^{-\nu_2, \mu_3}) \chi(\lambda_1, \nu_2 + \nu_3) \\ &= \varepsilon(c_{f_1, v_1}) \varepsilon(c_{f_2, v_2}) \varepsilon(c_{f_3, v_3}) \chi(\lambda_2, \lambda_3) \chi(\lambda_1, \lambda_2 + \lambda_3). \end{aligned}$$

On the other hand,

$$\begin{aligned}
 & \sigma((c_{f_1, v_1})_{(1)}, (c_{f_2, v_2})_{(1)}) \sigma((c_{f_1, v_1})_{(2)}, (c_{f_2, v_2})_{(2)}, c_{f_3, v_3}) \\
 &= \sum_{\nu_1, \nu_2} \sigma(c_{f_1, v_1}^{-\lambda_1, \nu_1}, c_{f_2, v_2}^{-\lambda_2, \nu_2}) \sigma(c_{f_1, v_1}^{-\nu_1, \mu_1} c_{f_2, v_2}^{-\nu_2, \mu_2}, c_{f_3, v_3}) \\
 &= \sum_{\nu_1, \nu_2} \varepsilon(c_{f_1, v_1}^{-\lambda_1, \nu_1}) \varepsilon(c_{f_2, v_2}^{-\lambda_2, \nu_2}) \chi(\lambda_1, \lambda_2) \varepsilon(c_{f_1, v_1}^{-\nu_1, \mu_1}) \varepsilon(c_{f_2, v_2}^{-\nu_2, \mu_2}) \varepsilon(c_{f_3, v_3}) \chi(\nu_1 + \nu_2, \lambda_3) \\
 &= \varepsilon(c_{f_1, v_1}) \varepsilon(c_{f_2, v_2}) \varepsilon(c_{f_3, v_3}) \chi(\lambda_1, \lambda_2) \chi(\lambda_1 + \lambda_2, \lambda_3).
 \end{aligned}$$

Thus, σ is a 2-cocycle on $\mathcal{O}_q(G)$. We now prove that it satisfies the equation given in Lemma 2.7. It actually follows by a direct computation using that $\chi(0, 0) = 1$, $\chi(\lambda, 0) = \chi(0, \lambda) = 1$, and

$$\sigma^{-1}(c_{f_1, v_1}, c_{f_2, v_2}) = \sigma(\mathcal{S}(c_{f_1, v_1}), c_{f_2, v_2}) = \varepsilon(c_{f_1, v_1}) \varepsilon(c_{f_2, v_2}) \chi(\mu_1, \lambda_2)$$

for $\Lambda_i \in P_+$, $v_i \in L(\Lambda_i)_{\mu_i}$, $f_i \in L(\Lambda_i)_{-\lambda_i}$, and $i = 1, 2$, and $\varepsilon(\mathcal{O}_q(G)_{\lambda, \mu}) = 0$ if $-\lambda \neq \mu$:

$$\begin{aligned}
 & m_\sigma(c_{f_1, v_1}, c_{f_2, v_2}) \\
 &= \sum_{\nu_1, \nu_2, \eta_1, \eta_2} \sigma(c_{f_1, v_1}^{-\lambda_1, \nu_1}, c_{f_2, v_2}^{-\lambda_2, \nu_2}) m(c_{f_1, v_1}^{-\nu_1, \eta_1} \otimes c_{f_2, v_2}^{\nu_2, \eta_2}) \sigma^{-1}(c_{f_1, v_1}^{-\eta_1, \mu_1} c_{f_2, v_2}^{-\eta_2, \mu_2}) \\
 &= \sum_{\nu_1, \nu_2, \eta_1, \eta_2} \varepsilon(c_{f_1, v_1}^{-\lambda_1, \nu_1}) \varepsilon(c_{f_2, v_2}^{-\lambda_2, \nu_2}) \chi(\lambda_1, \lambda_2) m(c_{f_1, v_1}^{-\nu_1, \eta_1} \otimes c_{f_2, v_2}^{\nu_2, \eta_2}) \\
 &\quad \times \varepsilon(c_{f_1, v_1}^{-\eta_1, \mu_1}) \varepsilon(c_{f_2, v_2}^{-\eta_2, \mu_2}) \chi(\mu_1, \eta_2) \\
 &= \chi(\lambda_1, \lambda_2) m(c_{f_1, v_1} \otimes c_{f_2, v_2}) \chi(\mu_1, -\mu_2) = \chi(\lambda_1, \lambda_2) \chi(\mu_1, \mu_2)^{-1} m(c_{f_1, v_1} \otimes c_{f_2, v_2}) \\
 &= q^{\frac{1}{2}((\varphi(\mu_1), \mu_2) - (\varphi(\lambda_1), \lambda_2))} m(c_{f_1, v_1} \otimes c_{f_2, v_2}).
 \end{aligned}$$

□

For more details on twisting, deformation, and r -matrices, see [HLT], [CV2, §2.2].

Definition 2.10. Let \mathcal{E}_φ be the full faithful subcategory in $\Gamma^\varphi(\mathfrak{g})$ -mod whose objects are the free R -modules of finite rank such that the elements K_i and $\binom{K_i; 0}{t}$ act by diagonal matrices with eigenvalues q_i^m and $\binom{m}{t}_{q_i}$, respectively. Define $R_q^\varphi[G]$ as the R -submodule of $\text{Hom}_R(\Gamma^\varphi(\mathfrak{g}), R)$ generated by the matrix coefficients of elements in \mathcal{E}_φ . Analogously, we define $R_q^\varphi[B_\pm]$ as the R -module generated by the matrix coefficients of elements of the full subcategories of $\Gamma^\varphi(\mathfrak{b}_+)$ -mod and $\Gamma^\varphi(\mathfrak{b}_-)$ -mod, respectively.

Since the categories are strict and tensorial, $R_q^\varphi[G]$ and $R_q^\varphi[B_\pm]$ are R -Hopf algebras. Moreover, by [CV2, §2.3], we have the isomorphisms

$$R_q^\varphi[B_\pm]' \simeq R_q^\varphi[B_\pm] \simeq R_q^\varphi[B_\pm]''.$$

Consider the linear map $\Gamma^\varphi(\mathfrak{b}_+) \otimes_R \Gamma^\varphi(\mathfrak{b}_-) \rightarrow \Gamma^\varphi(\mathfrak{g})$ given by the multiplication. The dual map composed with the isomorphism above gives the injection

$$(5) \quad \mu''_\varphi : R_q^\varphi[G] \rightarrow R_q^\varphi[B_+]'' \otimes_R R_q^\varphi[B_-]''.$$

Lemma 2.11 ([CV2, Lemma 2.5]). *The image of μ''_φ is contained in the R -subalgebra \mathbb{A}_φ'' generated by elements the $1 \otimes e_\alpha^\varphi$, $f_\alpha^\varphi \otimes 1$, and $K_{-(1+\varphi)\lambda} \otimes K_{(1-\varphi)\lambda}$ for $\lambda \in P$, $\alpha \in \Phi_+$.* □

Let $\lambda \in P_+$ and let $v_{\pm\lambda}$ be a highest (resp., lowest) weight vector of $L(\lambda)$ (resp., $L(-\lambda)$). Let $\phi_{\pm\lambda}$ be the unique element in $L(\pm\lambda)^*$ such that $\phi_{\pm\lambda}(v_{\pm\lambda}) = 1$ and vanishes over the complement $\Gamma(\mathfrak{h})$ -invariant of $\mathbb{Q}(q)v_{\pm\lambda} \subset L(\pm\lambda)$. Denote by $\psi_{\pm\lambda} = c_{\phi_{\pm\lambda}, v_{\pm\lambda}}$ the corresponding matrix coefficient.

As in [DL], we define for all $\alpha \in \Phi_+$, the matrix coefficient $\psi_{\pm\lambda}^{\pm\alpha}$ by

$$\begin{aligned}\psi_{\lambda}^{\alpha}(x) &= \phi_{\lambda}((E_{\alpha}x) \cdot v_{\lambda}), & \psi_{-\lambda}^{\alpha}(x) &= \phi_{-\lambda}((xE_{\alpha}) \cdot v_{-\lambda}), \\ \psi_{\lambda}^{-\alpha}(x) &= \phi_{\lambda}(xF_{\alpha} \cdot v_{\lambda}), & \psi_{-\lambda}^{-\alpha}(x) &= \phi_{-\lambda}(F_{\alpha}x \cdot v_{-\lambda}).\end{aligned}$$

Remark 2.12.

(a) Let $\lambda \in P_+$; then $\mu''_{\varphi}(\psi_{-\lambda}) = K_{-(1+\varphi)\lambda} \otimes K_{(1-\varphi)\lambda}$.

Indeed, evaluating both expressions in $EM \otimes FN$, where $EM = \xi_{m_1,0}\eta_{0,t_2}$ and $FN = \eta_{m_2,0}\xi_{0,t_1}$ for suitable m_1, t_2, m_2, t_1 of the basis of $\Gamma^{\varphi}(\mathfrak{b}_+)$ and $\Gamma^{\varphi}(\mathfrak{b}_-)$ (cf. Definition 2.5), respectively, and using [DL, Lemma 4.4 (iv)] we have that

$$\langle \mu''_{\varphi}(\psi_{-\lambda}), EM \otimes NF \rangle = \psi_{-\lambda}(EMNF) = \delta_{1,E}\delta_{1,F}MN(-\lambda),$$

where $M(\lambda) = \pi_0(K_{\lambda}, M)$ and $N(\lambda) = \bar{\pi}_0(K_{-\lambda}, N)$. Then, we have $MN(-\lambda) = \pi_0(K_{-\lambda}, MN) = \pi_0(K_{-\lambda}, M)\pi_0(K_{-\lambda}, N) = \pi_0(K_{-\lambda}, M)\bar{\pi}_0(K_{\lambda}, N) = M(-\lambda)N(-\lambda)$. Moreover, using (3) we have

$$\begin{aligned}\langle \mu''_{\varphi}(\psi_{\lambda}), EM \otimes NF \rangle &= \delta_{1,E}\delta_{1,F}M(-\lambda)N(-\lambda) \\ &= \delta_{1,E}\delta_{1,F}\pi''_{\varphi}(K_{-(1+\varphi)\lambda}, M)\bar{\pi}''_{\varphi}(K_{(1-\varphi)\lambda}, N).\end{aligned}$$

On the other hand, using the pairings π''_{φ} and $\bar{\pi}''_{\varphi}$ we have that

$$\langle K_{-(1+\varphi)\lambda} \otimes K_{(1-\varphi)\lambda}, EM \otimes NF \rangle = \delta_{1,E}\delta_{1,F}\pi''_{\varphi}(K_{-(1+\varphi)\lambda}, M)\bar{\pi}''_{\varphi}(K_{(1-\varphi)\lambda}, N),$$

and the claim follows.

(b) By [CV2, Propositions 1.9 & 2.7], for all $1 \leq i \leq n$ we have that

$$\begin{aligned}(6) \quad \mu''_{\varphi}(\psi_{-\omega_i}^{-\alpha_i}) &= q^{-(\tau_i, \omega_i)} f_{\alpha_i}^{\varphi} K_{-(1+\varphi)\omega_i} \otimes K_{(1-\varphi)\omega_i}, \\ \mu''_{\varphi}(\psi_{-\omega_i}^{\alpha_i}) &= q^{-(\tau_i, \omega_i)} K_{-(1+\varphi)\omega_i} \otimes K_{(1-\varphi)\omega_i} e_{\alpha_i}^{\varphi}.\end{aligned}$$

We check the first formula; the second follows similarly. Since $\mu''_{\varphi}(\psi_{-\omega_i}^{-\alpha_i}) = \mu''_0(\psi_{-\omega_i}^{-\alpha_i})$, and by [DL, Lemma 4.5 (vi)] it holds that $\mu''_0(\psi_{-\omega_i}^{-\alpha_i}) = f_{\alpha_i} K_{-\omega_i} \otimes K_{\omega_i}$, we have

$$\begin{aligned}\langle \mu''_{\varphi}(\psi_{-\omega_i}^{-\alpha_i}), EM \otimes NF \rangle &= \langle f_{\alpha_i} K_{-\omega_i} \otimes K_{\omega_i}, EM \otimes NF \rangle \\ &= \pi''_0(f_{\alpha_i} K_{-\omega_i}, EM)\bar{\pi}''_0(K_{\omega_i}, NF) \\ &= \pi''_0(f_{\alpha_i} K_{-\omega_i}, EM)\bar{\pi}''_0(K_{\omega_i}, N)\bar{\pi}''_0(K_{\omega_i}, F) \\ &= \pi''_0(f_{\alpha_i} K_{-\omega_i}, EM)N(-\omega_i)\delta_{1,F}.\end{aligned}$$

On the other hand, since $\pi''_{\varphi}(f_{\alpha_i}^{\varphi} K_{-(1+\varphi)\omega_i}, EM) = q^{(\tau_i, \omega_i)} \pi''_0(f_{\alpha_i}^0 K_{-\omega_i}, EM)$ by [CV2, Proposition 1.9] and [DL, (3.3)], using the definitions in (3), we obtain

$$\begin{aligned}\langle f_{\alpha_i}^{\varphi} K_{-(1+\varphi)\omega_i} \otimes K_{(1-\varphi)\omega_i}, EM \otimes NF \rangle &= \pi''_{\varphi}(f_{\alpha_i}^{\varphi} K_{-(1+\varphi)\omega_i}, EM)\pi''_{\varphi}(K_{(1-\varphi)\omega_i}, NF) \\ &= \pi''_{\varphi}(f_{\alpha_i}^{\varphi} K_{-(1+\varphi)\omega_i}, EM)N(-\omega_i)\delta_{1,F} \\ &= q^{(\tau_i, \omega_i)} \pi''_0(f_{\alpha_i}^0 K_{-\omega_i}, EM)N(-\omega_i)\delta_{1,F},\end{aligned}$$

and the assertion is proved.

The following lemma is a twisted version of [DL, Lemma 4.1].

Lemma 2.13. $R_q^\varphi[G]$ coincides with the R -Hopf subalgebra of $U_q^\varphi(\mathfrak{g})^\circ$ given by the set of all linear functions $f : \Gamma^\varphi(\mathfrak{g}) \rightarrow R$ such that there exists a cofinite ideal $I \subset \Gamma^\varphi(\mathfrak{g})$ and $N \in \mathbb{N}$ which satisfy that $f(I) = 0$ and $\prod_{p=-N}^N (K_i - q_i^p) \in I$ for all $1 \leq i \leq n$. Further, the induced Hopf pairing ρ between $R_q^\varphi[G]$ and $\Gamma^\varphi(\mathfrak{g})$ is nondegenerate.

