Schur complements in Krein spaces

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To the memory of Professor Mischa Cotlar

Abstract

The aim of this work is to generalize the notions of Schur complements and shorted operators to Krein spaces. Given a (bounded) J-selfadjoint operator A (with the unique factorization property) acting on a Krein space $\mathcal{H}$ and a suitable closed subspace $\mathcal{S}$ of $\mathcal{H}$, the Schur complement $A_{/[\mathcal{S}]}$ of A to $\mathcal{S}$ is defined. The basic properties of $A_{/[\mathcal{S}]}$ are developed and different characterizations are given, most of them resembling those of the shorted of (bounded) positive operators on a Hilbert space.

1 Introduction

Let $\mathcal{H}$ be a Hilbert space, $L(\mathcal{H})$ be the algebra of bounded linear operators on $\mathcal{H}$ and $L(\mathcal{H})^+$ be the cone of positive operators in $L(\mathcal{H})$. Given $A \in L(\mathcal{H})^+$ and a closed subspace $\mathcal{S}$ of $\mathcal{H}$, the Schur complement (or shorted operator) $A_{/\mathcal{S}}$ was defined by M. G. Krein [16] and W. N. Anderson and G. E. Trapp [2] as

$$A_{/\mathcal{S}} = \max_{\leq}\{X \in L(\mathcal{H})^+ \,:\, X \leq A,\ R(X) \subseteq \mathcal{S}^{\perp}\},$$

where the natural order $\leq$ in $L(\mathcal{H})^+$ is considered.

The notion of Schur complement was generalized to selfadjoint operators in Hilbert spaces, see [4], [9] ,[10], [17]. More generally, given Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, J. Antezana et. al. [6] defined the shorted operator for an arbitrary $A \in L(\mathcal{H}, \mathcal{K})$ with respect to a pair of suitable closed subspaces $\mathcal{S}$ and $\mathcal{T}$ of $\mathcal{H}$ ad $\mathcal{K}$, respectively.

If A is a positive operator, E. Pekarev [18] proved that

$$A_{/\mathcal{S}} = A^{1/2} P_{\mathcal{M}^{\perp}} A^{1/2}, \tag{1.1}$$

where $\mathcal{M} = \overline{A^{1/2}(\mathcal{S})}$ and $P_{\mathcal{M}^{\perp}}$ is the orthogonal projection onto $\mathcal{M}^{\perp}$. This paper is devoted to study the Schur complement of J-selfadjoint operators in Krein spaces, whose definition is inspired by Eq. (1.1).

Let $\mathcal{H}$ be a Krein space with fundamental symmetry J. Bognár-Kramli's theorem [8] states that, if $A \in L(\mathcal{H})$ is J-selfadjoint then there exist a Krein space $\mathcal{K}$ and a bounded injective operator $D \in L(\mathcal{K}, \mathcal{H})$ such that

$$A = DD^{\#},$$

where $D^{\#} \in L(\mathcal{K}, \mathcal{H})$ denotes the J-adjoint operator of D. However, this decomposition may not be unique (see [19]). A J-selfadjoint operator $A \in L(\mathcal{H})$ has the *unique factorization property* if, for any pair of decompositions $A = D_i D_i^{\#}$, $D_i \in L(\mathcal{K}_i, \mathcal{H})$, $N(D_i) = \{0\}$ $(i = 1, 2)$, there exists an isomorphism $U \in L(\mathcal{K}_1, \mathcal{K}_2)$ such that $D_1 = D_2 U$.

Consider a J-selfadjoint operator $A \in L(\mathcal{H})$ with the unique factorization property and suppose that $\mathcal{M} = \overline{D^{\#}(\mathcal{S})}$ is a Krein subspace of $\mathcal{K}$, then the *Schur complement of A to $\mathcal{S}$* is defined as

$$A_{/[\mathcal{S}]} = D P_{\mathcal{M}^{[\perp]}//\mathcal{M}} D^{\#}, \tag{1.2}$$

where $\mathcal{M}^{[\perp]}$ is the J-orthogonal subspace to $\mathcal{M}$ in the Krein space $\mathcal{K}$ and $P_{\mathcal{M}^{[\perp]}//\mathcal{M}} \in L(\mathcal{K})$ is the J-selfadjoint projection onto $\mathcal{M}^{[\perp]}$.

The main properties of shorted operators in Hilbert spaces, which where proved by M. G. Krein [16], W. N. Anderson and G. E. Trapp [2] and E. Pekarev [18], have a natural counterpart for Schur complements in Krein spaces.

The contents of the paper are the following: Section 2 introduces the basic notation and some known results in Krein spaces including topics such as Bognár-Kramli's theorem, the unique factorization property, and J-contractive projections. It also contains the definition and a summary of the properties of the shorting operation in Hilbert spaces.

In Section 3, the Schur complement of A to $\mathcal{S}$, $A_{/[\mathcal{S}]}$, and the $\mathcal{S}$-compression of A, $A_{[\mathcal{S}]}$, are defined for a given J-selfadjoint operator $A \in L(\mathcal{H})$ with the unique factorization property; also, the range and the nullspace of $A_{/[\mathcal{S}]}$ and $A_{[\mathcal{S}]}$ are characterized.

Section 4 is devoted to study the Schur complement for definite subspaces. In particular, it is proved that, if $\mathcal{M} = \overline{D^{\#}(\mathcal{S})}$ is a J-nonnegative subspace of $\mathcal{H}$, then

$$A_{/[\mathcal{S}]} = \max_{\leq_J}\{X \in \mathcal{I}(A) : \, X \leq_J A, \, R(X) \subseteq \mathcal{S}^{[\perp]}\},$$

where $\mathcal{I}(A) = \{X = EE^{\#} \, : \, E \in L(\mathcal{K}, \mathcal{H}), \, R(E) \subseteq R(D)\}$. Also, it is shown that

$$A_{/[\mathcal{S}]} = \inf_{\leq_J}\{Q^{\#}AQ \, : \, Q \in \mathcal{Q}(\mathcal{H}), \, N(Q) = \mathcal{S}\}.$$

Finally, in Section 5 the Schur complement for J-positive operators is described in detail. In this case $A_{/[\mathcal{S}]}$ is defined for every closed subspace $\mathcal{S}$ of $\mathcal{H}$ and it always has both extremal characterizations. Furthermore, the shorting operation of a J-positive operator A in a Krein space $\mathcal{H}$ is intimately related to the shorted of JA in the Hilbert space $|\mathcal{H}|$. This relationship allows to translate the classical results into the Krein space's context.

2 Preliminaries

Along this work $\mathcal{H}$ denotes either a (complex, separable) Hilbert space with inner product $\langle \, , \, \rangle$ or a (complex) Krein space with indefinite metric $[\, , \,]$, depending on the context. If $\mathcal{S}$ is a subspace of a Hilbert space $\mathcal{H}$, $\mathcal{S}^{\perp}$ is the orthogonal complement of $\mathcal{S}$. Analogously, if $\mathcal{S}$ is a subspace of a Krein space $\mathcal{H}$, the J-orthogonal subspace to $\mathcal{S}$ is the closed subspace of $\mathcal{H}$ defined by $\mathcal{S}^{[\perp]} = \{x \in \mathcal{H} \, : \, [x, y] = 0 \text{ for every } y \in \mathcal{S}\}$. Sometimes we use the notation $\mathcal{S}^{[\perp]_{\mathcal{H}}}$ instead of $\mathcal{S}^{[\perp]}$ to emphasize the Krein space considered.

Given two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, $L(\mathcal{H}, \mathcal{K})$ is the algebra of bounded linear operators from $\mathcal{H}$ into $\mathcal{K}$ and $L(\mathcal{H}) = L(\mathcal{H}, \mathcal{H})$. If $T \in L(\mathcal{H})$ then T^* denotes the adjoint operator of T, $R(T)$ stands for the range of T and $N(T)$ for its nullspace.

Given a Hilbert space $\mathcal{H}$, let $L(\mathcal{H})^+$ be the cone of (semidefinite) positive operators in $L(\mathcal{H})$ and denote by $\mathcal{Q}(\mathcal{H})$ the set of projections in $L(\mathcal{H})$, i.e., $\mathcal{Q}(\mathcal{H}) = \{Q \in L(\mathcal{H}) \, : \, Q^2 = Q\}$. If $\mathcal{S}$ and $\mathcal{T}$ are two (closed) subspaces of $\mathcal{H}$, denote by $\mathcal{S} \dot{+} \mathcal{T}$ the direct sum of $\mathcal{S}$ and $\mathcal{T}$. If $\mathcal{H} = \mathcal{S} \dot{+} \mathcal{T}$, the oblique projection onto $\mathcal{S}$ along $\mathcal{T}$, $P_{\mathcal{S}//\mathcal{T}}$, is the projection with $R(P_{\mathcal{S}//\mathcal{T}}) = \mathcal{S}$ and $N(P_{\mathcal{S}//\mathcal{T}}) = \mathcal{T}$. In particular, $P_{\mathcal{S}} = P_{\mathcal{S}//\mathcal{S}^{\perp}}$ is the orthogonal projection onto $\mathcal{S}$.

Krein spaces

In what follows we give some basic results on Krein spaces. For a complete exposition of the subject and the proofs of the results below see the books by J. Bognár [7] and T. Ya. Azizov and I. S. Iokhvidov [15], the monographs by T. Ando [3] and by M. Dritschel and J. Rovnyak [12] and the paper by J. Rovnyak [19].

Given a Krein space $\mathcal{H}$ and a fundamental decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, the direct sum of the Hilbert spaces $(\mathcal{H}_+, [\, , \,])$ and $(\mathcal{H}_-, -[\, , \,])$ is denoted by $|\mathcal{H}|$. If $\mathcal{H}$ and $\mathcal{K}$ are Krein spaces then $L(\mathcal{H}, \mathcal{K})$ (respectively $L(\mathcal{H})$) stands for $L(|\mathcal{H}|, |\mathcal{K}|)$ (respectively $L(|\mathcal{H}|)$). Given $T \in L(\mathcal{H}, \mathcal{K})$, the J-adjoint operator of T is denoted by $T^{\#}$. An operator $T \in L(\mathcal{H})$ is J-selfadjoint if $T = T^{\#}$.

The following theorem is due to J. Bognár and A. Krámli [8]. See also Theorem 1.1 in [12].

Theorem 2.1 (Bognár-Krámli)**.** *Let $\mathcal{H}$ be a Krein space with fundamental symmetry J. Any J-selfadjoint operator $T \in L(\mathcal{H})$ can be written in the form*

$$T = WW^{\#},$$

where $W \in L(\mathcal{K}, \mathcal{H})$ for some Krein space $\mathcal{K}$ and $N(W) = \{0\}$.

While factorizations as in Theorem 2.1 always exist, they are not in general unique.

Definition. Let $\mathcal{H}$ be a Krein space with fundamental symmetry J. A J-selfadjoint operator $T \in L(\mathcal{H})$ has the *unique factorization property* (UFP) if for any two factorizations

$$T = W_i W_i^{\#}, \quad W_i \in L(\mathcal{K}_i, \mathcal{H}), \quad N(W_i) = \{0\}, \quad i = 1, 2,$$

there is an isomorphism $U \in L(\mathcal{K}_1, \mathcal{K}_2)$ such that $W_1 = W_2 U$.

Remark 2.2. Let $T \in L(\mathcal{H})$ be a J-selfadjoint operator satisfying the UFP and suppose that $T = WW^{\#}$ where $W \in L(\mathcal{K}, \mathcal{H})$, $N(W) = \{0\}$ and $\mathcal{K}$ is a Krein space. Then,

1. if $T = DD^{\#}$ is another factorization of T as in Theorem 2.1 then $R(D) = R(W)$;
2. if $R(T)$ is closed then $R(D^{\#}) = \mathcal{K}$.

An operator $T \in L(\mathcal{H})$ is *J-positive* if $[\,Tx, x\,] \geq 0$ for every $x \in \mathcal{H}$. We denote it by $T \geq_J 0$. If T_1 and T_2 are J-selfadjoint operators, we say that $T_1 \geq_J T_2$ if $T_1 - T_2 \geq_J 0$. It is easy to show that $\geq_J$ is a partial order in the real vector space of J-selfadjoint operators.

The following theorem provides some examples of classes of operators with the UFP.

Theorem 2.3. *Let $\mathcal{H}$ be a Krein space with fundamental symmetry J, and let $T \in L(\mathcal{H})$ be a J-selfadjoint operator. Each of the following conditions is sufficient for T to have the unique factorization property:*

1. *$T \geq_J 0$;*
2. *$T^2 \leq_J T$.*

Given a Krein space $\mathcal{H}$, an operator $T \in L(\mathcal{H})$ is *J-contractive* if $[\,Tx, Tx\,] \leq [\,x, x\,]$ for every $x \in \mathcal{H}$. Therefore, T is J-contractive if and only if $T^{\#}T \leq_J I$. Analogously, an operator $T \in L(\mathcal{H})$ is *J-expansive* if $[\,Tx, Tx\,] \geq [\,x, x\,]$ for every $x \in \mathcal{H}$ (i.e. $T^{\#}T \geq_J I$).

We say that $\mathcal{S}$ is a *Krein subspace* of $\mathcal{H}$ if it is a Krein space with the indefinite metric of $\mathcal{H}$. It is well known that $\mathcal{S}$ is a Krein subspace of $\mathcal{H}$ if and only if $\mathcal{S} = R(Q)$ for some J-selfadjoint $Q \in \mathcal{Q}(\mathcal{H})$. Also, a subspace $\mathcal{S}$ of $\mathcal{H}$ is *J-nonnegative* (respectively *J-nonpositive*) if $[\,x, x\,] \geq 0$ (respectively $[\,x, x\,] \leq 0$) for every $x \in \mathcal{S}$.

S. Hassi and K. Nordström proved the following result, which characterizes those projections which are J-contractive (see [14, §3, Proposition 5]). A similar result holds for J-expansive projections.

Proposition 2.4. *If $Q \in \mathcal{Q}(\mathcal{H})$ then the following conditions are equivalent:*

1. *Q is J-contractive;*
2. *Q is J-selfadjoint and $N(Q)$ is J-nonnegative;*
3. *$I - Q$ is J-positive.*

Hassi and Nordström [14, §4, Theorem 2] also proved that every J-selfadjoint projection Q can be factored as follows.

Theorem 2.5. *Let Q be a J-selfadjoint projection in a Krein space $\mathcal{H}$. Then, Q can be represented as $Q = Q_+ Q_-$ where Q_+ and Q_- are two commuting projections such that Q_+ is J-contractive and Q_- is J-expansive.*

Shorted operators in Hilbert spaces

Definition (Krein [16], Anderson-Trapp [1] [2])**.** Let $\mathcal{H}$ be a Hilbert space. Given $A \in L(\mathcal{H})^+$ and a closed subspace $\mathcal{S}$ of $\mathcal{H}$, the *shorted operator of A to $\mathcal{S}$* is defined by

$$A_{/\mathcal{S}} = \max_{\leq}\{X \in L(\mathcal{H})^+ \,:\, X \leq A,\ R(X) \subseteq \mathcal{S}^\perp\},$$

where $\leq$ is the natural order given by the cone $L(\mathcal{H})^+$.

The following theorem collects many well known results about shorted operators. See [2], [18], [9], [10] for the proof of these facts.