Proof. Since $\Gamma^\varphi(\mathfrak{g}) = \Gamma(\mathfrak{g})$ as algebras, $R_q^\varphi[G]$ coincides with the set above by [DL, Lemma 4.1]. The Hopf algebra structure is the one induced from $\Gamma^\varphi(\mathfrak{g})^\circ$. The last claim follows from the fact that $\Gamma^\varphi(\mathfrak{g})$ has a PBW-basis and its dual basis lies in $R_q^\varphi[G]$. \square

2.5. Specializations at roots of one. In this subsection we recall the definition of the twisted quantum algebras at roots of unity and state some results that will be needed later. For all $Q \leq M \leq P$, we define

$$U_\epsilon^\varphi(\mathfrak{g}; M) = U_q^\varphi(\mathfrak{g}; M) \otimes_R \mathbb{Q}(\epsilon), \quad \Gamma_\epsilon^\varphi(\mathfrak{g}) := \Gamma^\varphi(\mathfrak{g}) \otimes_R \mathbb{Q}(\epsilon), \quad \mathcal{O}_\epsilon^\varphi(G)_{\mathbb{Q}(\epsilon)} := R_q^\varphi[G] \otimes_R \mathbb{Q}(\epsilon).$$

Note that $\Gamma_\epsilon^\varphi(\mathfrak{g}) \simeq \Gamma^\varphi(\mathfrak{g}) / [\chi_l(q) \Gamma^\varphi(\mathfrak{g})]$ and $\mathcal{O}_\epsilon^\varphi(G)_{\mathbb{Q}(\epsilon)} \simeq R_q^\varphi[G] / [\chi_\ell(q) R_q^\varphi[G]]$, where $R / [\chi_\ell(q) R] \simeq \mathbb{Q}(\epsilon)$. We denote $U_\epsilon^\varphi(\mathfrak{g}; P) := \check{U}_\epsilon^\varphi(\mathfrak{g})$ and $U_\epsilon^\varphi(\mathfrak{g}; Q) := U_\epsilon^\varphi(\mathfrak{g})$. For $r \in R$, denote by \bar{r} the image of the canonical projection $R \twoheadrightarrow \mathbb{Q}(\epsilon)$.

Lemma 2.14. $\mathcal{O}_\epsilon^\varphi(G)_{\mathbb{Q}(\epsilon)}$ is a 2-cocycle deformation of $\mathcal{O}_\epsilon(G)_{\mathbb{Q}(\epsilon)}$.

Proof. Let $\sigma : \mathcal{O}_q(G) \otimes \mathcal{O}_q(G) \rightarrow \mathbb{Q}(q)$ denote the 2-cocycle defined in Corollary 2.9. Then, the map $\bar{\sigma} : \mathcal{O}_\epsilon^\varphi(G)_{\mathbb{Q}(\epsilon)} \otimes \mathcal{O}_\epsilon^\varphi(G)_{\mathbb{Q}(\epsilon)} \rightarrow \mathbb{Q}(\epsilon)$ given by

$$\bar{\sigma}(\bar{x}, \bar{y}) = \overline{\sigma(x, y)} \quad \text{for all } x, y \in \mathcal{O}_q(G),$$

is a well-defined 2-cocycle for $\mathcal{O}_\epsilon^\varphi(G)_{\mathbb{Q}(\epsilon)}$, where \bar{x} denotes the image of $x \in \mathcal{O}_q(G)$ under the canonical projection $\mathcal{O}_q(G) \twoheadrightarrow \mathcal{O}_\epsilon^\varphi(G)_{\mathbb{Q}(\epsilon)}$. \square

Remark 2.15. The relations $E_i^\ell = 0$, $F_i^\ell = 0$, $K_{\alpha_i}^\ell = 1$ hold in $\Gamma_\epsilon^\varphi(\mathfrak{g})$ for all $1 \leq i \leq n$. Indeed, we have that $\prod_{s=1}^\ell (K_{\alpha_i} q^{(-s+1)} - 1) = \prod_{s=1}^\ell (q^s - 1) \binom{K_{\alpha_i}; 0}{\ell}$ in $\Gamma^\varphi(\mathfrak{g})$. If we specialize q at ϵ , then we have $\prod_{s=1}^\ell (K_{\alpha_i} \epsilon^{(-s+1)} - 1) = 0$. Since $K_{\alpha_i}^\ell - 1 = \prod_{s=0}^{\ell-1} (K_{\alpha_i} - \epsilon^s) = \epsilon^{\frac{(\ell-1)\ell}{2}} \prod_{s=0}^{\ell-1} (K_{\alpha_i} \epsilon^{-s+1} - 1)$, we have that $K_{\alpha_i}^\ell = 1$, as desired. The other two relations follow from the fact that $(\ell)_\epsilon = 0$.

The following lemma is analogue to [DL, Lemma 6.1].

Lemma 2.16. There exists a perfect Hopf pairing $\bar{\rho} : \mathcal{O}_\epsilon^\varphi(G)_{\mathbb{Q}(\epsilon)} \otimes_{\mathbb{Q}(\epsilon)} \Gamma_\epsilon^\varphi(\mathfrak{g}) \rightarrow \mathbb{Q}(\epsilon)$.

Proof. Let $\rho : R_q^\varphi[G] \otimes_R \Gamma^\varphi(\mathfrak{g}) \rightarrow R$ denote the pairing defined in Lemma 2.13. Then, we may define the pairing $\bar{\rho} : \mathcal{O}_\epsilon^\varphi(G)_{\mathbb{Q}(\epsilon)} \otimes_{\mathbb{Q}(\epsilon)} \Gamma_\epsilon^\varphi(\mathfrak{g}) \rightarrow \mathbb{Q}(\epsilon)$ via $\bar{\rho}(\bar{x}, \bar{u}) = \overline{(\rho(x, u))}$ for all $x \in R_q^\varphi[G]$ and $u \in \Gamma^\varphi(\mathfrak{g})$, where \bar{x} and \bar{u} denote the images of x and u under the canonical projections $R_q^\varphi[G] \rightarrow \mathcal{O}_\epsilon^\varphi(G)_{\mathbb{Q}(\epsilon)}$ and $\Gamma^\varphi(\mathfrak{g}) \rightarrow \Gamma_\epsilon^\varphi(\mathfrak{g})$, respectively. A direct computation shows that $\bar{\rho}$ is a well-defined map and it is a nondegenerate Hopf pairing. \square

Now we introduce the twisted quantum Frobenius map. For details, see [CV2, §3]. For $1 \leq i \leq n$, let e_i , f_i , and h_i denote the Chevalley generators of \mathfrak{g} and write $e_i^{(m)} := e_i/m!$, $f_i^{(m)} := f_i/m!$, $\binom{h_i}{m} := \frac{h_i(h_i-1)\cdots(h_i-m+1)}{m!}$ for all $m \geq 0$.

Lemma 2.17 ([CV2, §3.2 (i)]). *There is a unique Hopf algebra epimorphism $\text{Fr} : \Gamma_\epsilon^\varphi(\mathfrak{g}) \longrightarrow U(\mathfrak{g})_{\mathbb{Q}(\epsilon)}$ given for all $1 \leq i \leq n$ and $m > 0$ by*

$$\text{Fr}(E_i^{(m)}) = e_i^{(m/\ell)}, \quad \text{Fr}(F_i^{(m)}) = f_i^{(m/\ell)}, \quad \text{Fr}\left(\begin{smallmatrix} K_{\alpha_i}; 0 \\ m \end{smallmatrix}\right) = \begin{pmatrix} h_i \\ m/\ell \end{pmatrix}, \quad \text{Fr}(K_{\alpha_i}) = 1,$$

if $\ell \mid m$ or 0 otherwise. Its kernel is the ideal generated by the elements $K_{\alpha_i} - 1$, E_i , and F_i . In particular, there is a Hopf algebra monomorphism ${}^t\text{Fr} : \mathcal{O}(G)_{\mathbb{Q}(\epsilon)} \rightarrow \Gamma_\epsilon^\varphi(\mathfrak{g})^\circ$. \square

Let \mathbb{k} be a field extension of $\mathbb{Q}(\epsilon)$. We call $\mathcal{O}_\epsilon^\varphi(G)_{\mathbb{Q}(\epsilon)} \otimes_{\mathbb{Q}(\epsilon)} \mathbb{k}$ the \mathbb{k} -form of $\mathcal{O}_\epsilon^\varphi(G)_{\mathbb{Q}(\epsilon)}$. When $\mathbb{k} = \mathbb{C}$ we simply write $\mathcal{O}_\epsilon^\varphi(G)$.

Lemma 2.18 ([CV2, §3.3]). *$\mathcal{O}_\epsilon^\varphi(G)_{\mathbb{Q}(\epsilon)}$ contains a central Hopf subalgebra F_0 isomorphic to $\mathcal{O}(G)_{\mathbb{Q}(\epsilon)}$. Moreover, an element of $\mathcal{O}_\epsilon^\varphi(G)_{\mathbb{Q}(\epsilon)}$ belongs to F_0 if and only if it vanishes on I and*

$$F_0 = \mathbb{Q}(\epsilon) \langle \bar{c}_{f,v} \in \mathcal{O}_q^\varphi(G)_{\mathbb{Q}(\epsilon)} | f \in \overline{L}(\ell\Lambda)^*_{-\ell v}, v \in \overline{L}(\ell\Lambda)_{\ell\mu}; v, \mu \in P_+ \rangle,$$

where $\overline{L}(e\Lambda)$ is the $\Gamma^\varphi(\mathfrak{g})$ -module $\Gamma^\varphi(\mathfrak{g})_{v_{e\Lambda}}$ with $v_{e\Lambda}$ the highest weight vector of $L(e\Lambda)$. \square

Proposition 2.19. *$\mathcal{O}_\epsilon^\varphi(G)_{\mathbb{Q}(\epsilon)}$ is a free $\mathcal{O}(G)_{\mathbb{Q}(\epsilon)}$ -module of rank $\ell^{\dim \mathfrak{g}}$.*

Proof. The proof follows from [CV2, Proposition 3.5], [DL], and [BGS]. \square

Let $\overline{\mathcal{O}_\epsilon^\varphi(G)}$ be the quotient $\mathcal{O}_\epsilon^\varphi(G)/[\mathcal{O}(G)^+ \mathcal{O}_\epsilon^\varphi(G)]$ and let $\pi : \mathcal{O}_\epsilon^\varphi(G) \rightarrow \overline{\mathcal{O}_\epsilon^\varphi(G)}$ be the canonical projection. By Proposition 2.19, $\overline{\mathcal{O}_\epsilon^\varphi(G)}$ is a Hopf algebra of dimension $\ell^{\dim \mathfrak{g}}$. Moreover, since $\mathcal{O}_\epsilon^\varphi(G)$ is a free $\mathcal{O}(G)$ -module, it is faithfully flat. Then, by [Mo, Proposition 3.4.3], $\mathcal{O}_\epsilon^\varphi(G)$ fits into the short exact sequence of Hopf algebras.

$$1 \longrightarrow \mathcal{O}(G) \longrightarrow \mathcal{O}_\epsilon^\varphi(G) \longrightarrow \overline{\mathcal{O}_\epsilon^\varphi(G)} \longrightarrow 1.$$

3. TWISTED FROBENIUS–LUSZTIG KERNELS

In this section we define and study the twisted Frobenius–Lusztig kernels and the quotients of their duals. They are finite-dimensional pointed Hopf algebras which are twist deformations of the usual kernels.

Let Z_0^φ be the smaller $B_{\mathcal{W}}$ -invariant subalgebra of $U_\epsilon^\varphi(\mathfrak{g})$ that contains the elements $K_{\ell\alpha} = K_\alpha^\ell$, E_i^ℓ , F_i^ℓ for all $\alpha \in Q$ and $1 \leq i \leq n$.

Theorem 3.1.

- (i) Z_0^φ is a central Hopf subalgebra of $U_\epsilon^\varphi(\mathfrak{g})$.
- (ii) Z_0^φ is a polynomial ring in $\dim \mathfrak{g}$ generators, with n generators inverted.
- (iii) $U_\epsilon^\varphi(\mathfrak{g})$ is a free Z_0^φ -module of rank $\ell^{\dim \mathfrak{g}}$.

Proof. The proof follows the same lines as [BG, Theorem III.6.2], using the fact that the algebra W spanned by the elements $K_{\ell\alpha} = K_\alpha^\ell$, E_i^ℓ , F_i^ℓ for all $\alpha \in Q$ and $1 \leq i \leq n$ is a Hopf subalgebra, and this follows from a simple computation using the q -binomial formula. For example, $\Delta_\varphi(E_i^\ell) = (E_i \otimes K_{\tau_i} + K_{\alpha_i - \tau_i} \otimes E_i)^\ell = E_i^\ell \otimes K_{\ell\tau_i} + K_{\ell(\alpha_i - \tau_i)} \otimes E_i^\ell$, since $(E_i \otimes K_{\tau_i})(K_{\alpha_i - \tau_i} \otimes E_i) = \epsilon^{-2d_i}(K_{\alpha_i - \tau_i} \otimes E_i)(E_i \otimes K_{\tau_i})$. \square

Definition 3.2. The *twisted Frobenius–Lusztig kernel* is defined as the quotient

$$\mathbf{u}_\epsilon^\varphi(\mathfrak{g}) = U_\epsilon^\varphi(\mathfrak{g})/[(Z_0^\varphi)^+ U_\epsilon^\varphi(\mathfrak{g})].$$

By the theorem above, $\mathbf{u}_\epsilon^\varphi(\mathfrak{g})$ is a finite-dimensional pointed Hopf algebra of dimension $\ell^{\dim \mathfrak{g}}$ and $G(\mathbf{u}_\epsilon^\varphi(\mathfrak{g})) = \langle K_{\alpha_i} \mid 1 \leq i \leq n \rangle \simeq (\mathbb{Z}/\ell\mathbb{Z})^n$. We denote $G(\mathbf{u}_\epsilon^\varphi(\mathfrak{g})) = \mathbb{T}^\varphi$.