Theorem 2.6. *Let $\mathcal{S}$ be a closed subspace of a Hilbert space $\mathcal{H}$ and let $A \in L(\mathcal{H})^+$. Then:*

1. *If $\mathcal{M} = \overline{A^{1/2}(\mathcal{S})}$ then $A_{/\mathcal{S}} = A^{1/2}P_{\mathcal{M}^\perp}A^{1/2}$.*
2. *$R(A) \cap \mathcal{S}^\perp \subseteq R(A_{/\mathcal{S}}) \subseteq R(A^{1/2}) \cap \mathcal{S}^\perp$ and $N(A_{/\mathcal{S}}) = A^{-1/2}(\mathcal{M})$.*
3. *$R((A_{/\mathcal{S}})^{1/2}) = R(A^{1/2}) \cap \mathcal{S}^\perp$.*
4. *$A_{/\mathcal{S}} = \inf\{Q^*AQ \,:\, Q \in \mathcal{Q}(\mathcal{H}),\ N(Q) = \mathcal{S}\}$.*
5. *If $\mathcal{T}$ is a closed subspace of $\mathcal{H}$ such that $\mathcal{S} + \mathcal{T}$ is closed then $A_{/\mathcal{S}+\mathcal{T}} = (A_{/\mathcal{S}})_{/\mathcal{T}} = (A_{/\mathcal{T}})_{/\mathcal{S}}$.*

If $\mathcal{H}$ is a Hilbert space and $(A_n)_{n\in\mathbb{N}}$ is a sequence in $L(\mathcal{H})$ we say that $(A_n)_{n\in\mathbb{N}}$ converges in the SOT topology to $A \in L(\mathcal{H})$ (and denote it by $A_n \xrightarrow[n\to\infty]{\text{SOT}} A$) if $\|A_nx - Ax\| \xrightarrow[n\to\infty]{} 0$ for every $x \in \mathcal{H}$. Moreover, if $(A_n)_{n\in\mathbb{N}}$ and A are selfadjoint operators, we say that $A_n \overset{\text{SOT}}{\searrow} A$ if $A_n \xrightarrow[n\to\infty]{\text{SOT}} A$ and $A_n \geq A_{n+1}$ $(\geq A)$ for every $n \in \mathbb{N}$.

The following are some results about the continuity of the shorting operation, see [2], [5].

Proposition 2.7. *Let A_n $(n \in \mathbb{N})$ and A be operators in $L(\mathcal{H})^+$ such that $A_n \overset{SOT}{\searrow} A$ as $n \to \infty$. Then, $(A_n)_{/\mathcal{S}} \overset{SOT}{\searrow} A_{/\mathcal{S}}$ as $n \to \infty$, for every closed subspace $\mathcal{S}$ of $\mathcal{H}$.*

Proposition 2.8. *Let $\mathcal{S}_n$ $(n \in \mathbb{N})$ and $\mathcal{S}$ be closed subspaces such that $P_{\mathcal{S}_n} \overset{SOT}{\nearrow} P_{\mathcal{S}}$ as $n \to \infty$. Then, $A_{/\mathcal{S}_n} \overset{SOT}{\searrow} A_{/\mathcal{S}}$ as $n \to \infty$, for every $A \in L(\mathcal{H})^+$.*

The following example shows that $P_{\mathcal{S}_n} \overset{\text{SOT}}{\searrow} P_{\mathcal{S}}$ is not a sufficient condition to imply the convergence of the sequence $(A_{/\mathcal{S}_n})_{n\in\mathbb{N}}$ to $A_{/\mathcal{S}}$.

Example 2.9. *Let $A \in L(\mathcal{H})^+$ such that $N(A) = \{0\}$ and $R(A)$ is not closed. Consider a dense subspace $\mathcal{T}$ of $\mathcal{H}$ such that $\mathcal{T} \cap R(A^{1/2}) = \{0\}$ and let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{H}$ contained in $\mathcal{T}$.*

Let $\mathcal{S}_n = \overline{\operatorname{span}\{e_k \,:\, k \geq n\}}$ for $n \geq 1$. Then, $P_{\mathcal{S}_n} \overset{SOT}{\searrow} 0$. Furthermore, $A_{/\mathcal{S}_n} = 0$ because

$$R((A_{/\mathcal{S}_n})^{1/2}) = R(A^{1/2}) \cap \mathcal{S}_n^\perp = R(A^{1/2}) \cap \operatorname{span}\{e_1, \ldots, e_n\} = \{0\}.$$

But $A_{/\{0\}} = A \neq 0$.

3 Schur complements in Krein spaces

Let $\mathcal{H}$ be a Krein space with fundamental symmetry J and $A \in L(\mathcal{H})$ be a J-selfadjoint operator satisfying the UFP. Suppose that $A = DD^\#$, where $\mathcal{K}$ is a Krein space and $D \in L(\mathcal{K}, \mathcal{H})$ with $N(D) = \{0\}$. Given a closed subspace $\mathcal{S}$ of $\mathcal{H}$, consider $\mathcal{M} = \overline{D^\#(\mathcal{S})}$ and suppose that $\mathcal{M}$ is a Krein subspace of $\mathcal{K}$.

Definition. Under the above hypothesis, the *Schur complement* of A to $\mathcal{S}$ is defined by

$$A_{/[\mathcal{S}]} = DP_{\mathcal{M}^{[\perp]}//\mathcal{M}}D^{\#},$$

and the $\mathcal{S}$*-compression of* A is $A_{[\mathcal{S}]} = DP_{\mathcal{M}//\mathcal{M}^{[\perp]}}D^{\#}$.

The operators $A_{[\mathcal{S}]}$ *and* $A_{/[\mathcal{S}]}$ *are well defined:* by the UFP of A, if $A = D_iD_i^{\#}$ where $D_i \in L(\mathcal{K}_i, \mathcal{H})$ and $N(D_i) = \{0\}$ for $i = 1, 2$, there exists an isomorphism $U \in L(\mathcal{K}_1, \mathcal{K}_2)$ such that $D_1 = D_2U$. Given the subspaces $\mathcal{M}_i = \overline{D_i^{\#}(\mathcal{S})}$, for $i = 1, 2$, observe that $\mathcal{M}_1$ is a Krein subspace of $\mathcal{K}_1$ if and only if $\mathcal{M}_2 = U(\mathcal{M}_1)$ is a Krein subspace of $\mathcal{K}_2$, and in this case $UP_{\mathcal{M}_1//\mathcal{M}_1^{[\perp]}}U^{\#} = P_{\mathcal{M}_2//\mathcal{M}_2^{[\perp]}}$. Then,

$$D_1P_{\mathcal{M}_1//\mathcal{M}_1^{[\perp]}}D_1^{\#} = D_2(UP_{\mathcal{M}_1//\mathcal{M}_1^{[\perp]}}U^{\#})D_2^{\#} = D_2P_{\mathcal{M}_2//\mathcal{M}_2^{[\perp]}}D_2^{\#}.$$

Also, the following properties hold for the Schur complement and the $\mathcal{S}$-compression:

i. $A_{[\mathcal{S}]}, A_{/[\mathcal{S}]} \in L(\mathcal{H})$,

ii. $A_{[\mathcal{S}]}, A_{/[\mathcal{S}]}$ are $J_{\mathcal{H}}$-selfadjoint operators (because $P_{\mathcal{M}//\mathcal{M}^{[\perp]}}$ and $P_{\mathcal{M}^{[\perp]}//\mathcal{M}}$ are $J_{\mathcal{K}}$-selfadjoint),

iii. $A_{[\mathcal{S}]} + A_{/[\mathcal{S}]} = A$.

Let us characterize the range and the nullspace of $A_{[\mathcal{S}]}$ and $A_{/[\mathcal{S}]}$. The lemma below is well known and its proof is straightforward.

Lemma 3.1. *Let* $\mathcal{H}$ *and* $\mathcal{K}$ *be Krein spaces. If* $T \in L(\mathcal{H}, \mathcal{K})$ *then,*

1. $N(T^{\#}) = R(T)^{[\perp]_{\mathcal{K}}}$.
2. $T^{\#}(\mathcal{S})^{[\perp]_{\mathcal{H}}} = T^{-1}(\mathcal{S}^{[\perp]_{\mathcal{K}}})$ *for every subspace* $\mathcal{S}$ *of* $\mathcal{K}$.