Lemma 3.3. *Let $\widehat{U}_\epsilon^\varphi(\mathfrak{g})$ be the Hopf subalgebra of $\Gamma_\epsilon^\varphi(\mathfrak{g})$ generated by the elements E_i, F_i, K_{α_i} with $1 \leq i \leq n$. Then $\widehat{U}_\epsilon^\varphi(\mathfrak{g})$ and $\mathbf{u}_\epsilon^\varphi(\mathfrak{g})$ are isomorphic as Hopf algebras.*

Proof. By definition, there exists a Hopf epimorphism $\widehat{U}_\epsilon^\varphi(\mathfrak{g}) \twoheadrightarrow \mathbf{u}_\epsilon^\varphi(\mathfrak{g})$ given by $E_i \mapsto E_i$, $F_i \mapsto F_i$ and $K_{\alpha_i} \mapsto K_{\alpha_i}$ for all $1 \leq i \leq n$. Since by Remark 2.15, $\dim \widehat{U}_\epsilon^\varphi(\mathfrak{g}) \leq \ell^{\dim \mathfrak{g}}$, the claim follows. \square

Adapting the proof of [BG, Theorem III.7.10], we have the following.

Theorem 3.4. *The Hopf algebras $\overline{\mathcal{O}_\epsilon^\varphi(G)}$ and $\mathbf{u}_\epsilon^\varphi(\mathfrak{g})^*$ are isomorphic.*

Proof. The pairing defined in Lemma 2.16 induces a perfect Hopf pairing $\overline{\mathcal{O}_\epsilon^\varphi(G)} \otimes_{\mathbb{Q}(\epsilon)} \widehat{U}_\epsilon^\varphi(\mathfrak{g}) \rightarrow \mathbb{Q}(\epsilon)$. In particular, we have a Hopf algebra monomorphism $\overline{\mathcal{O}_\epsilon^\varphi(G)} \hookrightarrow \widehat{U}_\epsilon^\varphi(\mathfrak{g})^*$. Since both algebras have the same dimension, the assertion follows by Lemma 3.3. \square

As a consequence of the theorem above, the following sequence of Hopf algebras is exact:

$$1 \longrightarrow \mathcal{O}(G)_{\mathbb{Q}(\epsilon)} \xrightarrow{\iota} \mathcal{O}_\epsilon^\varphi(G)_{\mathbb{Q}(\epsilon)} \xrightarrow{\pi} \mathbf{u}_\epsilon^\varphi(\mathfrak{g})^* \longrightarrow 1.$$

Proposition 3.5. $\mathbf{u}_\epsilon^\varphi(\mathfrak{g}) \simeq \mathbf{u}_\epsilon(\mathfrak{g})^J$ for a twist $J \in \mathbb{Q}(\epsilon)[\mathbb{T}^\varphi \times \mathbb{T}^\varphi]$.

Proof. By Lemma 2.14, $\mathcal{O}_\epsilon^\varphi(G)$ is a 2-cocycle deformation of $\mathcal{O}_\epsilon(G)$. Denote this cocycle by $\bar{\sigma}$. Then, if we denote $\mathcal{I} = \mathcal{O}(G)^+ \mathcal{O}_\epsilon^\varphi(G)$, it holds that $\bar{\sigma}|_{\mathcal{I} \otimes \mathcal{O}_\epsilon^\varphi(G) + \mathcal{O}_\epsilon^\varphi(G) \otimes \mathcal{I}} = 0$, and by Remark 1.1, we have that $\mathbf{u}_\epsilon^\varphi(\mathfrak{g})^*$ is a 2-cocycle deformation of $\mathbf{u}_\epsilon(\mathfrak{g})^*$, where the cocycle is given by the formula $\hat{\sigma}(\pi(x), \pi(y)) = \bar{\sigma}(x, y)$ for all $x, y \in \mathcal{O}_\epsilon(G)$. We may consider $\hat{\sigma} : \mathbf{u}_\epsilon(\mathfrak{g})^* \otimes \mathbf{u}_\epsilon(\mathfrak{g})^* \rightarrow \mathbb{Q}(\epsilon)$ as an element in $\mathbf{u}_\epsilon(\mathfrak{g}) \otimes \mathbf{u}_\epsilon(\mathfrak{g})$, say $J = \sum_i u_i \otimes u^i$. Then,

$$\begin{aligned} \hat{\sigma}(\pi(c_{f_1, v_1}) \otimes \pi(c_{f_2, v_2})) &= \langle J, \pi(c_{f_1, v_1}) \otimes \pi(c_{f_2, v_2}) \rangle = \sum_i f_1(u_i \cdot v_1) f_2(u^i \cdot v_2) \\ &= \varepsilon(c_{f_1, v_1}) \varepsilon(c_{f_2, v_2}) \epsilon^{\frac{1}{2}(\varphi(\lambda_1), \lambda_2)} = f_1(v_1) f_2(v_2) \epsilon^{\frac{1}{2}(\varphi(\lambda_1), \lambda_2)}, \end{aligned}$$

for all $\Lambda_i \in P_+$, $v_i \in L(\Lambda_i)_{\mu_i}$, $f_i \in L(\Lambda_i)_{-\lambda_i}$, and $i = 1, 2$, where $\langle \cdot, \cdot \rangle$ is the perfect pairing given by the evaluation. Thus, the components of J must act diagonally, and consequently, $J \in \mathbb{Q}(\epsilon)[\mathbb{T}^\varphi \times \mathbb{T}^\varphi]$. \square

3.1. Subalgebras of $\mathbf{u}_\epsilon^\varphi(\mathfrak{g})$. In this subsection we discuss a parametrization of the Hopf subalgebras of $\mathbf{u}_\epsilon^\varphi(\mathfrak{g})$. Since $\mathbf{u}_\epsilon^\varphi(\mathfrak{g})$ is a pointed Hopf algebra, any Hopf subalgebra is also pointed, and in this case, it is generated by a subgroup of the group of group-like elements and a subset of skew-primitive elements.

Lemma 3.6. *The Hopf subalgebras of $\mathbf{u}_\epsilon^\varphi(\mathfrak{g})$ are parametrized by triples $(I_+, I_-, \Sigma^\varphi)$, where $I_\pm \subset \pm\Pi$ and Σ^φ is a subgroup of $G(\mathbf{u}_\epsilon^\varphi(\mathfrak{g}))$ such that $K_{(1 \mp \varphi)(\alpha_i)} \in \Sigma^\varphi$ if $\alpha_i \in I_\pm$. Denote $\tilde{E}_i := E_i K_{-\tau_i}$ and $\tilde{F}_j := K_{(\alpha_j + \tau_j)} F_j$. Then the Hopf subalgebra of $\mathbf{u}_\epsilon^\varphi(\mathfrak{g})$ corresponding to the triple $(I_+, I_-, \Sigma^\varphi)$ is the subalgebra generated by the set $\{g, \tilde{E}_i, \tilde{F}_j \mid g \in \Sigma^\varphi, \alpha_i \in I_+ \text{ and } \alpha_j \in I_-\}$.*

Proof. The proof follows from [AG, Corollary 1.12], since $\mathfrak{u}_\epsilon^\varphi(\mathfrak{g})$ is generated by group-like and skew-primitive elements. In particular, we have $\Delta_\varphi(\tilde{E}_i) = \tilde{E}_i \otimes 1 + K_{(1-\varphi)(\alpha_i)} \otimes \tilde{E}_i$, $\Delta_\varphi(\tilde{F}_j) = \tilde{F}_j \otimes 1 + K_{(1+\varphi)(\alpha_j)} \otimes \tilde{F}_j$. \square

Each pair (I_+, I_-) determines a regular Lie subalgebra \mathfrak{p} of \mathfrak{g} containing the fixed Cartan subalgebra \mathfrak{h} . Next we define the corresponding twisted quantum algebras.

Definition 3.7. For every pair (I_+, I_-) with $I_\pm \subset \pm\Pi$, we define $\Gamma^\varphi(\mathfrak{p})$ as the subalgebra of $\Gamma^\varphi(\mathfrak{g})$ generated by the elements

$$\begin{aligned} K_{\alpha_i}^{-1} & \quad (1 \leq i \leq n), \\ \binom{K_{\alpha_i}; 0}{m} &:= \prod_{s=1}^m \left(\frac{K_{\alpha_i} q_i^{-s+1} - 1}{q_i^s - 1} \right) \quad (m \geq 1, 1 \leq i \leq n), \\ E_j^{(m)} &:= \frac{E_j^m}{[m]_{q_j}!} \quad (m \geq 1, \alpha_j \in I_+), \\ F_k^{(m)} &:= \frac{F_k^m}{[m]_{q_k}!} \quad (m \geq 1, \alpha_k \in I_-). \end{aligned}$$

Proposition 3.8 ([AG, Proposition 2.3 (a)]). *Let $\Gamma_\epsilon^\varphi(\mathfrak{p}) := \Gamma^\varphi(\mathfrak{p})/[\chi_\ell(q)\Gamma^\varphi(\mathfrak{p})] \simeq \Gamma^\varphi(\mathfrak{p}) \otimes_R R/[\chi_\ell(q)R]$ denote the $\mathbb{Q}(\epsilon)$ -algebra given by the specialization. Then $\Gamma_\epsilon^\varphi(\mathfrak{p})$ is a Hopf subalgebra of $\Gamma_\epsilon^\varphi(\mathfrak{g})$.* \square

Next we define a family of regular twisted Frobenius–Lusztig kernels.

Definition 3.9. For every pair (I_+, I_-) with $I_\pm \subset \pm\Pi$, we define the *twisted regular Frobenius–Lusztig kernel* $\mathfrak{u}_\epsilon^\varphi(\mathfrak{p})$ as the subalgebra of $\Gamma_\epsilon^\varphi(\mathfrak{p})$ generated by the elements $\{K_{\alpha_i}, E_j, F_k : 1 \leq i \leq n, \alpha_j \in I_+, \alpha_k \in I_-\}$.

In the following propositions we collect some properties.

Proposition 3.10. $\mathfrak{u}_\epsilon^\varphi(\mathfrak{p})$ is the Hopf subalgebra of $\mathfrak{u}_\epsilon^\varphi(\mathfrak{g})$ given by $\Gamma_\epsilon^\varphi(\mathfrak{p}) \cap \mathfrak{u}_\epsilon^\varphi(\mathfrak{g}) = \mathfrak{u}_\epsilon^\varphi(\mathfrak{p})$. It corresponds to the triple $(I_+, I_-, \mathbb{T}^\varphi)$.

Proof. Follows from Lemmata 3.3 and 3.6. \square

Proposition 3.11.

- (i) Let $U(\mathfrak{p})_{\mathbb{Q}(\epsilon)} := \text{Fr}(\Gamma_\epsilon^\varphi(\mathfrak{p}))$ and denote $\text{Fr}_{res} = \text{Fr}|_{\Gamma_\epsilon^\varphi(\mathfrak{p})}$. Then the following diagram is commutative and all rows are exact sequences of Hopf algebras:

$$(7) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \mathfrak{u}_\epsilon^\varphi(\mathfrak{g}) & \longrightarrow & \Gamma_\epsilon^\varphi(\mathfrak{g}) & \xrightarrow{\text{Fr}} & U(\mathfrak{g})_{\mathbb{Q}(\epsilon)} \longrightarrow 1 \\ & & \uparrow & & \uparrow & & \uparrow \\ 1 & \longrightarrow & \mathfrak{u}_\epsilon^\varphi(\mathfrak{p}) & \longrightarrow & \Gamma_\epsilon^\varphi(\mathfrak{p}) & \xrightarrow{\text{Fr}_{res}} & U(\mathfrak{p})_{\mathbb{Q}(\epsilon)} \longrightarrow 1 \end{array}$$

- (ii) There is a surjective algebra map $\theta : \Gamma_\epsilon^\varphi(\mathfrak{p}) \rightarrow \mathfrak{u}_\epsilon^\varphi(\mathfrak{p})$ such that $\theta|_{\mathfrak{u}_\epsilon^\varphi(\mathfrak{p})} = \text{id}$.

Proof. (i) From [CV2, DL] and Lemma 2.17 it follows that $\text{Ker Fr} = \mathfrak{u}_\epsilon^\varphi(\mathfrak{g})^+ \Gamma_\epsilon^\varphi(\mathfrak{g})$. The proof that $\Gamma_\epsilon^\varphi(\mathfrak{g})^{\text{co Fr}} = \mathfrak{u}_\epsilon^\varphi(\mathfrak{g})$ follows from [A, Lemma 3.4.2], but by using the formula (4) instead of the formula (1.1.3) in [A]. So the first row is exact. To prove that the second row is exact, note that $\mathfrak{u}_\epsilon^\varphi(\mathfrak{p}) = \mathfrak{u}_\epsilon^\varphi(\mathfrak{g}) \cap \Gamma_\epsilon^\varphi(\mathfrak{p}) = \Gamma_\epsilon^\varphi(\mathfrak{g})^{\text{co Fr}} \cap \Gamma_\epsilon^\varphi(\mathfrak{p}) = \Gamma_\epsilon^\varphi(\mathfrak{p})^{\text{co Fr}_{res}}$ and $\text{Ker Fr}_{res} = \text{Ker Fr} \cap \Gamma_\epsilon^\varphi(\mathfrak{p}) = \mathfrak{u}_\epsilon^\varphi(\mathfrak{g})^+ \Gamma_\epsilon^\varphi(\mathfrak{g}) \cap \Gamma_\epsilon^\varphi(\mathfrak{p}) = \mathfrak{u}_\epsilon^\varphi(\mathfrak{p})^+ \Gamma_\epsilon^\varphi(\mathfrak{p})$.

- (ii) This follows from [AG, Lemma 1.10 & Proposition 2.6]. \square

Remark 3.12. Let \mathfrak{p} be the set of primitive elements in $U(\mathfrak{g})_{\mathbb{Q}(\epsilon)}$. Then, \mathfrak{p} is a regular Lie subalgebra of \mathfrak{g} , and $\mathfrak{u}_\epsilon^\varphi(\mathfrak{p})$ is the twisted Frobenius–Lusztig kernel associated with it.

Proposition 3.13. $\mathfrak{u}_\epsilon^\varphi(\mathfrak{p})$ is a twist deformation of $\mathfrak{u}_\epsilon(\mathfrak{p})$.