Proposition 3.2. *Let* $A = DD^{\#} \in L(\mathcal{H})$ *be a* J*-selfadjoint operator satisfying the UFP and* $\mathcal{S}$ *a closed subspace of* $\mathcal{H}$ *such that* $\mathcal{M} = \overline{D^{\#}(\mathcal{S})}$ *is a Krein subspace of* $\mathcal{K}$*. Then,*

1. $A(\mathcal{S}) \subseteq R(A_{[\mathcal{S}]}) \subseteq \overline{A(\mathcal{S})}$;
2. $N(A_{[\mathcal{S}]}) = A(\mathcal{S})^{[\perp]}$;
3. $R(A) \cap \mathcal{S}^{[\perp]} \subseteq R(A_{/[\mathcal{S}]}) \subseteq R(D) \cap \mathcal{S}^{[\perp]}$;
4. $N(A_{/[\mathcal{S}]}) = (D^{\#})^{-1}(\mathcal{M})$.

Proof. 1. It is easy to see that

$$A(\mathcal{S}) = D(D^{\#}(\mathcal{S})) = A_{[\mathcal{S}]}(\mathcal{S}) \subseteq R(A_{[\mathcal{S}]}) \subseteq D(\mathcal{M}) = D(\overline{D^{\#}(\mathcal{S})}) \subseteq \overline{DD^{\#}(\mathcal{S})} = \overline{A(\mathcal{S})}.$$

2. Since $N(D) = \{0\}$, it follows that

$$N(A_{[\mathcal{S}]}) = N(P_{\mathcal{M}//\mathcal{M}^{[\perp]}}D^{\#}) = (D^{\#})^{-1}(\mathcal{M}^{[\perp]}) = A^{-1}(\mathcal{S}^{[\perp]}) = A(\mathcal{S})^{[\perp]}.$$

3. First of all observe that, by Remark 2.2, $R(D)$ does not depend on the factorization. If $y \in R(A) \cap \mathcal{S}^{[\perp]}$ then there exists $x \in \mathcal{H}$ such that $y = Ax \in \mathcal{S}^{[\perp]}$. Note that $D^{\#}x \in \mathcal{M}^{[\perp]}$ and $A_{/[\mathcal{S}]}x = DP_{\mathcal{M}^{[\perp]}//\mathcal{M}}(D^{\#}x) = DD^{\#}x = y$. Thus, $R(A) \cap \mathcal{S}^{[\perp]} \subseteq R(A_{/[\mathcal{S}]})$. On the other hand, $R(A_{/[\mathcal{S}]}) \subseteq D(\mathcal{M}^{[\perp]}) = D(D^{-1}(\mathcal{S}^{[\perp]})) = \mathcal{S}^{[\perp]} \cap R(D)$.

4. As in item 2., notice that $N(A_{/[\mathcal{S}]}) = N(P_{\mathcal{M}^{[\perp]}//\mathcal{M}}D^{\#}) = (D^{\#})^{-1}(\mathcal{M})$. □

In general, the inclusions in items 1. and 3. of the above proposition are strict. See the examples in [2] and [10].

Proposition 3.3. *Let* $A \in L(\mathcal{H})$ *be a* J*-selfadjoint operator satisfying the UFP,* $A = DD^{\#}$*,* $D \in L(\mathcal{K}, \mathcal{H})$ *with* $N(D) = \{0\}$*, and* $\mathcal{S}$ *a closed subspace of* $\mathcal{H}$ *such that* $\mathcal{M} = \overline{D^{\#}(\mathcal{S})}$ *is a Krein subspace of* $\mathcal{K}$*. If* $\mathcal{T}$ *is a closed subspace of* $\mathcal{H}$ *such that* $\mathcal{S} \subseteq \mathcal{T} \subseteq (D^{\#})^{-1}(\mathcal{M})$ *then* $\overline{D^{\#}(\mathcal{T})} = \mathcal{M}$ *and*

$$A_{/[\mathcal{T}]} = A_{/[\mathcal{S}]}.$$

Proof. Let $\mathcal{T}$ be a closed subspace of $\mathcal{H}$ such that $\mathcal{S} \subseteq \mathcal{T} \subseteq (D^{\#})^{-1}(\mathcal{M})$, then applying $D^{\#}$ it follows that $D^{\#}(\mathcal{S}) \subseteq D^{\#}(\mathcal{T}) \subseteq D^{\#}((D^{\#})^{-1}(\mathcal{M})) \subseteq \mathcal{M}$. Therefore, $\overline{D^{\#}(\mathcal{T})} = \mathcal{M}$ and $A_{/[\mathcal{T}]} = A_{/[\mathcal{S}]}$. □

4 Extremal properties for definite subspaces

The main results in this section are stated for both J-nonnegative and J-nonpositive subspaces, but we only give the proofs for J-nonnegative ones. The proofs in the nonpositive case are similar.

Let $A \in L(\mathcal{H})$ be a J-selfadjoint operator satisfying the UFP. If $A = DD^{\#}$ where $\mathcal{K}$ is a Krein space and $D \in L(\mathcal{K}, \mathcal{H})$ with $N(D) = \{0\}$, consider the set

$$\mathcal{I}(A) = \{X = EE^{\#} \, : \, E \in L(\mathcal{K}, \mathcal{H}), \, R(E) \subseteq R(D)\}.$$

By Remark 2.2, the subspace $R(D)$ only depends on A, so that, the same is true for the set $\mathcal{I}(A)$.

If $\mathcal{S}$ is a closed subspace of $\mathcal{H}$, consider the subsets

$$\begin{aligned} \mathcal{M}^{-}(A, \mathcal{S}^{[\perp]}) &= \{X \in \mathcal{I}(A) : \, X \leq_J A, \, R(X) \subseteq \mathcal{S}^{[\perp]} \}, \\ \mathcal{M}^{+}(A, \mathcal{S}^{[\perp]}) &= \{X \in \mathcal{I}(A) : \, A \leq_J X, \, R(X) \subseteq \mathcal{S}^{[\perp]} \}. \end{aligned}$$

Observe that these sets can be empty.

First of all, consider the particular case $A = I$. Observe that $I \in L(\mathcal{H})$ has the UFP because it satisfies a sufficient condition: $I^2 = I \leq_J I$ (see Theorem 2.3). Furthermore, the unique factorization (up to isomorphism) is $I = DD^{\#}$, where $D = I \in L(\mathcal{H})$ and therefore $\mathcal{M}^{-}(I, \mathcal{S}^{[\perp]}) = \{X \in L(\mathcal{H}) : X \leq_J I, \, R(X) \subseteq \mathcal{S}^{[\perp]} \}$ and $\mathcal{M}^{+}(I, \mathcal{S}^{[\perp]}) = \{X \in L(\mathcal{H}) : I \leq_J X, \, R(X) \subseteq \mathcal{S}^{[\perp]} \}$.

Lemma 4.1. *Let $\mathcal{S}$ be a Krein subspace of $\mathcal{H}$ and $Q = P_{\mathcal{S}^{[\perp]}//\mathcal{S}}$. Then,*

1. $Q = \max_{\leq_J} \mathcal{M}^{-}(I, \mathcal{S}^{[\perp]})$ *if $\mathcal{S}$ is J-nonnegative.*

2. $Q = \min_{\leq_J} \mathcal{M}^{+}(I, \mathcal{S}^{[\perp]})$ *if $\mathcal{S}$ is J-nonpositive.*

Proof. Suppose that $\mathcal{S}$ is a J-nonnegative Krein subspace of $\mathcal{H}$. Then, Q is J-contractive (see Proposition 2.4) and $R(Q) = \mathcal{S}^{[\perp]}$. Therefore, $Q \in \mathcal{M}^{-}(I, \mathcal{S}^{[\perp]})$.