Proof. We know that $\mathfrak{u}_\epsilon^\varphi(\mathfrak{g}) \simeq \mathfrak{u}_\epsilon(\mathfrak{g})^J$ for a twist $J \in \mathbb{Q}(\epsilon)[\mathbb{T}^\varphi \times \mathbb{T}^\varphi]$. Thus, $J \in \mathfrak{u}_\epsilon(\mathfrak{p}) \otimes \mathfrak{u}_\epsilon(\mathfrak{p})$ and $\mathfrak{u}_\epsilon(\mathfrak{p})^J$ is the subalgebra of $\mathfrak{u}_\epsilon(\mathfrak{g})^J$ that is isomorphic to the Hopf subalgebra of $\mathfrak{u}_\epsilon^\varphi(\mathfrak{g})$ which corresponds to the triple $(I_+, I_-, \mathbb{T}^\varphi)$. Hence, $\mathfrak{u}_\epsilon(\mathfrak{p})^J \simeq \mathfrak{u}_\epsilon^\varphi(\mathfrak{p})$. \square

3.2. Quotients of $\mathfrak{u}_\epsilon^\varphi(\mathfrak{g})^*$. Denote the \mathbb{C} -form of the twisted Frobenius–Lusztig kernel by just $\mathfrak{u}_\epsilon^\varphi(\mathfrak{g})$. Let H be a Hopf algebra quotient of $\mathfrak{u}_\epsilon^\varphi(\mathfrak{g})^*$. Then, H^* is a Hopf subalgebra of $\mathfrak{u}_\epsilon^\varphi(\mathfrak{g})$ and whence, by Lemma 3.6, it is determined by a triple $(I_+, I_-, \Sigma^\varphi)$. Let $\mathfrak{u}_\epsilon^\varphi(\mathfrak{p})$ be the regular Frobenius–Lusztig kernel associated to the pair (I_+, I_-) . Then $H^* \hookrightarrow \mathfrak{u}_\epsilon^\varphi(\mathfrak{p}) \hookrightarrow \mathfrak{u}_\epsilon^\varphi(\mathfrak{g})$ as Hopf algebras, and consequently we have a sequence of Hopf algebra epimorphisms

$$\mathfrak{u}_\epsilon^\varphi(\mathfrak{g})^* \twoheadrightarrow \mathfrak{u}_\epsilon^\varphi(\mathfrak{p})^* \xrightarrow{\nu} H.$$

Let $I = I_+ \cup I_-$, $I' = I_+ \cap I_-$ and $I^c = (I_+ \cup I_-)^c = I_+^c \cap I_-^c$. We define the abelian subgroups \mathbb{T}_I^φ and $\mathbb{T}_{I'}^\varphi$ of Σ^φ as follows:

$$\begin{aligned} \mathbb{T}_I^\varphi &= \langle \bar{K}_i := K_{(1-\varphi)(\alpha_i)}, \bar{K}_j := K_{(1+\varphi)(\alpha_j)} : \text{ if } \alpha_i \in I_+, \alpha_j \in I_- \rangle, \\ \mathbb{T}_{I'}^\varphi &= \langle K_{\alpha_i} : \text{ if } \alpha_i \in I_+ \cap I_- \rangle. \end{aligned}$$

Note that if $\alpha_i \in I_+ \cap I_-$, then $K_{\alpha_i} \in \mathbb{T}_{I'}^\varphi$. Hence, $\mathbb{T}_{I'}^\varphi \subseteq \mathbb{T}_I^\varphi \subseteq \Sigma^\varphi \subseteq \mathbb{T}^\varphi$. Denote $\mathbb{T}_{I^c}^\varphi = \mathbb{T}^\varphi / \mathbb{T}_I^\varphi$ and $\Omega^\varphi = \Sigma^\varphi / \mathbb{T}_{I'}^\varphi$; so $\Omega^\varphi \subseteq \mathbb{T}_{I^c}^\varphi$.

Definition 3.14. For all $i \in \{1, \dots, n\}$ such that $\alpha_i \in (I_+ \cap I_-)^c$, we define the algebra homomorphism $D_i : \mathfrak{u}_\epsilon^\varphi(\mathfrak{p}) \rightarrow \mathbb{C}$ by

$$D_i(E_j) = 0 = D_i(F_k), \quad D_i(K_{\alpha_t}) = \epsilon^{\delta_{it}} \quad \text{for all } \alpha_j \in I_+, \alpha_k \in I_-, t \in \{1, \dots, n\}.$$

Remark 3.15.

(a) For $1 \leq i \leq n$, let $\hat{D}_i \in \widehat{\mathbb{T}^\varphi}$ be given by $\hat{D}_i(K_{\alpha_t}) = \epsilon^{\delta_{it}}$ for all $1 \leq t \leq n$. Then $\langle \hat{D}_i : 1 \leq i \leq n \rangle = \widehat{\mathbb{T}^\varphi}$ and we may identify $(\mathbb{Z}/\ell\mathbb{Z})^n \simeq \widehat{\mathbb{T}^\varphi}$ by $z \mapsto \hat{D}^z = \hat{D}_1^{z_1} \dots \hat{D}_n^{z_n}$. In particular, one has that $\hat{D}_i = D_i|_{\mathbb{T}^\varphi}$ for all $i \in (I_+ \cap I_-)^c$.

(b) Assume $(I_+ \cap I_-)^c = \{\alpha_{i_1}, \dots, \alpha_{i_m}\}$. For all $z \in (\mathbb{Z}/\ell\mathbb{Z})^m$, denote

$$D^z = D_{i_1}^{z_1} \dots D_{i_m}^{z_m} \in G(\mathfrak{u}_\epsilon^\varphi(\mathfrak{p})^*).$$

If $f \in G(\mathfrak{u}_\epsilon^\varphi(\mathfrak{p})^*)$, then $f = D^z$ for some $z \in (\mathbb{Z}/\ell\mathbb{Z})^m$. In particular, we may identify

$$G(\mathfrak{u}_\epsilon^\varphi(\mathfrak{p})^*) \simeq \widehat{\mathbb{T}^\varphi / \mathbb{T}_{I'}^\varphi} \simeq (\mathbb{Z}/\ell\mathbb{Z})^m.$$

(c) Since $\mathbb{T}_{I'}^\varphi \subseteq \mathbb{T}_I^\varphi$, there is a group monomorphism $\widehat{\mathbb{T}_{I^c}^\varphi} \hookrightarrow \widehat{\mathbb{T}^\varphi / \mathbb{T}_{I'}^\varphi} = G(\mathfrak{u}_\epsilon^\varphi(\mathfrak{p})^*)$ given for any $f \in \widehat{\mathbb{T}_{I^c}^\varphi}$ by the composition $\mathbb{T}^\varphi / \mathbb{T}_{I'}^\varphi \twoheadrightarrow \mathbb{T}_{I^c}^\varphi \xrightarrow{f} \mathbb{C}$.

(d) The inclusions $\mathbb{T}_I^\varphi \hookrightarrow \Sigma^\varphi \xrightarrow{j} \mathbb{T}^\varphi$ induce the surjective maps $\widehat{\mathbb{T}^\varphi} \xrightarrow{t_j} \widehat{\Sigma^\varphi}$ with $\text{Ker } t_j = \{f \in \widehat{\mathbb{T}^\varphi} : f(\Sigma^\varphi) = 1\}$ and $\widehat{\mathbb{T}_{I^c}^\varphi} \xrightarrow{t_{j_s}} \widehat{\Omega^\varphi}$ with $N^\varphi = \text{Ker } t_{j_s} = \{f \in \widehat{\mathbb{T}_{I^c}^\varphi} : f(\Omega^\varphi) = 1\}$. In particular, we have

$$(8) \quad |\Sigma^\varphi| = |\mathbb{T}_I^\varphi| |\Omega^\varphi| = |\mathbb{T}_I^\varphi| \frac{|\mathbb{T}_{I^c}^\varphi|}{|N^\varphi|} = |\mathbb{T}_I^\varphi| \frac{|\mathbb{T}^\varphi|}{|\mathbb{T}_I^\varphi| |N^\varphi|} = \frac{\ell^n}{|N^\varphi|}.$$

Moreover, one has that $\text{Ker } {}^t(j\iota) \simeq \widehat{\mathbb{T}_{I_c}^\varphi}$, since there is a group monomorphism $\text{Ker } {}^t(j\iota) \rightarrow \widehat{\mathbb{T}_{I_c}^\varphi}$ and $|\text{Ker } {}^t(j\iota)| = |\mathbb{T}_{I_c}^\varphi|$. Hence, in what follows we identify the elements of $\widehat{\mathbb{T}_{I_c}^\varphi}$ and $\text{Ker } {}^t(j\iota)$. On the other hand, if we denote $\hat{D}^z = \hat{D}_1^{z_1} \cdots \hat{D}_n^{z_n}$ for all $z = (z_1, \dots, z_n) \in (\mathbb{Z}/\ell\mathbb{Z})^n$, then

$$\text{Ker } {}^t(j\iota) = \{\hat{D}^z \mid \hat{D}^z(\bar{K}_i) = 1 = \hat{D}^z(\tilde{K}_j), \text{ for } i \in I_+, j \in I_-, z \in (\mathbb{Z}/\ell\mathbb{Z})^n\} \simeq \widehat{\mathbb{T}_{I_c}^\varphi}.$$

Therefore, if $\hat{D}^z \in \text{Ker } {}^t(j\iota)$, then

$$(9) \quad 1 = \hat{D}^z(\bar{K}_i) = \hat{D}^z(K_{(1-\varphi)(\alpha_i)}) = \hat{D}^z \left(K_{\alpha_i} \prod_{j=1}^n K_{\alpha_j}^{-2y_{ji}} \right) = \epsilon^{z_i} \prod_{j=1}^n \epsilon^{-2y_{ji}z_j},$$

$$(10) \quad 1 = \hat{D}^z(\tilde{K}_j) = \hat{D}^z(K_{(1+\varphi)(\alpha_j)}) = \hat{D}^z \left(K_{\alpha_j} \prod_{k=1}^n K_{\alpha_k}^{2y_{kj}} \right) = \epsilon^{z_j} \prod_{k=1}^n \epsilon^{2y_{kj}z_k},$$

for all $i \in I_+$ and $j \in I_-$. Thus, to find the generators of $\text{Ker } {}^t(j\iota)$ it suffices to solve a linear system over $\mathbb{Z}/\ell\mathbb{Z}$. Indeed, if $I_+ = \{\alpha_{i_1}, \dots, \alpha_{i_s}\}$ and $I_- = \{\alpha_{j_1}, \dots, \alpha_{j_r}\}$, by (9) and (10) we have a system of linear equations over $\mathbb{Z}/\ell\mathbb{Z}$ whose matrix S_ℓ^φ is given by

$$\begin{pmatrix} 2y_{i_1 1} & \cdots & 2y_{i_1 i_s} & \cdots & 2y_{i_1 j_1} & \cdots & 2y_{i_1 j_l} & \cdots & 2y_{i_1 i_1} - 1 & \cdots & 2y_{i_1 n} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 2y_{i_s 1} & \cdots & 2y_{i_s i_s} + 1 & \cdots & 2y_{i_s j_1} & \cdots & 2y_{i_s j_l} & \cdots & 2y_{i_s i_1} & \cdots & 2y_{i_s n} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 2y_{j_1 1} & \cdots & 2y_{j_1 i_s} & \cdots & 2y_{j_1 j_1} - 1 & \cdots & 2y_{j_1 j_l} & \cdots & 2y_{j_1 i_1} & \cdots & 2y_{j_1 n} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 2y_{j_l 1} & \cdots & 2y_{j_l i_s} & \cdots & 2y_{j_l j_1} & \cdots & 2y_{j_l j_l} + 1 & \cdots & 2y_{j_l i_1} & \cdots & 2y_{j_l n} \end{pmatrix}.$$

In particular, $|\text{Ker } {}^t(j\iota)| = |\widehat{\mathbb{T}_{I_c}^\varphi}| = \ell^{n-\text{rk } S_\ell^\varphi}$. Analogously, it is possible to characterize in the same way the kernel N^φ . In this case we have to consider the system of linear equations determined by the conditions $\hat{D}^z(\Omega^\varphi) = 1$ for all $\hat{D}^z \in \widehat{\mathbb{T}_{I_c}^\varphi}$.

Example 3.16. Assume \mathfrak{g} is of type C_3 with associated Cartan matrix $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}$. Then the multiparametric matrix Y is given by

$$Y = \begin{pmatrix} a + b/2 & -a + c/2 & -b/2 - c/2 \\ 2a + b & -a + c & -b/2 - c \\ 2a + 3b/2 & -a + 3c/2 & -b/2 - c \end{pmatrix},$$

where $a \in \mathbb{Z}$, and $b, c \in 2\mathbb{Z}$. Set $a = 1$, $b = 2$, $c = 0$, and $\ell = 11$. Then, $\varphi(\alpha_1) = 4\alpha_1 + 8\alpha_2 + 10\alpha_3$, $\varphi(\alpha_2) = -2\alpha_1 - 2\alpha_2 - 2\alpha_3$ and $\varphi(\alpha_3) = -2\alpha_1 - 2\alpha_2 - 2\alpha_3$.

(a) If we choose $I_+ = \{\alpha_2\}$ and $I_- = \{\alpha_1\}$, then $S_{11}^\varphi = \begin{pmatrix} 5 & 8 & 10 \\ 2 & 3 & 2 \end{pmatrix} \sim_{11} \begin{pmatrix} 1 & 0 & 8 \\ 0 & 1 & 10 \end{pmatrix}$ and $\widehat{\mathbb{T}_{I_c}^\varphi} = \langle \hat{D}_1^3 \hat{D}_2 \hat{D}_3 \rangle \simeq \mathbb{Z}/11\mathbb{Z}$. If we take $\Sigma^\varphi = \langle K_{(1-\varphi)(\alpha_2)}, K_{(1+\varphi)(\alpha_1)}, K_{\tau_3}, K_{\tau_2} \rangle$, then we have that $\Sigma^\varphi = \mathbb{T}^\varphi \simeq (\mathbb{Z}/11\mathbb{Z})^3$ and N^φ is trivial.

(b) If we choose $I_+ = \{\alpha_2\}$, $I_- = \emptyset$ and $\Sigma^\varphi = \langle K_{(1+\varphi)(\alpha_1)}, K_{(1-\varphi)(\alpha_2)} \rangle$, then we have that $\widehat{\mathbb{T}_{I_c}^\varphi} = \langle \hat{D}^{(1,0,10)}, \hat{D}^{(0,1,4)} \rangle \simeq (\mathbb{Z}/11\mathbb{Z})^2$, $\Omega^\varphi \simeq \langle K_{(1+\varphi)(\alpha_1)} \rangle$ and $N^\varphi = \langle \hat{D}^{(3,1,1)} \rangle$.

The following proposition states that the elements in $\widehat{\mathbb{T}_{I_c}^\varphi}$ are central in $\mathbf{u}_\epsilon^\varphi(\mathfrak{p})^*$.

Proposition 3.17. *The subgroup $\widehat{\mathbb{T}_{f_c}^\varphi}$ of $G(\mathbf{u}_\epsilon^\varphi(\mathfrak{p})^*)$ consists of central group-like elements.*

Proof. Let $z \in (\mathbb{Z}/\ell\mathbb{Z})^m$ and $D^z \in G(\mathbf{u}_\epsilon^\varphi(\mathfrak{p})^*)$ such that $D^z \in \widehat{\mathbb{T}_{f_c}^\varphi}$. Then $D^z(\overline{K}_i) = 1 = D^z(\overline{K}_j)$ for all $i \in I_+$ and $j \in I_-$. We show that D^z is central in $\mathbf{u}_\epsilon^\varphi(\mathfrak{p})^*$.

By [L, Theorem 6.7] and [AG, Lemma 2.14], $\mathbf{u}_\epsilon^\varphi(\mathfrak{p})$ has a basis

$$\left\{ \prod_{\beta \geq 0} F_\beta^{n_\beta} \prod_{i=1}^n K_{\alpha_i}^{t_i} \prod_{\alpha \geq 0} E_\alpha^{m_\alpha} : 0 \leq n_\beta, t_i, m_\alpha \leq \ell, 1 \leq i \leq n, \beta \in Q_{I_-}, \alpha \in Q_{I_+} \right\}.$$

The hypothesis on D^z ensures that $D^z f(E_i) = f D^z(E_i)$ and $D^z f(F_j) = f D^z(F_j)$ for all $f \in \mathbf{u}_\epsilon^\varphi(\mathfrak{p})^*$, $z \in (\mathbb{Z}/\ell\mathbb{Z})^m$, $i \in I_+$, and $j \in I_-$. Moreover, since the elements $K_{\alpha_t} \in \mathbf{u}_\epsilon^\varphi(\mathfrak{p})$ are group-like for all $1 \leq t \leq n$, $D^z f(K_{\alpha_t}) = f D^z(K_{\alpha_t})$. As D^z is a group-like element in $\mathbf{u}_\epsilon^\varphi(\mathfrak{p})^*$, we have that $D^z f(MN) = f D^z(MN)$ for $M, N \in \{K_{\alpha_t}, E_i, F_j : i \in I_+, j \in I_-\}$, since $D^z f(MN) = (D^z f)_{(1)}(M)(D^z f)_{(2)}(N) = D^z f_{(1)}(M) D^z f_{(2)}(N) = f_{(1)} D^z(M) f_{(2)} D^z(N) = f D^z(MN)$. Analogously, using an inductive argument one may prove that D^z and f commute when evaluated on every element of the basis. \square

The following proposition gives a characterization of all quotients of $\mathbf{u}_\epsilon^\varphi(\mathfrak{g})^*$.

Proposition 3.18. *Let H be a Hopf algebra quotient of $\mathbf{u}_\epsilon^\varphi(\mathfrak{g})^*$ such that H^* is determined by the triple $(I_+, I_-, \Sigma^\varphi)$ and $\mathbf{u}_\epsilon^\varphi(\mathfrak{p})$ the twisted regular Frobenius–Lusztig kernel associated to (I_+, I_-) . Then $H = \mathbf{u}_\epsilon^\varphi(\mathfrak{p})^* / \langle D^z - 1 : D^z \in N^\varphi \rangle$.*

Proof. If $(I_+ \cap I_-)^c = \{\alpha_{i_1}, \dots, \alpha_{i_m}\}$ and we write $D^z = D_{i_1}^{z_1} \cdots D_{i_m}^{z_m}$, then by Remark 3.15(b), $G(\mathbf{u}_\epsilon^\varphi(\mathfrak{p})^*) = \{D^z \mid z \in (\mathbb{Z}/\ell\mathbb{Z})^m\}$. By Proposition 3.17, we know that the elements of $\widehat{\mathbb{T}_{f_c}^\varphi}$ are central in $\mathbf{u}_\epsilon^\varphi(\mathfrak{p})^*$. Since $N^\varphi \subseteq \widehat{\mathbb{T}_{f_c}^\varphi}$, the two-sided ideal \mathcal{I} of $\mathbf{u}_\epsilon^\varphi(\mathfrak{p})^*$ generated by the elements $\{D^z - 1 : D^z \in N^\varphi\}$ is a Hopf ideal and whence $\mathbf{u}_\epsilon^\varphi(\mathfrak{p})^* / \mathcal{I}$ is a Hopf algebra.

On the other hand, we know that H^* is determined by the triple $(I_+, I_-, \Sigma^\varphi)$, and, consequently, H^* is included in $\mathbf{u}_\epsilon^\varphi(\mathfrak{p})$. If we denote by $\nu : \mathbf{u}_\epsilon^\varphi(\mathfrak{p})^* \rightarrow H$ the epimorphism induced by this inclusion, we have that $\text{Ker } \nu = \{f \in \mathbf{u}_\epsilon^\varphi(\mathfrak{p})^* : f(h) = 0 \text{ for all } h \in H^*\}$. Since by Remark 3.15(c), $D^z(g) = 1$ for all $g \in \Sigma^\varphi$ and $D^z \in N^\varphi$, we have that $D^z - 1 \in \text{Ker } \nu$ and whence there is a Hopf algebra epimorphism $\mathbf{u}_\epsilon^\varphi(\mathfrak{p})^* / \mathcal{I} \twoheadrightarrow H$. But by (8) we have that

$$\dim H = |\Sigma^\varphi| \ell^{|I_+| + |I_-|} = \frac{\ell^n}{|N^\varphi|} \ell^{|I_+| + |I_-|} = \dim \mathbf{u}_\epsilon^\varphi(\mathfrak{p})^* / \mathcal{I},$$

which implies that the epimorphism is indeed an isomorphism. \square

Example 3.19. Let φ be the twisting map defined in Example 3.16 over $\mathfrak{g} = \mathfrak{sp}_6$. If we take $I_+ = \{\alpha_2\}$, $I_- = \{\alpha_1\}$ and $\Sigma^\varphi = \langle K_{(1-\varphi)(\alpha_2)}, K_{(1+\varphi)(\alpha_1)}, K_{\tau_3}, K_{\tau_2} \rangle$, then $\Sigma^\varphi = \mathbb{T}^\varphi \simeq (\mathbb{Z}/11\mathbb{Z})^3$ and N^φ is trivial. On the other hand, if we set $\varphi = 0$, then $\Sigma = \langle K_{\alpha_1}, K_{\alpha_2} \rangle$ and N is not trivial. This implies that the quotient $\mathbf{u}_\epsilon^\varphi(\mathfrak{p})^* / \langle D^z - 1 : D^z \in N^\varphi \rangle$ cannot be a 2-cocycle deformation of $\mathbf{u}_\epsilon(\mathfrak{p})^* / \langle D^z - 1 : D^z \in N \rangle$, since they have different dimension.

4. QUANTUM SUBGROUPS

In this section we determine all quantum subgroups of the twisted quantum group G_ϵ^φ . We first construct a family of quantum subgroups using the root datum

associated to $\mathfrak{g} = \text{Lie}(G)$ and an algebraic subgroup Γ of G . Then we prove that any quantum subgroup of G_ϵ^φ is isomorphic to one constructed in this way. We end the section with a parametrization of the isomorphism classes.

From now on, we work with the complex form of all quantum algebras introduced above.

4.1. Twisted quantum regular subgroups. Let $I_\pm \subseteq \pm\Pi$. Let $\Gamma_\epsilon^\varphi(\mathfrak{p})$ be the Hopf algebra associated to the pair (I_+, I_-) as in Definition 3.7, and let \mathfrak{p} be the regular Lie subalgebra of \mathfrak{g} given by Remark 3.12. In this subsection we construct the twisted quantum function algebras related to the pair (I_+, I_-) .

Denote by $\text{Res} : \Gamma_\epsilon^\varphi(\mathfrak{g})^\circ \rightarrow \Gamma_\epsilon^\varphi(\mathfrak{p})^\circ$ the Hopf algebra map induced by the inclusion $\Gamma_\epsilon^\varphi(\mathfrak{p}) \hookrightarrow \Gamma_\epsilon^\varphi(\mathfrak{g})$. Using Lemma 2.16, we know that $\mathcal{O}_\epsilon^\varphi(G) \subseteq \Gamma_\epsilon^\varphi(\mathfrak{g})^\circ$.

Definition 4.1. We define the twisted quantum function algebra associated to the regular Lie subalgebra \mathfrak{p} of \mathfrak{g} as the Hopf algebra given by

$$\mathcal{O}_\epsilon^\varphi(P) := \text{Res}(\mathcal{O}_\epsilon^\varphi(G)).$$

If $\varphi = 0$, we have that $\mathcal{O}_\epsilon^0(P) = \mathcal{O}_\epsilon(P)$; see [AG, §2.3.1]. Since $\mathcal{O}(G)$ is a central Hopf subalgebra of $\mathcal{O}_\epsilon^\varphi(G)$, $\text{Res}(\mathcal{O}(G))$ is a central Hopf subalgebra of $\mathcal{O}_\epsilon^\varphi(P)$. Thus, there exists P an algebraic subgroup of G such that $\text{Res}(\mathcal{O}(G)) = \mathcal{O}(P)$. Since $\mathcal{O}(P)$ is a central Hopf subalgebra of $\mathcal{O}_\epsilon^\varphi(P)$, the quotient

$$\overline{\mathcal{O}_\epsilon^\varphi(P)} := \mathcal{O}_\epsilon^\varphi(P) / [\mathcal{O}(P)^+ \mathcal{O}_\epsilon^\varphi(P)]$$

is a Hopf algebra, which is in fact isomorphic to $\mathbf{u}_\epsilon^\varphi(\mathfrak{p})^*$.

Proposition 4.2.

- (i) P is a connected algebraic group and $\text{Lie}(P) = \mathfrak{p}$.
- (ii) The following sequence of Hopf algebras is exact:

$$1 \longrightarrow \mathcal{O}(P) \longrightarrow \mathcal{O}_\epsilon^\varphi(P) \longrightarrow \overline{\mathcal{O}_\epsilon^\varphi(P)} \longrightarrow 1.$$

- (iii) There exists a Hopf algebra epimorphism $\overline{\text{Res}} : \mathbf{u}_\epsilon^\varphi(\mathfrak{g})^* \rightarrow \overline{\mathcal{O}_\epsilon^\varphi(P)}$ making the following diagram commutative:

$$(11) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \mathcal{O}(G) & \xrightarrow{\iota} & \mathcal{O}_\epsilon^\varphi(G) & \xrightarrow{\pi} & \mathbf{u}_\epsilon^\varphi(\mathfrak{g})^* \longrightarrow 1 \\ & & \downarrow \text{res} & & \downarrow \text{Res} & & \downarrow \overline{\text{Res}} \\ 1 & \longrightarrow & \mathcal{O}(P) & \xrightarrow{\iota_P} & \mathcal{O}_\epsilon^\varphi(P) & \xrightarrow{\pi_P} & \overline{\mathcal{O}_\epsilon^\varphi(P)} \longrightarrow 1 \end{array}$$

- (iv) $\mathcal{O}_\epsilon^\varphi(P)$ and $\overline{\mathcal{O}_\epsilon^\varphi(P)}$ are 2-cocycle deformations of $\mathcal{O}_\epsilon(P)$ and $\overline{\mathcal{O}_\epsilon(P)}$, respectively.
- (v) $\overline{\mathcal{O}_\epsilon^\varphi(P)} \simeq \mathbf{u}_\epsilon^\varphi(\mathfrak{p})^*$ as Hopf algebras.

Proof. (i), (ii), (iii) follow *mutatis mutandis* from [AG, Propositions 2.7 & 2.8].

(iv) By Lemma 2.14, we know that $\mathcal{O}_\epsilon^\varphi(G)$ is a 2-cocycle deformation of $\mathcal{O}_\epsilon(G)$, say by the cocycle $\bar{\sigma}$. Since the kernel \mathcal{I} of the Hopf algebra map $\text{Res} : \mathcal{O}_\epsilon(G) \rightarrow \mathcal{O}_\epsilon(P)$ is spanned by matrix coefficients that vanish when restricted to $\Gamma_\epsilon(\mathfrak{p})$, using the definition of $\bar{\sigma}$ we see that $\bar{\sigma}|_{\mathcal{I} \otimes \mathcal{O}_\epsilon(G) + \mathcal{O}_\epsilon(G) \otimes \mathcal{I}} = 0$. Thus by Remark 1.1, Res induces a 2-cocycle $\hat{\sigma}$ on $\mathcal{O}_\epsilon(G)/\mathcal{I}$ and we have that $\mathcal{O}_\epsilon^\varphi(P) = \text{Res}((\mathcal{O}_\epsilon(G))_{\bar{\sigma}}) = (\mathcal{O}_\epsilon(G)/\mathcal{I})_{\hat{\sigma}} = (\mathcal{O}_\epsilon(P))_{\hat{\sigma}}$. The same argument applies for $\overline{\mathcal{O}_\epsilon^\varphi(P)}$ and $\overline{\mathcal{O}_\epsilon(P)}$, since $\mathcal{O}(P)$ is a central Hopf subalgebra of $\mathcal{O}_\epsilon(P)$ and the cocycle $\hat{\sigma}$ is trivial on it.