Moreover, if $X \in \mathcal{M}^{-}(I, \mathcal{S}^{[\perp]})$ then $X \leq_J Q$: $R(X) \subseteq \mathcal{S}^{[\perp]}$ implies that $QX = X$, and $QXQ = (QX)Q = XQ = QX = X$ because X and Q are J-selfadjoint. Then, if $x \in \mathcal{H}$,

$$[\,(Q - X)x, x\,] = [\,Q(I - X)Qx, x\,] = [\,(I - X)Qx, Qx\,] \geq 0,$$

i.e. $X \leq_J Q$. Therefore, $Q = \max_{\leq_J} \mathcal{M}^{-}(I, \mathcal{S}^{[\perp]})$. □

Corollary 4.2. *Let $\mathcal{S}$ be a Krein subspace of $\mathcal{H}$. If $Q = P_{\mathcal{S}^{[\perp]}//\mathcal{S}}$ then there exist two Krein subspaces $\mathcal{S}_+$ and $\mathcal{S}_-$ of $\mathcal{H}$ such that $\mathcal{S} = \mathcal{S}_+ \dot{+} \mathcal{S}_-$ and*

$$Q = \max_{\leq_J} \mathcal{M}^{-}(I, \mathcal{S}_+^{[\perp]}) \min_{\leq_J} \mathcal{M}^{+}(I, \mathcal{S}_-^{[\perp]}).$$

Proof. If $\mathcal{S}$ is a Krein subspace of $\mathcal{H}$ then, by Theorem 2.5, $Q = Q_+Q_-$, where Q_+ and Q_- are commuting projections such that Q_+ is J-contractive and Q_- is J-expansive. Also $(I - Q_+)(I - Q_-) = 0$ (see the proof in [14]) so that $I - Q = (I - Q_+) + (I - Q_-)$ and $\mathcal{S} = N(Q) = N(Q_+) \dot{+} N(Q_-)$.

By Lemma 4.1, $Q_+ = \max_{\leq_J} \mathcal{M}^{-}(I, R(Q_+))$ and $Q_- = \min_{\leq_J} \mathcal{M}^{+}(I, R(Q_-))$. Therefore, taking $\mathcal{S}_\pm = N(Q_\pm)$, the proof is complete.

□

The following theorem is an extremal characterization of the Schur complement similar to the one given by Anderson-Trapp [2, Theorem 1].

Theorem 4.3. *Let $\mathcal{M} = \overline{D^{\#}(\mathcal{S})}$ be a Krein subspace of $\mathcal{K}$. Then:*

1. $A_{/[\mathcal{S}]} = \max_{\leq_J} \mathcal{M}^{-}(A, \mathcal{S}^{[\perp]})$ *if $\mathcal{M}$ is J-nonnegative.*

2. $A_{/[\mathcal{S}]} = \min_{\leq_J} \mathcal{M}^{+}(A, \mathcal{S}^{[\perp]})$ *if $\mathcal{M}$ is J-nonpositive.*

Proof. Let $Q = P_{\mathcal{M}^{[\perp]}//\mathcal{M}}$ and suppose that $\mathcal{M}$ is J-nonnegative (i.e. Q is J-contractive). Notice that $A_{/[\mathcal{S}]} = (DQ)(DQ)^{\#}$ and $R(DQ) \subseteq R(D)$, then $A_{/[\mathcal{S}]} \in \mathcal{I}(A)$. Since $Q \leq_J I$ we have that $A_{/[\mathcal{S}]} = DQD^{\#} \leq_J DD^{\#} = A$ and, by Proposition 3.2, $R(A_{/[\mathcal{S}]}) \subseteq \mathcal{S}^{[\perp]}$. Therefore, $A_{/[\mathcal{S}]} \in \mathcal{M}^{-}(A, \mathcal{S}^{[\perp]})$.

Moreover, $A_{/[\mathcal{S}]} = \max_{\leq_J} \mathcal{M}^{-}(A, \mathcal{S}^{[\perp]})$. Indeed, if $X = EE^{\#} \in \mathcal{M}^{-}(A, \mathcal{S}^{[\perp]})$ then $R(E) \subseteq R(D)$ and, by Douglas' theorem [11, Theorem 1], the equation $DY = E$ admits a bounded solution in $L(\mathcal{K})$. If $Z \in L(\mathcal{K})$ is a solution of the above equation, then $X = DZZ^{\#}D^{\#}$. Since $X \leq_J A$, given $x \in \mathcal{H}$,

$$[\,(I_{\mathcal{K}} - ZZ^{\#})D^{\#}x, D^{\#}x\,]_{\mathcal{K}} = [\,D(I - ZZ^{\#})D^{\#}x, x\,]_{\mathcal{H}} = [\,(A - X)x, x\,]_{\mathcal{H}} \geq 0,$$

so $[\,(I_{\mathcal{K}} - ZZ^{\#})y, y\,]_{\mathcal{K}} \geq 0$ for every $y \in \overline{R(D^{\#})} = N(D)^{[\perp]_{\mathcal{K}}} = \mathcal{K}$. Hence, $ZZ^{\#} \leq_J I_{\mathcal{K}}$. Since $R(X) \subseteq \mathcal{S}^{[\perp]}$ we have that $R(ZZ^{\#}D^{\#}) \subseteq D^{-1}(\mathcal{S}^{[\perp]}) = \mathcal{M}^{[\perp]}$. Moreover, $R(ZZ^{\#}) = ZZ^{\#}(\overline{R(D^{\#})}) \subseteq \overline{R(ZZ^{\#}D^{\#})} \subseteq \mathcal{M}^{[\perp]}$. Therefore, $ZZ^{\#} \in \mathcal{M}^{-}(I, \mathcal{M}^{[\perp]})$ and, by Lemma 4.1, $ZZ^{\#} \leq_J Q$ (notice that the Krein space considered here is $\mathcal{K}$). Then,

$$X = DZZ^{\#}D^{\#} \leq_J DQD^{\#} = A_{/[\mathcal{S}]},$$

i.e. $A_{/[\mathcal{S}]} = \max_{\leq_J} \mathcal{M}^{-}(A, \mathcal{S}^{[\perp]})$. □

Corollary 4.4. *Let $\mathcal{H}$ be a Krein space and $A \in L(\mathcal{H})$ a J-selfadjoint operator with the UFP. Consider a factorization $A = DD^{\#}$ where $\mathcal{K}$ is a Krein space and $D \in L(\mathcal{K}, \mathcal{H})$ with $N(D) = \{0\}$. If A has closed range and $\mathcal{S}$ is a closed subspace of $\mathcal{H}$ such that $\mathcal{M} = \overline{D^{\#}(\mathcal{S})}$ is a Krein subspace of $\mathcal{K}$, then there exist two closed subspaces $\mathcal{S}_+$ and $\mathcal{S}_-$ of $\mathcal{H}$ such that $\mathcal{S}_+ \dot{+} \mathcal{S}_- = (D^{\#})^{-1}(\mathcal{M})$ and*

$$A_{/[\mathcal{S}]} = \max_{\leq_J} \mathcal{M}^{-}(A, \mathcal{S}_+^{[\perp]}) + \min_{\leq_J} \mathcal{M}^{+}(A, \mathcal{S}_-^{[\perp]}) - A.$$

Proof. Suppose that $\mathcal{M}$ is a Krein subspace of $\mathcal{K}$ and let $Q = P_{\mathcal{M}^{[\perp]}//\mathcal{M}}$. By Theorem 2.5, there exist commuting projections Q_+ and Q_- such that $Q = Q_+Q_-$, where Q_+ is J-contractive, Q_- is J-expansive and $N(Q) = N(Q_+) \dot{+} N(Q_-)$ (see the proof in [14]).