(v) Dualizing diagram (7) we get

$$\begin{array}{ccccccc} 1 & \longrightarrow & U(\mathfrak{g})^\circ & \xrightarrow{\quad {}^t\text{Fr} \quad} & \Gamma_\epsilon^\varphi(\mathfrak{g})^\circ & \xrightarrow{\quad \alpha \quad} & \mathfrak{u}_\epsilon^\varphi(\mathfrak{g})^* \longrightarrow 1 \\ & & \downarrow & & \downarrow \text{Res} & & \downarrow \\ 1 & \longrightarrow & U(\mathfrak{p})^\circ & \xrightarrow{\quad {}^t\text{Fr}_{\text{res}} \quad} & \Gamma_\epsilon^\varphi(\mathfrak{p})^\circ & \xrightarrow{\quad \beta \quad} & \mathfrak{u}_\epsilon^\varphi(\mathfrak{p})^* \longrightarrow 1 \end{array}$$

Since \mathfrak{g} is simple, we have that $\mathcal{O}(G) \simeq U(\mathfrak{g})^\circ$. Thus, as $\mathcal{O}(P) = \text{Res}(\mathcal{O}(G))$ and $\mathcal{O}_\epsilon^\varphi(P) = \text{Res}(\mathcal{O}_\epsilon^\varphi(G))$, we have that ${}^t\text{Fr}_{\text{res}}(\mathcal{O}(P)) \subseteq U(\mathfrak{p})^\circ$, and consequently $\mathcal{O}(P)^+ \subseteq \text{Ker } \beta$. Moreover, since $\alpha(\mathcal{O}_\epsilon^\varphi(G)) = \pi(\mathcal{O}_\epsilon^\varphi(G)) = \mathfrak{u}_\epsilon^\varphi(\mathfrak{g})^*$, we have that $\mathfrak{u}_\epsilon^\varphi(\mathfrak{p})^* = \beta \text{Res}(\mathcal{O}_\epsilon^\varphi(G)) = \beta(\mathcal{O}_\epsilon^\varphi(P))$. Hence, there exists a surjective Hopf algebra map $\gamma : \overline{\mathcal{O}_\epsilon^\varphi(P)} \rightarrow \mathfrak{u}_\epsilon^\varphi(\mathfrak{p})^*$. But by (iv), [AG, Proposition 2.8 (c)] and Proposition 3.13, we have that $\dim \overline{\mathcal{O}_\epsilon^\varphi(P)} = \dim \overline{\mathcal{O}_\epsilon(P)} = \dim \mathfrak{u}_\epsilon(\mathfrak{p}) = \dim \mathfrak{u}_\epsilon^\varphi(\mathfrak{p})$ and the epimorphism is in fact an isomorphism. \square

Remark 4.3. By the proposition above, we know that $\mathcal{O}_\epsilon^\varphi(P)$ fits into the central exact sequence of Hopf algebras $\mathcal{O}(P) \xrightarrow{\iota_P} \mathcal{O}_\epsilon^\varphi(P) \xrightarrow{\pi_P} \mathfrak{u}_\epsilon^\varphi(\mathfrak{p})^*$ and that $\mathcal{O}_\epsilon^\varphi(P)$ is a 2-cocycle deformation of $\mathcal{O}_\epsilon(P)$, where the 2-cocycle $\hat{\sigma}$ is given by the formula $\hat{\sigma}(\text{Res}(x), \text{Res}(y)) = \bar{\sigma}(x, y)$ for all $x, y \in \mathcal{O}_\epsilon(G)$. On the other hand, by Propositions 3.5 and 3.13 we know that $\mathfrak{u}_\epsilon^\varphi(\mathfrak{p})^* = (\mathfrak{u}_\epsilon(\mathfrak{p})^*)_\tau$ for the 2-cocycle τ given by $\tau(\overline{\text{Res}(\pi(x))}, \overline{\text{Res}(\pi(y))}) = \bar{\sigma}(x, y)$. Since diagram (11) for $\varphi = 0$ is commutative, the pullback of the cocycle τ coincides with the cocycle $\hat{\sigma}$.

4.2. Quantum subgroups from classical subgroups. In this subsection we construct a Hopf algebra quotient of $\mathcal{O}_\epsilon^\varphi(G)$ associated to the pair (I_+, I_-) and an algebraic subgroup of G included in P . This is based in the *pushout construction*, which is a general method for constructing Hopf algebras from central exact sequences.

The following proposition follows from the arguments in [AG, §2.2]. If $\gamma : \Gamma \rightarrow G$ is a homomorphism of algebraic groups, then ${}^t\gamma : \mathcal{O}(G) \rightarrow \mathcal{O}(\Gamma)$ denotes the corresponding algebra map between the coordinate algebras.

Proposition 4.4. *Let Γ be an algebraic group and $\gamma : \Gamma \rightarrow G$ an injective homomorphism of algebraic groups such that $\gamma(\Gamma) \subseteq P$. Let \mathcal{J} denote the two-sided ideal of $\mathcal{O}_\epsilon^\varphi(P)$ generated by $\iota(\text{Ker } {}^t\gamma)$. Then $\mathcal{A}_{\epsilon, \mathfrak{p}, \gamma}^\varphi = \mathcal{O}_\epsilon^\varphi(P)/\mathcal{J}$ is a Hopf algebra and*

there exist a Hopf algebra monomorphism $j : \mathcal{O}(\Gamma) \hookrightarrow A_{\epsilon, \mathbf{p}, \gamma}^{\varphi}$ and a Hopf algebra epimorphism $\bar{\pi} : A_{\epsilon, \mathbf{p}, \gamma}^{\varphi} \rightarrow \mathbf{u}_{\epsilon}^{\varphi}(\mathbf{p})^*$ such that $A_{\epsilon, \mathbf{p}, \gamma}^{\varphi}$ fits into the exact sequence of Hopf algebras

$$1 \longrightarrow \mathcal{O}(\Gamma) \xrightarrow{j} A_{\epsilon, \mathbf{p}, \gamma}^{\varphi} \xrightarrow{\bar{\pi}} \mathbf{u}_{\epsilon}^{\varphi}(\mathbf{p})^* \longrightarrow 1.$$

If in addition $|\Gamma|$ is finite, then $\dim A_{\epsilon, \mathbf{p}, \gamma}^{\varphi} = |\Gamma| \dim \mathbf{u}_{\epsilon}^{\varphi}(\mathbf{p})$. Moreover, the following diagram is commutative:

$$(12) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \mathcal{O}(G) & \xrightarrow{\iota} & \mathcal{O}_{\epsilon}^{\varphi}(G) & \xrightarrow{\pi} & \mathbf{u}_{\epsilon}^{\varphi}(\mathbf{g})^* \longrightarrow 1 \\ & & \downarrow \text{res} & & \downarrow \text{Res} & & \downarrow \overline{\text{Res}} \\ 1 & \longrightarrow & \mathcal{O}(P) & \xrightarrow{\iota_P} & \mathcal{O}_{\epsilon}^{\varphi}(P) & \xrightarrow{\pi_P} & \mathbf{u}_{\epsilon}^{\varphi}(\mathbf{p})^* \longrightarrow 1 \\ & & \downarrow {}^t\gamma & & \downarrow \psi & & \downarrow \text{id} \\ 1 & \longrightarrow & \mathcal{O}(\Gamma) & \xrightarrow{j} & A_{\epsilon, \mathbf{p}, \gamma}^{\varphi} & \xrightarrow{\bar{\pi}} & \mathbf{u}_{\epsilon}^{\varphi}(\mathbf{p})^* \longrightarrow 1 \end{array}$$

□

Proposition 4.5. $A_{\epsilon, \mathbf{p}, \gamma}^{\varphi}$ is a 2-cocycle deformation of $A_{\epsilon, \mathbf{p}, \gamma}$.

Proof. By Proposition 4.2(iv), we know that $\mathcal{O}_{\epsilon}^{\varphi}(P)$ is a 2-cocycle deformation of $\mathcal{O}_{\epsilon}(P)$, say by the cocycle $\hat{\sigma}$; see Remark 4.3 above. Then, by Remark 1.1 it is enough to check that $\hat{\sigma}|_{\mathcal{O}_{\epsilon}^{\varphi}(P) \otimes \mathcal{J} + \mathcal{J} \otimes \mathcal{O}_{\epsilon}^{\varphi}(P)} = 0$. Since $\mathcal{J} = \mathcal{O}_{\epsilon}^{\varphi}(P) \iota_P(\text{Ker } {}^t\gamma)$ and $\text{Ker } {}^t\gamma$ is generated by matrix coefficients $c_{f,v}$ in $\mathcal{O}(P)$, of degree $(\ell\lambda, \ell\mu)$ for some $\lambda, \mu \in P$, we have that $\hat{\sigma}|_{\iota_P(\text{Ker } {}^t\gamma) \otimes \iota_P(\text{Ker } {}^t\gamma)} = \varepsilon \otimes \varepsilon = 0$ and whence $\hat{\sigma}|_{\mathcal{O}_{\epsilon}^{\varphi}(P) \otimes \mathcal{J} + \mathcal{J} \otimes \mathcal{O}_{\epsilon}^{\varphi}(P)} = 0$. Thus, we may define a 2-cocycle $\tilde{\sigma} : A_{\epsilon, \mathbf{p}, \gamma} \otimes A_{\epsilon, \mathbf{p}, \gamma} \rightarrow \mathbb{C}$ by $\tilde{\sigma}(\psi(x), \psi(y)) = \hat{\sigma}(x, y)$ for all $x, y \in \mathcal{O}_{\epsilon}(P)$ and $(A_{\epsilon, \mathbf{p}, \gamma})_{\tilde{\sigma}} = A_{\epsilon, \mathbf{p}, \gamma}^{\varphi}$. Note that $\tilde{\sigma}$ coincides with the pullback through $\bar{\pi}$ of the 2-cocycle τ on $\mathbf{u}_{\epsilon}(\mathbf{p})^*$. □

By Proposition 3.11(ii), we know that there exists an injective coalgebra map ${}^t\theta : \mathbf{u}_{\epsilon}^{\varphi}(\mathbf{p})^* \rightarrow \Gamma_{\epsilon}^{\varphi}(\mathbf{p})^{\circ}$, and since $\mathbf{u}_{\epsilon}^{\varphi}(\mathbf{p})^* \simeq \widehat{\mathcal{O}_{\epsilon}^{\varphi}(P)}$ by Proposition 4.2, we have that $\text{Im } {}^t\theta \subseteq \mathcal{O}_{\epsilon}^{\varphi}(P)$. Thus, the image of the central subgroup $\widehat{\mathbb{T}_{I_c}^{\varphi}}$ of $G(\mathbf{u}_{\epsilon}^{\varphi}(\mathbf{p})^*)$ is a subgroup of $G(\mathcal{O}_{\epsilon}^{\varphi}(P))$. Denote $d^z = {}^t\theta(D^z)$ for $z \in (\mathbb{Z}/\ell\mathbb{Z})^m$, $D^z \in \widehat{\mathbb{T}_{I_c}^{\varphi}}$.

Lemma 4.6. *There exists a subgroup $\mathcal{A} = \{\partial^z = \psi({}^t\theta(D^z)) : D^z \in \widehat{\mathbb{T}_{I_c}^{\varphi}}\}$ of $G(A_{\epsilon, \mathbf{p}, \gamma}^{\varphi})$ isomorphic to $\widehat{\mathbb{T}_{I_c}^{\varphi}}$ consisting of central elements. In particular, $|\mathcal{A}| = \ell^{n-\text{rk } S_{\ell}^{\varphi}}$.*

Proof. Using the same argument as in the proof of Proposition 3.17, one sees that the elements d^z are central in $\mathcal{O}_{\epsilon}^{\varphi}(P)$. Indeed, if $f \in \mathcal{O}_{\epsilon}^{\varphi}(P)$, then $d^z f(M) = f d^z(M)$ for every generator M of $\Gamma_{\epsilon}^{\varphi}(\mathbf{p})$ from Definition 3.7. For example, let $i \in I_+$ and $m \geq 0$. Then by (4) we have

$$\begin{aligned} d^z f(E_i^{(m)}) &= \sum_{r+s=m} q_i^{-rs} d^z(E_i^{(r)} K_{s(\alpha_i - \tau_i)}) f(E_i^{(s)} K_{r\tau_i}) \\ &= \sum_{r+s=m} q_i^{-rs} d^z(E_i^{(r)}) d^z(K_{s(\alpha_i - \tau_i)}) f(E_i^{(s)} K_{r\tau_i}) \\ &= d^z(K_{m(\alpha_i - \tau_i)}) f(E_i^{(m)}) \text{ and} \end{aligned}$$

$$\begin{aligned}
fd^z \left(E_i^{(m)} \right) &= \sum_{r+s=m} q_i^{-rs} f \left(E_i^{(r)} K_{s(\alpha_i - \tau_i)} \right) d^z \left(E_i^{(s)} K_{r\tau_i} \right) \\
&= \sum_{r+s=m} q_i^{-rs} f \left(E_i^{(r)} K_{s(\alpha_i - \tau_i)} \right) d^z \left(E_i^{(s)} \right) d^z (K_{r\tau_i}) \\
&= f \left(E_i^{(m)} \right) d^z (K_{m\tau_i}).
\end{aligned}$$

Since $d^z(\overline{K_i}) = d^z(K_{\alpha_i - 2\tau_i}) = D^z(\theta(K_{\alpha_i - 2\tau_i})) = 1$, we have that $d^z(K_{m(\alpha_i - \tau_i)}) = d^z(K_{m\tau_i})$ for all $m \geq 0$, and then $d^z f \left(E_i^{(m)} \right) = fd^z \left(E_i^{(m)} \right)$. Analogously, using that $1 = d^z(\widetilde{K_j})$ for all $j \in I_-$, we have that $d^z f \left(F_j^{(m)} \right) = fd^z \left(F_j^{(m)} \right)$ for all $m \geq 0$. The equality on the generators $K_{\alpha_i}^{-1}$ and $(K_m^{\alpha_i; 0})$ follows easily since the coproduct is cocommutative on them. Applying an inductive argument on monomials on the generators we have that d^z is central in $\mathcal{O}_\epsilon^\varphi(P)$. Since $\psi : \mathcal{O}_\epsilon^\varphi(P) \rightarrow A_{\epsilon, \mathbf{p}, \gamma}^\varphi$ is surjective, the group-like elements ∂^z are also central in $A_{\epsilon, \mathbf{p}, \gamma}^\varphi$.

Now we show that $\mathcal{A} \simeq \widehat{\mathbb{T}_{I_c}^\varphi}$ as groups. By construction, we have that $\psi \circ {}^t\theta : \widehat{\mathbb{T}_{I_c}^\varphi} \rightarrow \mathcal{A}$ is a group epimorphism. As the diagram

$$\begin{array}{ccc}
\mathcal{O}_\epsilon^\varphi(P) & \xrightarrow{\pi_P} & \mathbf{u}_\epsilon^\varphi(\mathbf{p})^* \\
\psi \downarrow & \nearrow \overline{\pi} & \\
A_{\epsilon, \mathbf{p}, \sigma}^\varphi & &
\end{array}$$

is commutative by (12), we have that $\overline{\pi}(\mathcal{A}) = \overline{\pi}(\psi({}^t\theta(\widehat{\mathbb{T}_{I_c}^\varphi}))) = \pi_P({}^t\theta(\widehat{\mathbb{T}_{I_c}^\varphi})) = \widehat{\mathbb{T}_{I_c}^\varphi}$, which implies that $\psi \circ {}^t\theta$ is indeed an isomorphism. \square

4.3. Quantum subgroups from subalgebras of twisted Frobenius–Lusztig kernels. In this subsection we construct Hopf algebras from a Hopf subalgebra of $\mathbf{u}_\epsilon^\varphi(\mathfrak{g})$ and an algebraic subgroup of G .

Let L be a Hopf subalgebra of $\mathbf{u}_\epsilon^\varphi(\mathfrak{g})$. By Lemma 3.6, it is determined by a triple $(I_+, I_-, \Sigma^\varphi)$ where Σ^φ is a subgroup of $G(\mathbf{u}_\epsilon^\varphi(\mathfrak{g}))$ and $I_\pm \subset \pm\Pi$ are such that $K_{(1 \mp \varphi)(\alpha_i)} \in \Sigma^\varphi$ if $\alpha_i \in I_\pm$. If $H = L^*$, then by Proposition 3.18, $H = \mathbf{u}_\epsilon^\varphi(\mathbf{p})^* / \langle D^z - 1 : D^z \in N^\varphi \rangle$, where N^φ is determined by Σ^φ as in Remark 3.15(d). Let P be the regular subgroup of G determined by the pair (I_+, I_-) with $\mathbf{p} = \text{Lie}(P)$ and let $\mathcal{O}_\epsilon^\varphi(P)$, $\mathbf{u}_\epsilon^\varphi(\mathbf{p})$ be the corresponding twisted quantum algebras.

Proposition 4.7. *Let Γ be an algebraic group and let $\gamma : \Gamma \rightarrow G$ be an injective morphism of algebraic groups such that $\gamma(\Gamma) \subseteq P$. For every group homomorphism $\delta : N^\varphi \rightarrow \widehat{\Gamma}$, the two-sided ideal J_δ of $A_{\epsilon, \mathbf{p}, \gamma}^\varphi$ generated by the elements $\delta(D^z) - \partial^z$ for $\partial^z \in \mathcal{A}$ and D^z in N^φ is a Hopf ideal and the Hopf algebra $A_{\epsilon, \mathbf{p}, \gamma}^\varphi / J_\delta$ fits into the central exact sequence*

$$1 \longrightarrow \mathcal{O}(\Gamma) \xrightarrow{\hat{\iota}} A_{\epsilon, \mathbf{p}, \gamma}^\varphi / J_\delta \xrightarrow{\hat{\pi}} H \longrightarrow 1.$$

If in addition $|\Gamma|$ is finite, then $\dim A_{\epsilon, \mathfrak{p}, \gamma}^\varphi / J_\delta = |\Gamma| \dim H$. Moreover, the following diagram is commutative:

$$(13) \quad \begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{O}(G) & \xrightarrow{\iota} & \mathcal{O}_\epsilon^\varphi(G) & \xrightarrow{\pi} & \mathbf{u}_\epsilon^\varphi(\mathfrak{g})^* & \longrightarrow & 1 \\ & & \text{res} \downarrow & & \downarrow \text{Res} & & \downarrow \overline{\text{Res}} & & \\ 1 & \longrightarrow & \mathcal{O}(P) & \xrightarrow{\iota_P} & \mathcal{O}_\epsilon^\varphi(P) & \xrightarrow{\pi_P} & \mathbf{u}_\epsilon^\varphi(\mathfrak{p})^* & \longrightarrow & 1 \\ & & \downarrow \iota_\gamma & & \downarrow \psi & & \downarrow \text{id} & & \\ 1 & \longrightarrow & \mathcal{O}(\Gamma) & \xrightarrow{j} & A_{\epsilon, \mathfrak{p}, \gamma}^\varphi & \xrightarrow{\bar{\pi}} & \mathbf{u}_\epsilon^\varphi(\mathfrak{p})^* & \longrightarrow & 1 \\ & & \downarrow \text{id} & & \downarrow & & \downarrow \nu & & \\ 1 & \longrightarrow & \mathcal{O}(\Gamma) & \xrightarrow{\hat{\iota}} & A_{\epsilon, \mathfrak{p}, \gamma}^\varphi / J_\delta & \xrightarrow{\hat{\pi}} & H & \longrightarrow & 1 \end{array}$$

Proof. The proof follows by the proof of [AG, Theorem 2.17]. We reproduce the first part here to give an idea. By Lemma 4.6, we know that the group-like elements $\partial^z \in \mathcal{A}$ are central in $A_{\epsilon, \mathfrak{p}, \gamma}^\varphi$. Since $\delta(D^z) \in \mathcal{O}(\Gamma)$ for all $D^z \in N^\varphi$, the ideal J_δ in $A_{\epsilon, \mathfrak{p}, \gamma}^\varphi$ generated by the elements $\delta(D^z) - \partial^z$ is a Hopf ideal, and whence $A_{\epsilon, \mathfrak{p}, \gamma}^\varphi / J_\delta$ is a Hopf algebra. If we write $\mathcal{J}_\delta = J_\delta \cap \mathcal{O}(\Gamma)$, then $A_{\epsilon, \mathfrak{p}, \gamma}^\varphi / J_\delta$ fits into the central exact sequence

$$1 \longrightarrow \mathcal{O}(\Gamma) / \mathcal{J}_\delta \longrightarrow A_{\epsilon, \mathfrak{p}, \gamma}^\varphi / J_\delta \longrightarrow \mathbf{u}_\epsilon^\varphi(\mathfrak{p})^* / \bar{\pi}(J_\delta) \longrightarrow 1.$$

Since $\bar{\pi}(\delta(D^z)) = 1$ and $\bar{\pi}(\partial^z) = D^z$ by the proof of Lemma 4.6, it follows that $\bar{\pi}(J_\delta) = \langle D^z - 1 : D^z \in N^\varphi \rangle$. Thus, by Proposition 3.18(ii), we have that $\mathbf{u}_\epsilon^\varphi(\mathfrak{p})^* / \bar{\pi}(J_\delta) = H$. The proof that $\mathcal{J}_\delta = 0$ and that $A_{\epsilon, \mathfrak{p}, \gamma}^\varphi / J_\delta$ fits into the commutative diagram follow the same arguments used in loc. cit. \square

4.4. Parametrization of quantum subgroups. In this subsection we parametrize the Hopf algebra quotients of $\mathcal{O}_\epsilon^\varphi(G)$ by a 6-tuple called *twisted subgroup datum*. We show first that there is a 1-1 correspondance between Hopf algebra quotients of $\mathcal{O}_\epsilon^\varphi(G)$ and twisted subgroup data, and then we classify these quotients up to isomorphism.

Definition 4.8. A *twisted subgroup datum* is a collection $\mathcal{D}^\varphi := (I_+, I_-, N^\varphi, \Gamma, \gamma, \delta)$ where

- ▷ $I_\pm \subset \pm \prod$. Let $\Psi_\pm = \{\alpha \in \Phi : \text{Supp } \alpha \in I_\pm\}$, $\mathfrak{p} = \sum_{\alpha \in \Psi_\pm} \mathfrak{g}_\alpha$ and $\mathfrak{p} = \mathfrak{p}_+ \oplus \mathfrak{h} \oplus \mathfrak{p}_-$. Let P be the connected Lie subgroup of G with $\text{Lie}(P) = \mathfrak{p}$.
- ▷ N^φ a subgroup of $\widehat{\mathbb{T}_{f_c}^\varphi}$; see Remark 3.15(d).
- ▷ Γ is an algebraic group.
- ▷ $\gamma : \Gamma \rightarrow P$ is a injective algebraic group homomorphism.
- ▷ $\delta : N^\varphi \rightarrow \widehat{\Gamma}$ is a group homomorphism.

If Γ is finite, we call \mathcal{D}^φ a *finite twisted subgroup datum*.

Summarizing the previous results we obtain the first main result of the paper.

Theorem 4.9. *Let $\mathcal{D}^\varphi = (I_+, I_-, N^\varphi, \Gamma, \gamma, \delta)$ be a twisted subgroup datum. Then there exists a Hopf algebra $A_{\mathcal{D}^\varphi} = A_{\epsilon, \mathfrak{p}, \gamma} / J_\delta$ of $\mathcal{O}_\epsilon^\varphi(G)$ that fits into the central exact sequence*

$$1 \longrightarrow \mathcal{O}(\Gamma) \xrightarrow{\hat{i}} A_{\mathcal{D}^\varphi} \xrightarrow{\hat{\pi}} H \longrightarrow 1.$$

In particular, if $|\Gamma|$ is finite, then $\dim A_{\mathcal{D}^\varphi} = |\Gamma| \dim H$.

Proof. By Lemma 3.6 and Remark 3.15(d), the triple (I_+, I_-, N^φ) determines a quotient H of $\mathbf{u}_\epsilon^\varphi(\mathfrak{g})^*$. Besides, by Proposition 4.2, the pair (I_+, I_-) determines a regular subgroup P of G , a regular Lie subalgebra \mathfrak{p} of \mathfrak{g} , and the quantum algebras $\mathcal{O}_\epsilon^\varphi(P)$ and $\mathbf{u}_\epsilon^\varphi(\mathfrak{p})$, which makes the upper part of diagram (13) commutative. Then by Proposition 4.4, the morphism $\gamma : \Gamma \rightarrow P \subset G$ gives rise to the Hopf algebra $A_{\epsilon, \mathfrak{p}, \gamma}^\varphi$ through the pushout construction. Finally, by Proposition 4.7 the group homomorphism $\delta : N^\varphi \rightarrow \hat{\Gamma}$ defines the Hopf ideal J_δ of $A_{\epsilon, \mathfrak{p}, \gamma}^\varphi$ and the Hopf algebra $A_{\mathcal{D}^\varphi} = A_{\epsilon, \mathfrak{p}, \sigma}^\varphi / J_\delta$ fits into the commutative diagram (13). \square

The next theorem establishes the converse of Theorem 4.9. We give its proof in several lemmata.

Theorem 4.10. *Let $\kappa : \mathcal{O}_\epsilon^\varphi(G) \rightarrow A$ be a surjective Hopf algebra morphism. Then there exists a twisted subgroup datum \mathcal{D}^φ such that $A \simeq A_{\mathcal{D}^\varphi}$ as Hopf algebras.* \square

Lemma 4.11. *There exists an algebraic group Γ and an injective homomorphism of algebraic groups $\gamma : \Gamma \rightarrow G$ such that $\mathcal{O}(\Gamma)$ is a Hopf subalgebra of A and A fits into the central exact sequence of Hopf algebras $1 \longrightarrow \mathcal{O}(\Gamma) \xrightarrow{\hat{i}} A \xrightarrow{\hat{\pi}} H \longrightarrow 1$, where $H = A / \mathcal{O}(\Gamma)^+ A$. Moreover, the following diagram is commutative:*

$$(14) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \mathcal{O}(G) & \xrightarrow{\iota} & \mathcal{O}_\epsilon^\varphi(G) & \xrightarrow{\pi} & \mathbf{u}_\epsilon^\varphi(\mathfrak{g})^* \longrightarrow 1 \\ & & \downarrow t_\gamma & & \downarrow \kappa & & \downarrow \\ 1 & \longrightarrow & \mathcal{O}(\Gamma) & \xrightarrow{\hat{i}} & A & \xrightarrow{\hat{\pi}} & H \longrightarrow 1 \end{array}$$

Proof. Let $K = \kappa(\iota(\mathcal{O}(G)))$. Since $\mathcal{O}(G)$ is central in $\mathcal{O}_\epsilon^\varphi(G)$, K is central in A and there exists an algebraic group Γ and an algebraic group homomorphism $\gamma : \Gamma \rightarrow G$ such that $K = \mathcal{O}(\Gamma)$ and $t_\gamma : \mathcal{O}(G) \rightarrow \mathcal{O}(\Gamma)$ is the Hopf algebra epimorphism $\kappa \circ \iota|_{\mathcal{O}(G)}$. Moreover, if we set $H = A / K^+ A$, then the sequence

$$1 \longrightarrow \mathcal{O}(\Gamma) \xrightarrow{\hat{i}} A \xrightarrow{\hat{\pi}} H \longrightarrow 1$$

is exact and diagram (14) is commutative. \square

By the lemma above, H^* is a Hopf subalgebra of $\mathbf{u}_\epsilon^\varphi(\mathfrak{g})$. Thus, by Lemma 3.6 it is determined by a triple $(I_+, I_-, \Sigma^\varphi)$. Let P be the subgroup of G determined by the pair (I_+, I_-) , let $\mathfrak{p} = \text{Lie}(P)$, and let $\mathcal{O}_\epsilon^\varphi(P)$, $\mathbf{u}_\epsilon^\varphi(\mathfrak{p})$ be the quantum algebras given by Proposition 4.2. In particular, we have that $H^* \subseteq \mathbf{u}_\epsilon^\varphi(\mathfrak{p}) \subseteq \mathbf{u}_\epsilon^\varphi(\mathfrak{g})$, and by Proposition 3.18, $H \simeq \mathbf{u}_\epsilon^\varphi(\mathfrak{p})^* / \langle D^z - 1 : D^z \in N^\varphi \rangle$, where N^φ is determined by Σ^φ as in Remark 3.15(d). Denote by $\nu : \mathbf{u}_\epsilon^\varphi(\mathfrak{p}) \rightarrow H$ the corresponding epimorphism.

The next lemma follows from [AG, Lemma 3.1] but is adapted to the twisted case.