Let $\mathcal{S}_\pm = (D^{\#})^{-1}(N(Q_\pm))$ and define $\mathcal{M}_\pm = \overline{D^{\#}(\mathcal{S}_\pm)}$. Since $R(D^{\#}) = \mathcal{K}$ (see Remark 2.2), it follows that $\mathcal{M}_\pm = \overline{D^{\#}(\mathcal{S}_\pm)} = \overline{N(Q_\pm) \cap R(D^{\#})} = N(Q_\pm)$. Therefore, $A_{/[\mathcal{S}_\pm]} = DQ_\pm D^{\#}$ and

$$A_{[\mathcal{S}]} = D(I - Q)D^{\#} = D((I - Q_+) + (I - Q_-))D^{\#} = A_{[\mathcal{S}_+]} + A_{[\mathcal{S}_-]}.$$

As a consequence of Proposition 2.4, the subspaces $\mathcal{M}_+$ and $\mathcal{M}_-$ are J-nonnegative and J-nonpositive, respectively. Then, by Theorem 4.3,

$$\begin{aligned} A_{/[\mathcal{S}]} &= A - A_{[\mathcal{S}]} = A - (A_{[\mathcal{S}_+]} + A_{[\mathcal{S}_-]}) = A_{/[\mathcal{S}_+]} + A_{/[\mathcal{S}_-]} - A = \\ &= \max_{\leq_J} \mathcal{M}^{-}(A, \mathcal{S}_+^{[\perp]}) + \min_{\leq_J} \mathcal{M}^{+}(A, \mathcal{S}_-^{[\perp]}) - A. \end{aligned}$$

□

Theorem 4.5. *Let $\mathcal{S}$ be a closed subspace of $\mathcal{H}$. Suppose that $A \in L(\mathcal{H})$ is J-selfadjoint and satisfies the UFP. If $A = DD^{\#}$ with $D \in L(\mathcal{K}, \mathcal{H})$, $N(D) = \{0\}$, suppose that $\mathcal{M} = \overline{D^{\#}(\mathcal{S})}$ is a Krein subspace of $\mathcal{K}$. Then:*

1. *$A_{/[\mathcal{S}]} = \inf_{\leq_J} \{Q^{\#}AQ : Q \in \mathcal{Q}(\mathcal{H}),\ N(Q) = \mathcal{S}\}$ if $\mathcal{M}$ is J-nonnegative.*

2. *$A_{/[\mathcal{S}]} = \sup_{\leq_J} \{Q^{\#}AQ : Q \in \mathcal{Q}(\mathcal{H}),\ N(Q) = \mathcal{S}\}$ if $\mathcal{M}$ is J-nonpositive.*

Proof. Suppose that $\mathcal{M}$ is J-nonnegative and consider $P = P_{\mathcal{M}^{[\perp]}//\mathcal{M}}$. Then, for every $x \in \mathcal{K}$,

$$[\,Px, Px\,]_{\mathcal{K}} = \min_{m \in \mathcal{M}} [\,x - m, x - m\,]_{\mathcal{K}}.$$

Indeed, given $x \in \mathcal{K}$ and $m \in \mathcal{M}$,

$$[\,x - m, x - m\,] = [\,Px + (I-P)x - m, Px + (I-P)x - m\,] = [\,Px, Px\,] + [\,(I-P)x - m, (I-P)x - m\,] \geq [\,Px, Px\,].$$

Furthermore, observe that $R(D^{\#})$ is dense in $\mathcal{K}$ because $N(D)=\{0\}$. Then, if $y\in\mathcal{H}$,

$$\begin{aligned}[A_{/[\mathcal{S}]}y,y]_{\mathcal{H}} &= [PD^{\#}y,PD^{\#}y]_{\mathcal{K}}=\min_{m\in\mathcal{M}}[D^{\#}y-m,D^{\#}y-m]_{\mathcal{K}}=\inf_{s\in\mathcal{S}}[D^{\#}(y-s),D^{\#}(y-s)]_{\mathcal{K}}=\\ &= \inf_{s\in\mathcal{S}}[A(y-s),y-s]_{\mathcal{H}}.\end{aligned}$$

If $Q\in\mathcal{Q}(\mathcal{H})$ with $N(Q)=\mathcal{S}$, given $x\in\mathcal{H}$,

$$[Q^{\#}AQx,x]_{\mathcal{H}} = [AQx,Qx]_{\mathcal{H}}=[A(x-(I-Q)x),x-(I-Q)x]_{\mathcal{H}}\geq[A_{/[\mathcal{S}]}x,x]_{\mathcal{H}}$$

because $(I-Q)x\in\mathcal{S}$. Then, $A_{/[\mathcal{S}]}\leq_J Q^{\#}AQ$ for every $Q\in\mathcal{Q}(\mathcal{H})$ with $N(Q)=\mathcal{S}$ i.e. $A_{/[\mathcal{S}]}$ is a lower bound of the set $\{Q^{\#}AQ : Q\in\mathcal{Q}(\mathcal{H}),\ N(Q)=\mathcal{S}\}$.

Let C be any lower bound of the set $\{Q^{\#}AQ : Q\in\mathcal{Q}(\mathcal{H}),\ N(Q)=\mathcal{S}\}$, we are going to show that $C\leq_J A_{/[\mathcal{S}]}$. Fixed $x\in\mathcal{H}$, if $x\notin\mathcal{S}$, observe that for every $s\in\mathcal{S}$ there exists $Q\in\mathcal{Q}(\mathcal{H})$ with $N(Q)=\mathcal{S}$ such that $(I-Q)x=s$. Therefore,

$$[A(x-s),x-s]_{\mathcal{H}}=[AQx,Qx]_{\mathcal{H}}\geq[Cx,x]_{\mathcal{H}}$$

for every $s\in\mathcal{S}$. Then, $[A_{/[\mathcal{S}]}x,x]_{\mathcal{H}}\geq[Cx,x]_{\mathcal{H}}$. On the other hand, if $x\in\mathcal{S}$ then $Q^{\#}AQx=0$ for every $Q\in\mathcal{Q}(\mathcal{H})$ with $N(Q)=\mathcal{S}$. Therefore,

$$[Cx,x]_{\mathcal{H}}\leq[Q^{\#}AQx,x]_{\mathcal{H}}=0.$$

But $A_{/[\mathcal{S}]}x=DP_{\mathcal{M}^{[\perp]}//\mathcal{M}}D^{\#}x=0$ because $D^{\#}x\in\mathcal{M}$. Thus, $[A_{/[\mathcal{S}]}x,x]_{\mathcal{H}}=0\geq[Cx,x]_{\mathcal{H}}$. Since $x\in\mathcal{H}$ was arbitrary, $A_{/[\mathcal{S}]}\geq_J C$. So,

$$A_{/[\mathcal{S}]}=\inf_{\leq_J}\{Q^{\#}AQ : Q\in\mathcal{Q}(\mathcal{H}),\ N(Q)=\mathcal{S}\}.$$

□

5 Schur complements of J-positive operators in Krein spaces

By Theorem 2.3, J-positive operators have the unique factorization property. Furthermore, it is easy to see that, given a factorization as in Theorem 2.1, the vector space $\mathcal{K}$ acting as the domain of the factor can be chosen to be a Hilbert space (see Theorem 1.1 in [12]).

Let $\mathcal{H}$ be a Krein space and $A\in L(\mathcal{H})$ be J-positive. Along this section, we are going to use the following factorization of A: if $|A|=JA\in L(|\mathcal{H}|)^+$, consider the Hilbert space $\mathcal{K}=J(N(A)^{\perp})$ and $D=J|A|^{1/2}J|_{\mathcal{K}}\in L(\mathcal{K},\mathcal{H})$. Then, $N(D)=\{0\}$, $D^{\#}=J|A|^{1/2}\in L(\mathcal{H},\mathcal{K})$ and $DD^{\#}=A$.