Lemma 4.12. *Diagram (14) factorizes through the central exact sequence*

$$1 \longrightarrow \mathcal{O}(P) \xrightarrow{\iota_P} \mathcal{O}_\epsilon^\varphi(P) \xrightarrow{\pi_P} \mathbf{u}_\epsilon^\varphi(\mathfrak{p})^* \longrightarrow 1.$$

Proof. We want to show that A fits into the commutative diagram

$$(15) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \mathcal{O}(G) & \xrightarrow{\iota} & \mathcal{O}_\epsilon^\varphi(G) & \xrightarrow{\pi} & \mathbf{u}_\epsilon^\varphi(\mathfrak{g})^* \longrightarrow 1 \\ & & \downarrow \text{res} & & \downarrow \text{Res} & & \downarrow \overline{\text{Res}} \\ 1 & \longrightarrow & \mathcal{O}(P) & \xrightarrow{\iota_P} & \mathcal{O}_\epsilon^\varphi(P) & \xrightarrow{\pi_P} & \mathbf{u}_\epsilon^\varphi(\mathfrak{p})^* \longrightarrow 1 \\ & & \downarrow {}^t\zeta & & \downarrow \psi & & \downarrow \nu \\ 1 & \longrightarrow & \mathcal{O}(\Gamma) & \xrightarrow{\hat{\iota}} & A & \xrightarrow{\hat{\pi}} & H \longrightarrow 1 \end{array}$$

To prove this, it suffices to show that $\text{Ker Res} \subseteq \text{Ker } \kappa$. In order to do so, we realize $\mathcal{O}_\epsilon^\varphi(G)$ as a subalgebra of $\mathbb{A}_\epsilon^\varphi = \mathbb{A}_\epsilon'' \otimes_R \mathbb{Q}(\epsilon)$; see [CV2, §3.6] and Lemma 2.11.

Let $\mu_\epsilon^\varphi : \mathcal{O}_\epsilon^\varphi(G) \rightarrow \check{U}_\epsilon^\varphi(\mathfrak{b}_-)^{\text{cop}} \otimes \check{U}_\epsilon^\varphi(\mathfrak{b}_+)^{\text{cop}}$ be the complexification of the injective algebra map μ_ϵ'' given by (5). Then by Lemma 2.11, $\mu_\epsilon^\varphi(\mathcal{O}_\epsilon^\varphi(G)) \subseteq \mathbb{A}_\epsilon^\varphi$, which is the algebra generated by $f_\alpha^\varphi \otimes 1$, $1 \otimes e_\alpha^\varphi$ and $K_{-(1+\varphi)\lambda} \otimes K_{(1-\varphi)\lambda}$ for $\lambda \in P$ and $\alpha \in \Phi_+$.

The proof follows by showing that $\mu_\epsilon^\varphi(\text{Ker Res}) \subseteq \mu_\epsilon^\varphi(\text{Ker } \kappa)$. First note that $\mu_\epsilon^\varphi(\text{Ker Res})$ is the two-sided ideal \mathcal{I} generated by $\{1 \otimes e_k^\varphi, f_i^\varphi \otimes 1 \mid \alpha_k \notin I_-, \alpha_j \notin I_+\}$. Indeed, by [CV2, Proposition 2.7], Remark 2.12(a), and (6) we have

$$\begin{aligned} \mu_\epsilon^\varphi(\psi_{-\omega_i}^{-\alpha_i} \psi_{\omega_i}) &= \left(\left(\epsilon^{-(\tau_i, \omega_i)} f_{\alpha_i}^\varphi \right) K_{-(1+\varphi)(\omega_i)} \otimes K_{(1-\varphi)(\omega_i)} \right) (K_{(1+\varphi)(\omega_i)} \otimes K_{-(1-\varphi)(\omega_i)}) \\ &= \epsilon^{-(\tau_i, \omega_i)} f_{\alpha_i}^\varphi \otimes 1. \end{aligned}$$

Analogously, we have $\mu_\epsilon^\varphi(\psi_{\omega_i} \psi_{-\omega_i}^{\alpha_k}) = \epsilon^{-(\tau_i, \omega_i)} 1 \otimes e_\alpha^\varphi$. Since by definition $\psi_{-\omega_i}^{\alpha_k}$, $\psi_{-\omega_i}^{\alpha_k} \in \text{Ker Res}$ when $\alpha_k \notin I_-$, $\alpha_j \notin I_+$, we obtain that $1 \otimes e_k^\varphi, f_i^\varphi \otimes 1 \in \mu_\epsilon^\varphi(\text{Ker Res})$ for $\alpha_k \notin I_-$ and $\alpha_j \notin I_+$. Conversely, assume $f \in \text{Ker Res}$. Then $f|_{\Gamma_\epsilon^\varphi(\mathfrak{p})} = 0$ and $\langle \mu_\epsilon^\varphi(f), FM \otimes NE \rangle = f(FMNE) = 0$ for all elements $FMNE$ in a basis of $\Gamma_\epsilon^\varphi(\mathfrak{p})$. Using the perfect pairing (3) on ϵ , it follows that $\mu_\epsilon^\varphi(f) \subseteq \mathcal{I}$.

The proof that $\mathcal{I} \subseteq \mu_\epsilon^\varphi(\text{Ker } \kappa)$ is analogous to the proof of [AG, Lemma 3.1]. \square

Note that the map ${}^t\zeta : \mathcal{O}(P) \rightarrow \mathcal{O}(\Gamma)$ is given by the restriction $\psi|_{\mathcal{O}(P)}$. Hence, ${}^t\zeta \text{res} = {}^t\gamma$ and $\text{Im } \gamma \subseteq P$.

We end the proof of Theorem 4.10 with the following lemma. Its proof is analogous to the case $\varphi = 0$ and will be given without any detail.

Lemma 4.13 ([AG, Lemmata 3.2 & 3.3]). *There exists a group homomorphism $\delta : N^\varphi \rightarrow \hat{\Gamma}$ such that the two-sided ideal J_δ of $A_{\epsilon, \mathfrak{p}, \gamma}^\varphi$ generated by the elements $\delta(D^z) - \partial^z$ for D^z in N^φ is a Hopf ideal, $A \simeq A_{\mathcal{D}^\varphi} = A_{\epsilon, \mathfrak{p}, \gamma}^\varphi / J_\delta$ as Hopf algebras,*

and A fits into the commutative diagram

$$\begin{array}{ccccccc}
 1 & \longrightarrow & \mathcal{O}(G) & \xrightarrow{\iota} & \mathcal{O}_\epsilon^\varphi(G) & \xrightarrow{\pi} & \mathbf{u}_\epsilon^\varphi(\mathfrak{g})^* \longrightarrow 1 \\
 & & \downarrow \text{res} & & \downarrow \text{Res} & & \downarrow \overline{\text{Res}} \\
 1 & \longrightarrow & \mathcal{O}(P) & \xrightarrow{\iota_P} & \mathcal{O}_\epsilon^\varphi(P) & \xrightarrow{\pi_P} & \mathbf{u}_\epsilon^\varphi(\mathfrak{p})^* \longrightarrow 1 \\
 & & \downarrow t_\gamma & & \downarrow \psi & & \downarrow \text{id} \\
 1 & \longrightarrow & \mathcal{O}(\Gamma) & \xrightarrow{j} & A_{\epsilon, \mathfrak{p}, \gamma}^\varphi & \xrightarrow{\bar{\pi}} & \mathbf{u}_\epsilon^\varphi(\mathfrak{p})^* \longrightarrow 1 \\
 & & \downarrow \text{id} & & \downarrow & & \downarrow \nu \\
 1 & \longrightarrow & \mathcal{O}(\Gamma) & \xrightarrow{\hat{i}} & A & \xrightarrow{\hat{\pi}} & H \longrightarrow 1
 \end{array}$$

Proof. (Sketch) Using that $A_{\epsilon, \mathfrak{p}, \gamma}^\varphi$ is given by a pushout, one first shows that A fits into the commutative diagram above. Then, using the commutativity of the diagram, one proves that there exists a group homomorphism $\delta : N^\varphi \rightarrow \widehat{\Gamma}$ and a Hopf ideal J_δ such that $A \simeq A_{\epsilon, \mathfrak{p}, \gamma}^\varphi / J_\delta$. \square

4.4.1. Isomorphism classes of quantum subgroups. In this subsection we parametrize the Hopf algebra quotients of $\mathcal{O}_\epsilon^\varphi(G)$ up to isomorphism. To do so, we first define a partial order on the isomorphism classes of quotients of $\mathcal{O}_\epsilon^\varphi(G)$ and on the set of twisted subgroup data.

Let $\mathcal{Q}(\mathcal{O}_\epsilon^\varphi(G))$ be the category whose objects are surjective Hopf algebra maps $\kappa : \mathcal{O}_\epsilon^\varphi(G) \rightarrow A$. If $\kappa : \mathcal{O}_\epsilon^\varphi(G) \rightarrow A$ and $\kappa' : \mathcal{O}_\epsilon^\varphi(G) \rightarrow A'$ are such maps, then an arrow $\kappa \xrightarrow{\alpha} \kappa'$ in $\mathcal{Q}(\mathcal{O}_\epsilon^\varphi(G))$ is a Hopf algebra map $\alpha : A \rightarrow A'$ such that $\alpha\kappa = \kappa'$. In this language, a *quotient* of $\mathcal{O}_\epsilon^\varphi(G)$ is just an isomorphism class of objects in $\mathcal{Q}(\mathcal{O}_\epsilon^\varphi(G))$; let $[\kappa]$ denote the class of the map κ . There is a partial order in the set of quotients of $\mathcal{O}_\epsilon^\varphi(G)$, given by $[\kappa] \leq [\kappa']$ iff there exists an arrow $\kappa \xrightarrow{\alpha} \kappa'$ in $\mathcal{Q}(\mathcal{O}_\epsilon^\varphi(G))$. Note that $[\kappa] \leq [\kappa']$ and $[\kappa'] \leq [\kappa]$ imply $[\kappa] = [\kappa']$. Our goal is to describe the partial order in $\mathcal{Q}(\mathcal{O}_\epsilon^\varphi(G))$.

Let $I_\pm, I'_\pm \subseteq \pm\Pi$. If $I'_+ \subseteq I_+$ and $I'_- \subseteq I_-$, then $I' \subseteq I$ and $\mathbb{T}_{I'}^\varphi \subseteq \mathbb{T}_I^\varphi$. Thus, there exists an epimorphism $\mathbb{T}_{I'c}^\varphi \twoheadrightarrow \mathbb{T}_{Ic}^\varphi$ which induces a monomorphism $\eta : \widehat{\mathbb{T}_{I'c}^\varphi} \hookrightarrow \widehat{\mathbb{T}_{Ic}^\varphi}$.

Definition 4.14. Let $\mathcal{D}^\varphi = (I_+, I_-, N^\varphi, \Gamma, \gamma, \delta)$ and $\mathcal{D}^{\varphi'} = (I'_+, I'_-, N^{\varphi'}, \Gamma', \gamma', \delta')$ be twisted subgroup data with respect to $\mathcal{O}_\epsilon^\varphi(G)$. We say that $\mathcal{D}^\varphi \leq \mathcal{D}^{\varphi'}$ if and only if:

- ▷ $I'_+ \subseteq I_+, I'_- \subseteq I_-$.
- ▷ $\eta(N^\varphi) \subseteq N^{\varphi'}$.
- ▷ There exists an algebraic group homomorphism $\tau : \Gamma' \rightarrow \Gamma$ such that $\gamma\tau = \gamma'$.
- ▷ $\delta'\eta = \tau^t\delta$.

Moreover, we say that $\mathcal{D}^\varphi \sim \mathcal{D}^{\varphi'}$ if and only if $\mathcal{D}^\varphi \leq \mathcal{D}^{\varphi'}$ and $\mathcal{D}^{\varphi'} \leq \mathcal{D}^\varphi$. In particular, this implies that $I'_+ = I_+, I'_- = I_-, N^\varphi = N^{\varphi'}, \tau$ is an isomorphism, and $\delta' = \tau^t\delta$.

Our last theorem yields the parametrization of the quotients of $\mathcal{O}_\epsilon^\varphi(G)$ up to isomorphism. The proof is analogous to the case $\varphi = 0$ since it relies on the

commutativity of diagram (12) and general constructions of the successive quotients. For these reasons, it will be omitted. See [AG, Theorem 2.20].

Theorem 4.15. *Let \mathcal{D}^φ and $\mathcal{D}^{\varphi'}$ be twisted subgroup data and let $\kappa : \mathcal{O}_\epsilon^\varphi(G) \rightarrow A_{\mathcal{D}^\varphi}$, $\kappa' : \mathcal{O}_\epsilon^{\varphi'}(G) \rightarrow A_{\mathcal{D}^{\varphi'}}$ be the corresponding quotients. Then $[\kappa] \leq [\kappa']$ if and only if $\mathcal{D}^\varphi \leq \mathcal{D}^{\varphi'}$. \square*

4.4.2. Properties of the quotients. We end the paper with a list of some properties of the quotients. Apart from item (v), the proof is analogous to [AG2, Proposition 3.8].

Proposition 4.16. *Let $D^\varphi = (I_+, I_-, N^\varphi, \Gamma, \gamma, \delta)$ be a twisted subgroup datum.*

- (i) *If A_{D^φ} is pointed, then $I_+ \cap I_- = \emptyset$ and Γ is a subgroup of the group of upper triangular matrices of some size. In particular, if Γ is finite, then it is abelian.*
- (ii) *A_{D^φ} is semisimple if and only if $I_+ \cup I_- = \emptyset$ and Γ is finite.*
- (iii) *If $\dim A_{D^\varphi} < \infty$ and $A_{D^\varphi}^*$ is pointed, then $\gamma(\Gamma)$ is included in the fixed torus of G .*
- (iv) *If A_{D^φ} is co-Frobenius then Γ is reductive.*
- (v) *If φ and $(I_+, I_-, \Sigma^\varphi)$ are such that $\Sigma^\varphi = \mathbb{T}^\varphi$ but $\Sigma \neq \mathbb{T}$, then A_{D^φ} is not a 2-cocycle deformation of A_D .*

Proof. We prove only (v). If φ and $(I_+, I_-, \Sigma^\varphi)$ are such that $\Sigma^\varphi = \mathbb{T}^\varphi$ but $\Sigma \neq \mathbb{T}$, then $N^\varphi = 1$ and $N \neq 1$. Then, the quotient $H^\varphi = \mathbf{u}_\epsilon^\varphi(\mathfrak{p})^* / \langle D^z - 1 : D^z \in N^\varphi \rangle$ cannot be a 2-cocycle deformation of $H = \mathbf{u}_\epsilon(\mathfrak{p})^* / \langle D^z - 1 : D^z \in N \rangle$ since they have different dimension. If A_{D^φ} were a 2-cocycle deformation of A_D , then by a chasing diagram argument we would have that H^φ is a 2-cocycle deformation of H , a contradiction; see Example 3.19. \square

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