Observe that, if $\mathcal{K}$ is a Hilbert space and $\mathcal{S}$ is any closed subspace of $\mathcal{H}$, then the subspace $\mathcal{M}=\overline{D^{\#}(\mathcal{S})}$ is a closed subspace of $\mathcal{K}$ and therefore a "Krein subspace" of $\mathcal{K}$. Thus, the Schur complement $A_{/[\mathcal{S}]}$ is well defined for every closed subspace $\mathcal{S}$ of $\mathcal{H}$ and

$$\begin{aligned}A_{/[\mathcal{S}]} &= DP_{\mathcal{M}^{\perp}}D^{\#}=(J|A|^{1/2}J)P_{\mathcal{M}^{\perp}}(J|A|^{1/2})=J|A|^{1/2}(JP_{\mathcal{M}^{\perp}}J)|A|^{1/2}=\\ &= J|A|^{1/2}P_{J(\mathcal{M}^{\perp})}|A|^{1/2},\end{aligned}\tag{5.1}$$

where $P_{J(\mathcal{M}^{\perp})}\in L(\mathcal{K})$ is the orthogonal projection onto $J(\mathcal{M}^{\perp})$. Therefore, $A_{/[\mathcal{S}]}$ is J-positive. Furthermore, notice that the operator $E\in L(\mathcal{M}^{\perp},\mathcal{H})$ defined by $Ex=Dx=J|A|^{1/2}Jx$, $x\in\mathcal{M}^{\perp}$ satisfies

$$A_{/[\mathcal{S}]}=EE^{\#},\quad\text{and}\quad N(E)=\{0\}.$$

Therefore, it is the unique factorization (up to isomorphism) of $A_{/[\mathcal{S}]}$.

Remark 5.1. Observe that $J(\mathcal{M}^{\perp})=\overline{JD^{\#}(\mathcal{S})}^{\perp}=(|A|^{1/2}(\mathcal{S}))^{\perp}$. Thus, from Eq. (5.1) and item 1. of Theorem 2.6 follows that, if $A\in L(\mathcal{H})$ is J-positive then

$$A_{/[\mathcal{S}]}=J\left(|A|_{/\mathcal{S}}\right),\tag{5.2}$$

where $|A|_{/\mathcal{S}}$ is the shorted operator (in the Hilbert space sense) of $|A|$ to $\mathcal{S}$.

Therefore, the shorting operation of a J-positive operator A in a Krein space $\mathcal{H}$ is intimately related to the shorted of the positive operator JA in the Hilbert space $|\mathcal{H}|$. The following propositions translate the classical results of Schur complements into Krein space's context. First of all, we state Douglas' theorem for J-positive operators in Krein spaces.

Theorem 5.2. *Let $\mathcal{H}$ be a Krein space and consider J-positive operators $A, B \in L(\mathcal{H})$. If $A = DD^{\#}$, $D \in L(\mathcal{K}_1, \mathcal{H})$, $N(D) = \{0\}$ is any factorization of A as in Theorem 2.1 (resp. $B = EE^{\#}$, $E \in L(\mathcal{K}_2, \mathcal{H})$, $N(E) = \{0\}$) then the following conditions are equivalent:*

1. *equation $DX = E$ has a solution in $L(\mathcal{K}_2, \mathcal{K}_1)$;*
2. *$R(E) \subseteq R(D)$;*
3. *there exists $\lambda > 0$ such that $B \leq_J \lambda A$.*

In this case, there exists a unique $X \in L(\mathcal{K}_2, \mathcal{K}_1)$ such that $DX = E$. Moreover, $N(X) = N(E)$ and $\|X\| = \inf\{\lambda > 0 : B \leq_J \lambda A\}$.

Proof. Observe that if A (resp. B) is J-positive then $\mathcal{K}_1$ (resp. $\mathcal{K}_2$) is a Hilbert space. Therefore, $D^{\#} = D^*J$ and $E^{\#} = E^*J$. So, equation $A \leq_J \lambda B$ is equivalent to $DD^* \leq \lambda EE^*$ and the results follows by Douglas' theorem [11]. □

Proposition 5.3. *If $\mathcal{S}$ and $\mathcal{T}$ are closed subspaces of $\mathcal{H}$ and $A, B \in L(\mathcal{H})$ are J-positive, then*

1. *$A_{/[\mathcal{S}]} = \max_{\leq_J} \mathcal{M}^-(A, \mathcal{S}^{[\perp]}) = \max_{\leq_J} \{X \in L(\mathcal{H}) : \ 0 \leq_J X \leq_J A, \ R(X) \subseteq \mathcal{S}^{[\perp]}\}$;*
2. *$A_{/[\mathcal{S}]} = \inf_{\leq_J} \{Q^{\#}AQ \ : \ Q \in \mathcal{Q}(\mathcal{H}), \ N(Q) = \mathcal{S}\}$;*
3. *if $A \leq_J B$ then $A_{/[\mathcal{S}]} \leq_J B_{/[\mathcal{S}]}$;*
4. *if $\mathcal{T} \subseteq \mathcal{S}$ then $A_{/[\mathcal{S}]} \leq_J A_{/[\mathcal{T}]}$.*

Proof. 1. Given $A \in L(\mathcal{H})$ J-positive and $\mathcal{S}$ a closed subspace of $\mathcal{H}$, $A_{/[\mathcal{S}]} = \max_{\leq_J} \mathcal{M}^-(A, \mathcal{S}^{[\perp]})$ by Theorem 4.3 (recall that $\mathcal{K}$ is a Hilbert space). Furthermore,

$$\mathcal{M}^-(A, \mathcal{S}^{[\perp]}) = \{X \in L(\mathcal{H}) : \ 0 \leq_J X \leq_J A, \ R(X) \subseteq \mathcal{S}^{[\perp]}\}.$$

Let $\mathcal{A} = \{X \in L(\mathcal{H}) : \ 0 \leq_J X \leq_J A, \ R(X) \subseteq \mathcal{S}^{[\perp]}\}$. If $X \in \mathcal{A}$ then $X \geq_J 0$ and it admits a factorization $X = EE^{\#}$, where $E \in L(\mathcal{K}_1, \mathcal{H})$, $N(E) = \{0\}$ and $\mathcal{K}_1$ is a Hilbert space, but we can substitute $\mathcal{K}_1$ be the Hilbert space $\mathcal{K}$ appearing in the decomposition of A. Since $X \leq_J A$ it follows that $R(E) \subseteq R(D)$ by Theorem 5.2. Thus $X \in \mathcal{I}(A)$, and the conditions $X \leq_J A$ and $R(X) \subseteq \mathcal{S}^{[\perp]}$ implies that $X \in \mathcal{M}^-(A, \mathcal{S}^{[\perp]})$.

On the other hand, if $X \in \mathcal{M}^-(A, \mathcal{S}^{[\perp]})$ then there exists $E \in L(\mathcal{K}, \mathcal{H})$ such that $X = EE^{\#} = EE^*J$ because $\mathcal{K}$ is a Hilbert space. Then, $X \geq_J 0$ and, by the remaining conditions on X, $X \in \mathcal{A}$. Therefore, $\mathcal{M}^-(A, \mathcal{S}^{[\perp]}) \subseteq \mathcal{A}$.

3. If $A \leq_J B$ then $|A| = JA \leq JB = |B|$. By Theorem 2.6, $|A|_{/\mathcal{S}} \leq |B|_{/\mathcal{S}}$ and therefore $A_{/[\mathcal{S}]} = J(|A|_{/\mathcal{S}}) \leq_J J(|B|_{/\mathcal{S}}) = B_{/[\mathcal{S}]}$ (see Eq. (5.2)).

Items 2. and 4. follows analogously. □

The following proposition generalizes item 3. of Theorem 2.6:

Proposition 5.4. *Let $\mathcal{S}$ be a subspace of $\mathcal{H}$ and $A \in L(\mathcal{H})$ a J-positive operator. If $A = DD^{\#}$ (with $\mathcal{K}$ a Hilbert space, $D \in L(\mathcal{K}, \mathcal{H})$, $N(D) = \{0\}$) and $A_{/[\mathcal{S}]} = EE^{\#}$ (with $\mathcal{E}$ a Hilbert space, $E \in L(\mathcal{E}, \mathcal{H})$, $N(E) = \{0\}$) then*

$$R(E) = R(D) \cap \mathcal{S}^{[\perp]}.$$

Proof. If $A = DD^{\#}$ with $D \in L(\mathcal{K}, \mathcal{H})$, $N(D) = \{0\}$ then $A_{/[\mathcal{S}]} = FF^{\#}$ where $F \in L(\mathcal{M}^{\perp}, \mathcal{H})$ is defined by $Fx = Dx$ for $x \in \mathcal{M}^{\perp}$. Thus,

$$R(F) = R(DP_{\mathcal{M}^{\perp}}) = D(\mathcal{M}^{\perp}) = D(D^{-1}(\mathcal{S}^{[\perp]})) = R(D) \cap \mathcal{S}^{[\perp]},$$

and, by Remark 2.2, $R(E) = R(F) = R(D) \cap \mathcal{S}^{[\perp]}$. □

Proposition 5.5. *Let $\mathcal{H}$ be a Krein space and $A \in L(\mathcal{H})$ a J-positive operator. If $\mathcal{S}_1$ and $\mathcal{S}_2$ are closed subspaces of $\mathcal{H}$ such that $\mathcal{S}_1 + \mathcal{S}_2$ is closed then*

$$A_{/[\mathcal{S}_1+\mathcal{S}_2]} = (A_{/[\mathcal{S}_1]})_{/[\mathcal{S}_2]} = (A_{/[\mathcal{S}_2]})_{/[\mathcal{S}_1]}.$$

Proof. Suppose that $\mathcal{S}_1$ and $\mathcal{S}_2$ are closed subspaces of $\mathcal{H}$ such that $\mathcal{S}_1 + \mathcal{S}_2$ is closed. Consider $|A| = JA \in L(|\mathcal{H}|)^{+}$. Then, by item 4. of Theorem 2.6, $|A|_{/\mathcal{S}_1+\mathcal{S}_2} = (|A|_{/\mathcal{S}_1})_{/\mathcal{S}_2} = (|A|_{/\mathcal{S}_2})_{/\mathcal{S}_1}$. Therefore, by Eq. (5.2),

$$A_{/[\mathcal{S}_1+\mathcal{S}_2]} = J(|A|_{/\mathcal{S}_1+\mathcal{S}_2}) = J[(|A|_{/\mathcal{S}_1})_{/\mathcal{S}_2}] = (J(|A|_{/\mathcal{S}_1}))_{/[\mathcal{S}_2]} = (A_{/[\mathcal{S}_1]})_{/[\mathcal{S}_2]}.$$

Analogously, $A_{/[\mathcal{S}_1+\mathcal{S}_2]} = (A_{/[\mathcal{S}_2]})_{/[\mathcal{S}_1]}$. □

In what follows, given a sequence $(T_n)_{n\in\mathbb{N}}$ of J-positive operators, the notation $T_n \overset{\text{J-SOT}}{\searrow} T$ stands for $T_n \xrightarrow[n\to\infty]{\text{SOT}} T$ and $T_n \geq_J T_{n+1} (\geq_J T)$ for every $n \in \mathbb{N}$.

Observe that, $T_n \overset{\text{J-SOT}}{\searrow} T$ if and only if $JT_n \overset{\text{SOT}}{\searrow} JT$: Indeed, if $T_n \overset{\text{J-SOT}}{\searrow} T$ then $T_n \xrightarrow[n\to\infty]{\text{SOT}} T$ and $T_n \geq_J T_{n+1}$ ($\geq_J T$). Equivalently, $JT_n \xrightarrow[n\to\infty]{\text{SOT}} JT$ (because J is invertible) and $JT_n \geq JT_{n+1}$ ($\geq JT$), i. e. $JT_n \overset{\text{SOT}}{\searrow} JT$.

The next proposition follows easily using the remark above and Propositions 2.7 and 2.8.

Proposition 5.6. *Let $\mathcal{H}$ be a Krein space.*

1. *If $(A_n)_{n\in\mathbb{N}}$ is a sequence of J-positive operators in $L(\mathcal{H})$ such that $A_n \overset{\text{J-SOT}}{\searrow} A$, then*

$$A_{n\,/[\mathcal{S}]} \overset{\text{J-SOT}}{\searrow} A_{/[\mathcal{S}]}.$$

2. *If $(\mathcal{S}_n)_{n\in\mathbb{N}}$ and $\mathcal{S}$ are closed subspaces of $\mathcal{H}$ such that $\mathcal{S}_n \subseteq \mathcal{S}_{n+1}$ for every $n \in \mathbb{N}$ and $\mathcal{S} = \overline{\bigcup_{n\in\mathbb{N}} \mathcal{S}_n}$, then $A_{/[\mathcal{S}_n]} \overset{\text{J-SOT}}{\searrow} A_{/[\mathcal{S}]}$ for every J-positive operator $A \in L(\mathcal{H})$.*

Remark 5.7. Example 2.9 can be modified to prove that item 2 of Proposition 5.6 is not true if $\mathcal{S}_n \supseteq \mathcal{S}_{n+1}$ for every $n \in \mathbb{N}$ and $\mathcal{S} = \bigcap_{n\in\mathbb{N}} \mathcal{S}_n$.

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References

[1] W. N. Anderson Jr., *Shorted operators*, SIAM J. Appl. Math. 20 (1971) 520–525.

[2] W. N. Anderson Jr. and G. E. Trapp, *Shorted operators II*, SIAM J. Appl. Math. 28 (1975) 60–71.

[3] T. Ando, *Linear operators on Krein spaces*, Hokkaido University, Sapporo, Japan, 1979.

[4] T. Ando, *Generalized Schur complements*, Linear Algebra Appl. 27 (1979), 173–186.

[5] J. Antezana, G. Corach and D. Stojanoff, *Spectral Shorted Operators*, Integ. equ. oper. theory 55 (2006), 169–188.

[6] J. Antezana, G. Corach and D. Stojanoff, *Bilateral shorted operators and parallel sums*, Linear Algebra Appl. 414 (2006), no. 2-3, 570–588.

[7] J. Bognár, *Indefinite inner product spaces*, Springer-Verlag, 1974.

[8] J. Bognár and A. Krámli, *Operators of the form C^*C in indefinite inner product spaces*, Acta Sci. Math. (Szeged) 29 (1968), 19–29.

[9] G. Corach, A. Maestripieri and D. Stojanoff, *Oblique projections and Schur complements*, Acta Sci. Math. (Szeged) 67 (2001), 337–256.

[10] G. Corach, A. Maestripieri and D. Stojanoff, *Generalized Schur complements and oblique projections*, Linear Algebra Appl. 341 (2002), 259–272.

[11] R. G. Douglas, *On majorization, factorization and range inclusion of operators in Hilbert space*, Proc. Amer. Math. Soc. 17 (1966), 413–416.

[12] M. A. Dritschel and J. Rovnyak, *Operators on indefinite inner product spaces*, Fields Institute Monographs no. 3, Amer. Math. Soc. Edited by Peter Lancaster 1996, 3, 141–232.

[13] M. A. Dritschel and J. Rovnyak, *Extension theorems for contraction operators on Krein spaces*, Oper. Theory Adv. Appl. 47 (1990), 221–305.

[14] S. Hassi and K. Nordström, *On projections in a space with an indefinite metric*, Linear Algebra Appl. 208-209 (1994), 401–417.

[15] I. S. Iokhvidov, T. Ya. Azizov, *Linear Operators in spaces with an indefinite metric*, John Wiley and sons, 1989.

[16] M. G. Krein, *The theory of self-adjoint extensions of semibounded Hermitian operators and its applications*, Mat. Sb. (N. S.) 20 (62) (1947), 431–495.

[17] P. Massey and D. Stojanoff, *Generalized Schur Complements and P-Complementable Operators*, Linear Algebra Appl. 393 (2004) 299–318.

[18] E. L. Pekarev, *Shorts of operators and some extremal problems*, Acta Sci. Math. (Szeged) 56 (1992) 147–163.

[19] J. Rovnyak, *Methods on Krein space operator theory*, Interpolation theory, systems theory and related topics (Tel Aviv/Rehovot, 1999), Oper. Theory Adv. Appl., 134 (2002), 31–66.